This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

VOL. XXII.

LONDON:
CHARLES & EDWIN LAYTON, 150 FLEET STREET.
PARIS: ARMAND ANGER, LIBRAIRIE DES ASSURANCES, 48 RUE LAFFITTE.
HAMBURG: W. MAUKE SÖHNE. LA HAGUE: MARTINUS NIJHOFF.

1881.

1521

[ENTERED AT STATIONERS' HALL.]

LONDON:
PRINTED BY CHARLES AND EDWIN LAYTON,
FLEET STREET.

CONTENTS OF VOL. XXII.

PAGE

A Suggestion towards adding a New Feature of Usefulness to the Institute of Actuaries of Great Britain and Ireland. By Cornelius Walford, F.I.A., F.S.S., F.R.H.S., Barrister-at-Law 1

Abstract of the Discussion on the preceding 12

Remarks by Mr. Sprague 17

Law Intelligence:—Report of the Judgment of the Master of the Rolls in the Case of The London Assurance v. Mansel 20

On the Adjustment of Mortality Tables with reference to the Weight of the Observations. By Prof. C. F. McCay 24

How should the Liability of a Life Office under its Policies be estimated? Being Remarks by Mr. J. A. Higham, Mr. A. H. Bailey, Mr. A. Baden, and Mr. T. B. Sprague 43

On the Application ov Lubbock's Formula ov Summation to find the Value ov Annuities and Assurances. By Mr. T. B. Sprague, M.A. 55

On the Rate ov Remarryge among Widoers. By T. B. Sprague, M.A., Manager ov the Scottish Equitable Life Assurance Society 77

Abstract of the Discussion on the preceding 98

Mr. Sprague's Reply 100

Fire Insurance. A Theory of Statistics. By Thomas Miller, F.I.A. 103

On the Probability that there will hereafter be Issue of a Marriage hitherto Childless. By Thomas Bond Sprague, M.A., F.R.S.E. 117

Abstract of the Discussion on the preceding 126

Mr. Sprague's Reply 128

What is a Policy legally worth to the Insured? (Reprinted from the *New York Insurance Monitor)* 132

The Position of Life and other Insurance Associations in relation to their Local Medical Examiners—Is it satisfactory? By Cornelius Walford, F.I.A., F.S.S., F.R.H.S., Barrister-at-Law 153

Abstract of the Discussion on the preceding 170

Extracts from Correspondence in the *Insurance Record* on the preceding 180

Notes on the Practical Application of Mr. Makeham's Formula to the Graduation of Mortality Tables. By George King, of the Alliance Assurance Company, F.I.A., and George F. Hardy, of the British Empire Mutual Assurance Company, A.I.A. 191

Abstract of the Discussion on the preceding 226

Remarks by Mr. Sorley 230

P

Remarks as to the Influence of Marriage on the Death Rates of Males and Females; abridged from Dr. James Stark's Letters to the Registrar-General of Births, Deaths, and Marriages in Scotland 233

Form of Life Policy in the year 1721 248

On the Reserve that should be made for Policies on Recently Assured Lives. By Richard Teece, of the Australian Mutual Provident Society 250

Abstract of the Discussion on the preceding 260

Remarks by Mr. Sprague 264

Explanation ov a New Formula for Interpolation. By T. B. Sprague, M.A., Manager ov the Scottish Equitable Life Assurance Society 270

Bonuses—how Earned and how Distributed. By Archibald Hewat, F.F.A., A.I.A., Hon. Member Inst. of Accountants and Actuaries; Secretary at Glasgow to The Edinburgh Life Assurance Company 286

Observations on the Graduation of Mortality Tables; with special reference to the Conditions under which certain Methods are to be preferred. By James Sorley, of the Life Association of Scotland, Edinburgh, Fellow of the Institute of Actuaries 309

Abstract of the Discussion on the preceding 340

Note on the Calculation of Logarithms to a large number of places by means of Factor Tables. By J. W. L. Glaisher, M.A., F.R.S., Fellow of Trinity College, Cambridge 346

Note on the preceding by Mr. Gray 350

On the Rates ov Marryge among Widos and Spinsters. By T. B. Sprague, M.A., F.I.A., F.R.S.E. 352

On the Tradition Question in Probabilities. By Alexander Macfarlane, M.A., D.Sc., F.R.S.E. 356

Note on the Probability that a Marriage entered into by a Man above Forty will be Fruitful. By Thomas Bond Sprague, M.A., F.R.S.E. 359

Scheme for the Encouragement of Life Insurance among the Employees of Life Offices 365

On the Application of the Arithmometer to the Construction of Tables of the Values of Endowment Assurance Policies. By David Carment, of the Australian Mutual Provident Society, Sydney, N.S.W.; Fellow of the Institute of Actuaries, and of the Faculty of Actuaries in Scotland 368

Opening Address by the President, A. H. Bailey, Esq. 381

On the Construction and Use ov a Series ov Select Mortality Tabls to be employd in combination with the Institute $H^{M(5)}$ Tabl. By T. B. Sprague, M.A., Manager ov the Scottish Equitable Life Assurance Society.—Part II—Construction (*continued*); Part III—Use 391

Abstract of the Discussion on the preceding 441

Mr. Sprague's Reply 446

ORIGINAL TABLES:—

Letter from Mr. H. Ambrose Smith on Annual and Single Endowment Assurance Premiums—Institute H^M 3 per-cent 143

HOME AND FOREIGN INTELLIGENCE:— PAGE

Friendly Societies in New Zealand 449

ACTUARIAL NOTES:—

On the Factorial Notation in Boole's Finite Differences. By Mr. Richard Wilding 60

On the Market Value of an Annuity. By Mr. Richard Wilding 61

Smart's Interest Tables. Do. 129

Formulas for Policy-Values in terms of Premiums. Do. 131

On a General Expression for the Value of an Annuity on the last *r* Survivors of *m* Lives—$\left(a^{\;r}_{\overline{wxyz\ldots(m)}}\right)$. By George King, F.I.A. 293

On Reversionary Annuities. Do. 296

CORRESPONDENCE:—

Letter from Mr. T. B. Macaulay on Retirants in the Eighth Year 134

" Mr. Wm. G. Walton on the Relation between the Value of a Policy and the Rate of Interest 135

" Major-Gen. J. Hannyngton on Mr. Gray's Methods of Constructing Life Tables 136

" Mr. T. E. Young—Can a Law of Mortality be represented in a Mathematical Form? 139

" Mr. William Robertson, M.D., on Death Rates among Infants in Scotland 140

" Mr. E. L. De Forest—Formula for the Force of Mortality 231

" Mr. F. W. Frankland—Suggestion as to Statistics of Individual Sickness 454

INSTITUTE OF ACTUARIES:—

Preliminary Examination Paper, 1879 63

Intermediate " " 1879 65

Proceedings of the Institute, Session 1878–9 67

Annual Report 69

Income and Expenditure for the Year ending 31 March 1879, and Balance Sheet, 31 March 1879 72

List of all Members who have passed the Examinations of the Institute 73

Preliminary Examination Paper, 1880 298

Intermediate " " " 300

Final " " " 302

Proceedings of the Institute, Session 1879-80 304

Annual Report 305

Income and Expenditure for the Year ending 31 March 1880, and Balance Sheet, 31 March 1880 308

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

A Suggestion towards adding a New Feature of Usefulness to the Institute of Actuaries of Great Britain and Ireland. *By* CORNELIUS WALFORD, F.I.A., F.S.S., F.R.H.S., *Barrister-at-Law.*

[Read before the Institute of Actuaries, 28 April 1879.]

THAT the Institute of Actuaries has, during the one-third of a century of its existence (1847-79), been of real utility to the Life Offices—directly by the training of experts, and by the compilation of the Actuaries Experience Tables; and indirectly by lending practical aid to the development of the science of life contingencies—is a fact which will readily be admitted by all who are conversant with its history. That another function may be added to its utility it is my purpose on the present occasion to show.

The business of life insurance has, during the present generation, undergone a very complete transformation. The change began in the year 1844 with the passing of the Joint Stock Companies' Registration Act, 7 & 8 Vict., c. 110. Under this Act facilities were given for the formation of Joint Stock Insurance Companies such as had never before existed. The life insurance associations existing at that date were institutions of good repute; were, in fact, regarded as money-making projects of the highest class, and it was certain, therefore, that promoters would turn their attention in this direction. The failures which had resulted from the Joint Stock Company Epidemic of 1824-5 had passed out of remembrance; the course was clear for a new spurt in the same direction. Hence, a large number of new life offices were, in fact, founded; and in their struggle for existence, an amount of competition was engendered, which (while productive of good in widening the area of the insuring class, by bringing the practice down to the middle, and, gradually, to the lower ranks of the population) has eventuated in the evils whereof I here propose to speak, and concerning which it is my design to attempt a remedy.

The business of insuring lives is of necessity a conservative one: the office has ever to be on the alert against the temptation to outsiders to select against it. When Medical Selection was first brought into play, about the commencement of the present century,* it was in view of counteracting this danger of adverse selection, as against the office. In process of time, Local Medical Examiners were appointed. But, at this early period, the local agents of insurance offices were usually professional men of good repute, or traders of substance; while the medical officers were men of known position in their respective districts. Here was a double guarantee against mischief. The persons proposing for insurance were known to the agent personally; and were very frequently known also by the examining surgeon in their own locality. There was thus something like a practical assurance of good faith; and probably no better class of lives were ever selected than those introduced to the life offices during the first half of the present century.†

The effect of the competition consequent upon the rush of young life offices which sprung into existence coeval with the commencement of this second half of the present century was to bring into the field another class of agents, and, almost of necessity, another class of local medical referees. The new class of agents were those who were to make life insurance a special business—a means of gaining a livelihood. The agents desired to have for medical referees men whose interests were not forestalled in other directions, but the rather men who would identify themselves more exclusively with the new company, and with the new crusade which the aggregate of the new companies were, in fact, waging against the old conservative mode of conducting life insurance business in this country.

---

* Previously to this time, it was the practice of the life offices to request persons seeking insurance to attend before the Board, when they were accepted or rejected, according as the Directors took a favourable view or otherwise.

† The mortality of the period here spoken of may be considered as fairly represented by the [illegible]teen *Offices' Experience* (Experience Table No. 1).

I am one of those who always looked with a species of sympathy upon the advent of the new companies. My personal knowledge of a great many of the men concerned in their management, and the means of estimating in some degree the difficulties of the task in which they were engaged, combined with a strong personal desire to see the range of the insuring class widened, constituted the groundwork of this sympathy. But the very interest I took in the substantial progress of life insurance caused me to watch the progress of the experiment then under initiation narrowly, and to note the results carefully.

It was not in the nature of things that the results which were to flow from this altered mode of working the agencies of life offices should declare themselves at once. The agents would enter upon their work with honest intentions. They would not possess the knowledge required to constitute themselves ready, not to say accomplished, mediums of fraud upon the offices. This latter faculty must be reserved for a later state of development. It was so reserved. The lives introduced to the majority of the British life offices during the third quarter of the present century—which is the precise period to which the last preceding observations point—were, on the whole, satisfactory, and, if taken in the mass, would not present a greatly inferior vitality to that of the general population of the Kingdom. It will be here understood that I am speaking upon general knowledge only, based indeed upon a somewhat extended observation.*

I now arrive, in point of time and circumstance, at the phase of the business of life insurance with which it is the more especial object of this paper to deal. A change has come over the scene. The preceding review has a necessary bearing upon the circumstances which have induced this change; but there are other elements which have not yet been, but must now, be noticed. The number of the progressive life companies has been steadily decreasing for the past ten years. The agents who represented these have been drafted into the companies originally known to be of the conservative order. A general race in the direction of a *large new business* has resulted. The desire has been, and still is, that this large new business should be annually progressive. To secure this end the scale of agency commission has been steadily augmented until it has, speaking generally, reached a point beyond which it must not go. The circumstances of the country during the period immediately under review—1850-75—have been such as greatly to favour the expansion of insurance amongst the middle and operative classes. Never at any period of our history had wealth been so rapidly accumulated; and the increase of wages and other circumstances have caused an unusual proportion of such wealth to find its way into the pockets of those classes, who I believe have mainly augmented the ranks of insured lives.*

---

* The Institute of Actuaries' H^M Table (Experience Table No. 2) fairly represents the mortality of this period.

But, as already intimated, a change has come over the scene. Since 1875 there has been a steady decline of our manufacturing and industrial prosperity. An abatement of wages is following, as of necessity. The depression amongst the middle and artizan classes is beginning to be very severely felt. If the prosperity of these classes helped forward the returns of new business obtained by the life offices, the admitted adversity must operate in the contrary direction. True, there are not so many offices to share the business. The aggregate return of insurances effected in the nation may be less, while the individual returns of certain leading offices may be more. The increase of population will always imply (other things being equal) an increase in the aggregate of new insurances. The actual figures, as returned by *White's Register*, are, in fact, as follows:—

Comparative Table of new Life Insurances in the United Kingdom in the Five Years 1874-8.†

Year.	No. of Life Insurance Companies in operation.	No. of Companies contributing to the results in following columns.	No. of New Insurances.	Amount of New Insurances.	New Premiums thereon.
				£	£
1874	120	62	47,516	23,300,856	777,035
1875	114	63	48,779	21,832,643	715,178
1876	(?) 110	59	49,283	23,394,061	781,909
1877	109	62	53,293	24,221,973	808,249
1878	107	59	52,528	23,369,021	774,503

---

* *Vide* paper by Mr. Robt. Giffin, "Recent Accumulations of Capital in the United Kingdom," *Journal of Statistical Society*, vol. xli, p. 1.

† The Life Assurance Companies Act, 1870, omitting to require any return of the new business yearly transacted by the British Life Offices, renders the obtaining of complete details very difficult. It appears that it is mostly the same offices which make the returns of new business for each year embraced in the comparison.

These figures, on the whole, show a less falling off than the circumstances of the country might reasonably have led one to expect, but they only present one aspect of the case. The progressive companies have whipped up their agents to increased exertions; but I have said that in this respect I think the limit has been reached—perhaps even over reached, and there may be a necessary reaction here. The real point is, that while the aggregate of the business has been so well kept up, *what of its character?* Has there been any lowering of the standard of selection during the last few years, either voluntarily or involuntarily? That there are several ways whereby the standard of selection may be lowered, I think may be made plain in a few sentences.

The first, but least likely, way of lowering the standard of selection, is by the directors and manager voluntarily instructing the chief medical officer to be less exacting, less scrutinizing than before. The only motive, as the only justification for this, would be the expectation that an increased volume of business might result in producing an equally good average of lives. But I do not believe that in practice this would ever be taken, and I will not therefore dwell upon it further. A more likely means of lowering the standard of selection may arise by the operation of an almost involuntary, and therefore more insidious process. In the general desire for an increased volume of business, it may be that lives come to be accepted, which under previously existing circumstances would have been certainly rejected. This would be accomplished in the first instance by "rating up." But by whatever means brought about, the local agents are quick to detect the change; the local medical examiners equally so. The chief medical adviser is almost certain to draw his inspiration from the management; and thus the entire process may come about by a progress so gradual as hardly to be perceived in any of its intermediate stages. The third process may be quite involuntary, as far as the office is concerned, and accomplished entirely by the fraud of the agent, usually and perhaps necessarily, connived at by the local medical examiner. This state of things will usually follow a laxity in the head office in the matter of selection, but not always so.

The results ensuing from the second and third causes last enumerated will only become apparent in the more rapid development of claims upon the funds than had previously been the case; and hence it may not be revealed in its full force for several years—until indeed the next, or even the next succeeding periodical investigation. *In the case of fraudulent insurances, they are usually* *designed to mature early, and so the office may be put on its guard, happily, before any serious mischief has arisen to its funds;* but in the case of laxity of selection, the process is more slow, and hence the mischief may be more widespread before its consequences are discovered.

It is worth while here to consider the means by which the frauds of agents may be, and are, most readily accomplished. These are—(1) by the process of concealing habits; (2) of misrepresenting family history; and (3) by suppressing the fact of a previous attempt to insure. There is indeed (4) personation; but I am inclined to think that this is very rarely resorted to in England. In the south of Ireland it has often been heard of. Each of the three first heads may be briefly considered:—

1. *Habits.*—An unscrupulous agent soon learns how, in filling up proposal forms, to use general words such as may put the officials of the company entirely off the scent of bad habits; and if he be present when the local medical examiner has the life under examination (a most irregular, but too constant practice), he can perpetuate the misrepresentation by a clever prompting of replies; and even if he be not present he can "coach" the proposer beforehand, so that he may avoid committing himself.

2. *Family history.*—The vague filling in of details as to cause of, and ages at, death; the substitution of innocent causes, as "accident", "drowned", &c., for vital causes, afford a never-ending opportunity to the skilful agent to practice fraud against his company (especially, as already said, where he can be present at the medical examination)—an opportunity sadly too much availed of; but there is happily growing up a means of checking this form of irregularity, of which I shall hereafter speak.

3. *Previously declined lives.*—That the fact of a life having been previously rejected is often concealed at the instance of the agent, I have had many opportunities of knowing. Of course, if there is any present attempt at concealment regarding habits or family history, it is of the first consequence to prevent reference to the papers in the case of any previous proposal—and this, too, even where a life has been previously accepted; that is, where detrimental causes, as hernia, gout, &c., have developed subsequently. Hence it is only too general to declare off hand that a life has not been previously proposed; this wards off all reason for inquiry.

It may be answered that there are the "Private Friends' Reports." Apart from the fact that many offices have come to regard these of so little value as practically to discard their use, there is the certainty that the agent who is bent upon getting a life through at all odds will see the private friends, and prepare them for giving consistent answers. The old practice of sending to the private friends direct from the office, and receiving the replies there, is now too generally disused.

I ought, perhaps, here to say that while I speak of all such practices as those last enumerated as "Fraudulent"—for they *are* fraudulent, to the extent that they either do, or are designed to, impose lives upon the office, which, if the real facts were known, would not be accepted—many agents practice them, not so much with a directly fraudulent design, as for the purpose of making up a given amount of "completed business" within some given period, in respect of which a given rate, or amount of commission, is guaranteed by the office. This pressure too often leads, I believe, to the first inception of irregular practices—and these, once began, go on with ever-increasing ingenuity.

It is not, I believe, at all an unusual occurrence for a person to accept a sub-agency under a pushing district agent, for the sole purpose of getting his life insured, when it presents any departure from the true standard of health, and in such case (as also if the life be declined) his own is his first and last proposal.

PROPOSED REMEDY.

And now I have to detail the means by which I propose to remedy the evils herein treated of, and these will be enumerated under several heads.

1. *Declined Lives to be Registered.*—That every life insurance association in the United Kingdom be invited to fill in a *white* card (to be prepared and supplied to it under conditions hereafter to be named) as to every case of a *declined life* for any sum of £100 and upwards, containing (*inter alia*) the following details:—

1. Name and address of life proposed for insurance.
2. Amount £ , and class of insurance [as whole life, short term, half credit, &c.]
3. Name of agent or other person proposing the life for insurance.
4. The reasons shortly stated for declining or *suspending* the acceptance of the life, as "habits," "family history," &c., &c.
5. The proposed rating-up by the office, if declined by the proposer.
6. Name or number of contributing office.

Here it is seen that the "suspending" the acceptance for a year or any other period, say beyond 60 days, shall be regarded as equivalent to declining the life; also that where a proposal is made to rate up a life, and the proposer declines to accept within 14 days, this also shall be regarded as making it a "declined life," for the purposes herein contemplated.

2. *Misconduct of Agents.*—That every insurance association shall fill up a *blue* card (to be supplied for the purpose) stating (*inter alia*) the following details:—

1. Name and address of any agent or other person who has been discovered in the attempt to defraud an insurance office by the introduction of bad lives.
2. Name and address of agent or other person who has, either in the opinion of an office, or by direct legal proof, dishonourably broken any contract entered into with such company.
3. The name or number of the contributing company.

[NOTE.—This department I should be glad to see extended to agents defaulting in their accounts.]

3. *Legal proceedings in relation to Fraudulent Claims.*—That every insurance company shall fill in a *red* card (to be supplied for the purpose) containing (*inter alia*) the following details:—

1. Names of life insured, and person insuring, in relation to which the office is being sued for resisting payment of claim.
2. The amount insured £ , and when insurance was effected.
3. The name of the company's own solicitors defending.
4. The name of the solicitors suing for the plaintiff.
5. The name or number of the contributing office.

This last department should be, I think, extended even to reinsurances—but I do not press this. It will naturally arise for consideration in practice.

The cards for the preceding purposes I propose should be prepared and supplied by the Council of the Institute of Actuaries, or by a committee of that body specially appointed for that purpose. They should be supplied as of right to any insurance association admitted into fellowship, under conditions to be stated by the council. The cards, as filled up, to be returned to the Institute within a certain specified number of days in each case from the happening of the events therein indicated—any instance of irregularity in this respect to be deemed a breach of faith, to be visited first by fine, and upon repetition, by exclusion from further membership. The cards to be kept at the Institute in suitable cabinets, under lock and key, by an officer to be specially appointed for the purpose; and to be open to inspection within certain prescribed hours daily (Sundays and holidays excepted), by the officers and clerks of all the contributing offices. The fee charged to such contributing offices to be determined by scale as, say—

For first 3,000 policyholders . .	£2	2	0	per annum.
For each additional 3,000 policies .	1	1	0	„
No fee in the aggregate to exceed, say	6	6	0	„

An efficient officer, as I have said, would be required to take charge of these cards, and to carry out the mechanical details of their classification (with the daily additions promptly made), and to superintend their inspection by persons representing the contributing offices. The mechanical details (including a record of every card which comes in) may be kept in a most simple form, regarding which I can supply details hereafter.

I have suggested that the contributing offices may be designated by *numbers*, instead of names—the "key" to the numbers being annually given to all contributing societies. All that would be required of any searcher would thus be limited to the production of the membership card. In this manner I think the fear of any legal consequences on the part of persons whose wrong doings had become exposed might be averted. If any person obtained a view of any of the cards wrongfully, *i.e.*, without being a member of the association, he would have no key to the signification of the numbers. In practice I do not think any difficulties would arise. The various trade-protection societies carry on their useful operations with very small hindrance.

I shall, of course, be asked if I have any especial reason for deeming such a scheme of mutual protection, on the part of the life offices, of value at this particular juncture. I will frankly say that, after some twenty years of professional experience as Legal Assessor, Arbitrator, or Adviser, I have never known so many fraudulent claims against life and accident companies as the last twelve months have developed; and this, too, not in any one company or set of companies, but, as far as my experience goes, all round,—in each division of the United Kingdom. I also know, too, that concerted action would be a most material aid to the offices in checking fraud. Now, very frequently the money obtained wrongfully from one company (too prompt to pay, without adequate inquiry) is used for the purpose of fighting, and thus coercing, the more prudent company into payment.

I am fully aware that some parts of the scheme here presented are open to the charge of being crude and undeveloped; but the suggestion must be regarded as a tentative one. If the principle be accepted, I have in my mind all the necessary details; but in the practical elaboration of these, the experience of many minds may be brought to bear with advantage. It may be, too, that certain parts only of the scheme will be thought practicable at starting.

I confess that I look forward to the time when the organization herein proposed may be extended further, and when the ample materials for the investigation of family histories afforded by the rapidly growing records at the office of the General Registrar of Births, Marriages and Deaths, may be made more available than they now are for checking attempts at fraud. I have on various occasions indicated the uses of these records with well-told effect in relation to particular cases whereon I have had to advise. It will not be expected that I should be more precise here. But in the cases of obscure "Family History", why should such ready means of verification or check be overlooked? The same trained staff of the Institute might in time be found available in this direction.

I know that this protective scheme cannot be brought into play unless a considerable proportion of the life offices of the Kingdom concur and co-operate in its adoption. The *German* Association of Life Office Managers have for several years exchanged cards with each other respectively as to declined lives, with the best possible effect. The number of life offices in operation in the United Kingdom renders the individual exchange of cards with all the offices almost impracticable. *

I feel and anticipate at least one very general ground of objection to my present proposal. It has been alike the pride and boast of the British Life Offices at all times rather to pay than to demur. This, it must be admitted, is an attractive, perhaps almost a noble, sentiment; but is it always just? Upon whom does the loss resulting from the payment of doubtful or fraudulent claims fall? Is it not upon the honest policy-holders? But surely *they* are entitled to the first consideration! Is it not certain, too, that if my surmise of a steadily increasing tendency towards fraud be a true one, that the offices *must* change their tactics in this regard? It is always wise to avoid litigation, either in private or public life, where possible; but directors of life offices are trustees for *all* the policyholders, as also for the shareholders in proprietary companies. Their forbearance must, therefore, clearly have a limit. Those who are familiar with the practice of life insurance in France or in Germany, will know that a contract of this class in either of these countries is regarded as *a contract equally binding on all parties to it;* while the practice of the *American* offices, too (fairly supported, I am glad to say, by the courts of that country), has recognized, and is more clearly recognizing, the idea that the contract of insurance is one (as enunciated by our own great Lord Mansfield) founded in good faith, and hence demanding good faith from all parties associated in it. I confess I would never willingly be a party (in any capacity) to resist payment of an otherwise well-founded claim, upon a merely technical plea,—such as misstatement of age,—or for an unintentional breach of the conditions of Foreign Travel, or even of Residence. I have, and would always, advocate a merciful consideration, even in the case of "suicide", where the policy had been sufficiently long in force to preclude the idea of an intentional fraud by such means. But, in the case of a downright, deliberately-contrived concealment of important facts (whether in collusion with the agent of the company or otherwise), I would make a firm, and, as I hold, an honourable stand; and I would adopt all means necessary and essential to this end. All I have or shall here contend for is in the direction of this most justifiable—may I not say, laudable—end.

---

* Among the Fire Offices of the United Kingdom, or at least such of them as belong to the Tariff Association, there is a mode of intimation by means of cards, which is of the highest significance.

Fully conscious of the shortcomings of this proposal to meet all that is now, or may hereafter be, needed in view of the efficient protection of the vast business of life insurance against fraudulent arts; deprecating, too, anything like panic remedies; prepared, also (as I always am), for a certain amount of misconstruction, not to say misrepresentation of motives; I submit this scheme for the consideration of the members of the Institute, as also of those at present outside our organization, but who, I think, by the means proposed, may be brought advantageously within the range of its protection.

DISCUSSION.

The **PRESIDENT** (Mr. A. H. Bailey), in inviting remarks, reminded the Members of the rule of the Institute, that they are never to introduce the name of any particular society, nor to introduce the name of any particular case with which they may be acquainted.

Mr. C. D. **HIGHAM** mentioned that some system of the nature suggested by Mr. Walford is in use among the fire offices; at all events, when he was in a fire office they used, every now and then, to get a card with the name of some distinguished assured on the back of it in red ink, which was entered in a book, and kept there for future reference. He believed also that Mr. Walford's suggestion about searching family history in Somerset House registers, is now carried out by one company at any rate, which spends a very large number of shillings in looking for the records of deaths of parents and relations of the lives proposed. It appeared to him that the paper fails of its logical conclusion. Mr. Walford had shown how, in the earlier years, life assurance did very well without agents, and without medical examiners, too, for that matter; and how, later on, they also did very well in days when Members of Parliament and country bankers and solicitors did not disdain to represent companies; and how, in after years, a system of fraudulent assurances had been attempted. Fraud seems to have sprung up amongst a class of offices which Mr. Walford politely calls "progressive", who, having now progressed into stages where their agents are of no use to them, have left those agents as a legacy to more "conservative" institutions. And here every one would have expected to find some caution as to the appointment of agents, or as to pushing for business; but instead of that Mr. Walford proposes to establish a sort of Criminal Investigation Department, to be attached to this Institute, founded, as it was, for scientific purposes, and as a scientific society. Surely this is to apply the remedy to effects rather than to causes. He thought that the cause may be sought even deeper than the agents, and be found in that extraordinary delight in large figures which is so prevalent nowadays, as if in millions of assurance there must be strength, and in a multitude of policyholders there must needs be safety. While a certain number of lives is required to get an average, the pushing for business must not be carried beyond the point at which the ordinary rules of political economy show that division of labour reduces expense. Instead of this, however, office is racing with office, and agent pitting one company against another for increased allowances, and sometimes using the same lives to obtain procuration fees from more than one company. Instead of setting up this new department which Mr. Walford proposes to establish, the remedy for this increased want of scrupulous dealing should be sought in a return to more primitive systems of life assurance, remembering that, however desirable the inculcation of provident habits may be among those who are not troubled with such proclivities, life assurance companies were not established as philanthropic societies, but as
ial speculations.

IESSENT felt that it would be a good thing for the life
this country if the managers met together sometimes to

talk over some of the subjects dealt with in the paper, but regretted that a question of this kind had been introduced into the Institute of Actuaries. The Members of the Institute are not all managers of life offices. The Institute has a specific object, which is not to enable the boards of life offices to conduct their business on the most safe and sound principles; and he feared that if they were, for one moment, to admit anything like trade principles into the objects of the Institute of Actuaries, its days would be very soon numbered. He had not personally had experience of many of the sensational cases that had been talked about during the last year or two, and did not quite believe in the number and extent of these reported frauds, or think that we should get a very great deal by these registrations. But if there could be any kind of common ground upon which the managers of life offices could meet and from time to time settle upon certain rules of practice and certain questions with regard to the business, it would be a very good thing. He thought that the question of fraud need not much enter into consideration.

Mr. Morris thought it desirable that the principle of registration, as illustrated in the *Register of Shipping*, should be, in some modified form, applied to life assurance. They would have to exclude from it, however, that information which would be most useful to life insurance companies,—that is, the declined lives. How far would they be morally justified in damnifying by registration a life which for some reason has been rejected by an office, and thus debarring that life practically from the privilege of getting itself insured at all? If an attempt were made by a person, after having been once declined, to obtain the acceptance of another society by withholding the fact of the declinature, the moral objection to registration would fall away. But that lives are sometimes declined upon inadequate grounds, must, in view of the fallibility of human nature, be admitted; and the evil would *à fortiori* still more exist as to lives which had been only postponed or rated up. No doubt life assurance societies are especially subjected to the insidious attacks of fraudulent persons. Nevertheless, it would be both impolitic and unfair to assume that a deviation from the standard of health *ipso facto* implies a deviation from the standard of honesty. There is a plan open to life offices which would involve some trouble, but would enable them to check the statements of proposers about previous proposals. This is simply to register the full name, date of birth, and age of every person making a proposal to every society, with the name and address, of course, of the office to which the proposal was made, but without any further particulars, and without any indication of the result of the proposal. Of course, in a vast majority of cases the reference to this register would not be necessary; but where necessary reference could be made, and then an application to the society to which the proposal had been made, for information on the subject, would, of course, meet the requirements of the case.

Mr. W. White had looked into the history of life insurance in the two years, 1874 and 1878, taking from the best source he could at the moment reach, the companies that give yearly the number of proposals received and the corresponding number of policies issued. He found that in the year 1874, 22 companies gave an aggregate of

25,127 proposals, and of that number 20,610 were completed. In the year 1878, 30,744 proposals were made to these same 22 companies; and of that number 25,194 were completed, the percentage completed being slightly less in 1878 than in 1874 (81·95 as against 82·02); in other words, that the power of selection was slightly greater in 1878 than in 1874. Mr. Walford's paper would have had very great additional interest if he had compared the proposals received and the policies completed in the five years before 1874, with the proposals received and the policies completed in the last quinquennium. This would show whether it is the case or not that the power of selection is being diminished in this country. He did not think so himself. Although there may be great competition, as there no doubt is, and competition is right, yet he believed that the general feeling throughout the country is that we are really and decidedly progressing in the morals of life assurance.

Mr. F. A. Curtis did not think Mr. White's test quite correct; for if a larger number of cases are rejected now as compared with the proposals made, it may be that the quality of the lives proposed is not so good as it was before. A better way of judging of the selection and of the standard of the lives, is to take out the mortality of the lives that have been recently assured. If this were done, he believed they would find that the standard of selection has not lowered. In his own experience he found that the lives recently assured are quite as good as the old lives, and he had no doubt that is the experience of most societies.

Mr. R. P. Hardy had detected a fraud that morning, and had not the slightest doubt that there is a very great deal of fraud about. He had received particulars of 20 cases of attempted and completed fraud in a single year's experience of an office not doing an unusually large amount of business; and these were frauds that were only discovered incidentally. By "frauds" he meant cases where there is a deliberate lie upon the papers. Whether there was ground for rating the life for five or ten years or not, the parties in question did not allow an opportunity of judging. When twenty cases of fraud are found in an office whose business is not a pushing business—not what is called a "progressive" business—what may not be the case in those offices which are very pushing indeed? There always will be frauds, although some gentlemen do not believe in them.

Mr. Humphreys said that Mr. Walford, in his paper, had spoken of the effects resulting from the action of unscrupulous agents, but the mischief resulting from the employment of such persons might be avoided by the directors and managers being very particular in appointing their agents. If they are very careful in making thorough investigations into the character of persons offering themselves as agents, they would not lay themselves open to the frauds of unscrupulous agents.

Mr. A. Baden had great sympathy with the substance of Mr. Walford's paper. He believed that the character of insurance business has within the last few years deteriorated, and thought Mr. Walford had not in the slightest degree overstated his case. He thought, however, that it is almost a matter of impossibility that the Institute should undertake the registration which Mr. Walford proposes. The doing so would be the means of throwing down an apple of discord which would eventuate in the final dissolution of the Institute. But if some other means could be found by which this great and growing evil, as he believed it to be, could be cured, such as by registration outside the walls of the Institute, or some intercommunication among the managers of the different offices, he should hail the establishment of some such means as that with very great pleasure.

Mr. G. **Cutcliffe** feared that it would not be practicable to carry out Mr. Walford's views; but, if practicable, it would be a very great benefit to the offices. There would be the great objection that if particulars of proposals were sent in, it might be actionable to make the result known. He had had thirty-eight years' experience and had had more than 25,000 proposals passed through his hands, and he said now deliberately that for the last three or four years, especially, the character of the business has become very much deteriorated. There have been great attempts at getting policies unfairly—more so than in previous years. He could bring forward cases just in the same way as Mr. Hardy had done. He had also a very strong opinion that of late years, especially perhaps the last ten or fifteen years, business had been taken by the offices more readily. Business which would have been formerly declined, is now taken sometimes at very low rates.

Mr. H. W. **Manly** said that the bankers have their association, which meets once a week to determine upon certain rates of interest and other matters in which all are interested; there is an association of fire managers, who have adopted plans like those now proposed in order to protect themselves against frauds; and he believed the same thing is carried out at Lloyds'. There appears to be a general concensus of opinion that a similar plan, if it could be carried out amongst the life offices, would certainly add to the protection of the companies and relieve the managers of a great deal of anxiety in examining proposals. But that cannot be carried out by the Institute of Actuaries. It must be a private association, not a public one. If there is a private detective department, it must be private. He did not think that there was any fear as to its being actionable for the offices to protect themselves against fraud by furnishing each other with names. He hoped that the question would not be allowed to drop, but that some of the more influential members of the Institute would exert themselves to start an association of this kind.

Mr. F. W. **White** said there is one point which must cause some amount of difference in the business taken now, compared with that taken years ago; and that is the different relation between the agent and the medical examiner. It is very difficult now, in many cases, to get an agent to take the proposers to an examiner if he can possibly manage to get the examiner to go to the proposers. If several gentlemen are appointed in a large city—one perhaps a gentleman of eminence, and the others younger and less experienced—the agent will take all the business to the younger, and sturdily refuse to take any proposal to the older examiner. He will say that the older examiner's hours are such that the proposers find it very difficult to get there. That must tend adversely to the office. Mr. Walford has alluded to the fact that sometimes agents are present during the examination. Not long ago a medical examiner was asked how it was his report stated that the proposer was not pregnant when in reality it was the case that she was so. The medical examiner went immediately to the agent, and said, "Here is a letter from the head office. You know all about it. You had better tell them privately." The agent confessed that the question was asked in his presence, and was answered in the negative, because the lady was too bashful to say anything else. With regard to the comparative tables of new life assurances, there is a very remarkable diminution in 1875. Whereas the new assurances exceeded those of 1874 in number, they fell short by £1,500,000 in amount. It seems as though there must be a disturbing cause there.

The PRESIDENT agreed with Mr. Walford that life assurance has undergone a complete transformation in the present generation. Formerly the public insured with the office, and now they insure with the agents, and that is a retrograde movement. Formerly the object was so to conduct the business as to get a reputation for the office. The agent took a pride in his own office, and his duty in the main was to be the representative of the office in his district. All persons in that district whose lives were proposed, were bound to appear before him; and the report of the agent was not only an important matter, but *the* important matter. Then after medical examinations were practised in London, they were not usual in the country. The agent's report was the main thing upon which the lives were selected, and so lately as even in his own experience of the London Assurance, a man proposed in the country and not introduced by the agent, was required to appear before the nearest agent, and that agent had a fee of 10*s*. 6*d*. paid to him for his report, to which great weight attached, and therefore great care was taken in appointing agents. In the present day the agent is purely and simply a touter for business, and the result has been that the business has deteriorated. If the rate of an agent's allowance increases with the amount of the new business, it is not to be wondered at that he should look upon the introduction of bad lives as a good piece of business. About the main subject of the paper he could not sufficiently express his strong concurrence in the opinion that, if the Institute of Actuaries attempts to be a business organization, it will fail. Any attempt to help offices in their competition with each other for getting proposals, or in any kind of business arrangements, is out of harmony with the objects of the Institute. He believed the Fire Managers' Association to be a trade union, pure and simple, to keep up rates, and nothing more. As regards the particular dangers that Mr. Walford points out, he did not believe that they exist in any material degree. Before the money can be obtained for a fraudulent insurance, the very strong measure has to be resorted to of killing the man. He did not, therefore, believe that any considerable amount of fraud exists in life assurance, though there are, and always will be, some fraudulent cases.

Mr. Sprague requests us to print the following remarks on the subject of Mr. Walford's paper:—

I heartily approve of Mr. Walford's suggestion, that the life insurance offices of the country should combine together, for the purpose of assisting each other to detect the attempts to defraud them which unscrupulous persons are constantly making; but I think that some improvement may be made upon the details of his proposed plan, and I entirely concur in the opinion expressed by so many speakers at the Institute of Actuaries, that the Institute is not a suitable body to have charge of the matter. Mr. Walford proposes that there should be a register of declined lives, and that every contributing office should have a right to examine the register. The object, no doubt, of this arrangement would be that, whenever the officials of an office had reason to suppose that a fraud was being attempted, they might be able immediately to send off and examine the register. This precaution, however, would not meet the case which unfortunately is of constant occurrence, and of which a glaring instance was recently exposed in the law-courts, in the case of the London Assurance Corporation *v.* Mansel. I believe that it is a common thing for persons who have been declined or charged an extra premium by one office, to suppress all mention of the fact in making a subsequent proposal to another office; for instances of this kind are constantly being discovered, although no systematic means is taken for the purpose. In such cases, where the papers are all satisfactory, there is nothing to suggest a fraud or to lead a company to send and examine the register of declined lives. It appears to me therefore, that, for the effectual detection of fraud, it would be necessary that the register should be searched weekly for every proposal made to each of the contributing companies.

The register should contain the result of each proposal not accepted at the ordinary rate, and should state whether it was accepted at an increased rate, delayed indefinitely, or declined. It would, I think, be unnecessary to have the further information proposed by Mr. Walford as to sum assured, class of insurance, name of agent or other person proposing the life; and I think it would be decidedly objectionable to register any reason for declining or rating up a life. If Office A has ascertained that a life proposed to it for insurance, has been previously declined or charged an entra premium by Office B, that is all the information which it can properly expect to receive from the register; and for any further information its officials should, as at present, apply direct to Office B. I do not think it would even be desirable, when a life is rated up, to state the addition made to the age or the premium; but I believe it would be better simply to enter in the return that the life was accepted at an extra premium.

The weekly search of the register could be made by each office for itself, if thought desirable; but I believe it would be more convenient that it should be made by the registrar on behalf of the offices that chose to take advantage of his services for the purpose. I would propose that each contributing office should send in weekly a sheet containing the names of the lives proposed for insurance and such other particulars as may be agreed upon, and that the registrar should copy out upon a card the name of each life not accepted at the ordinary rate of premium, and place the card in its proper alphabetical order among the other cards. In doing this he would at once see whether the life named in the card was already included in his register as declined or rated up, and he should note the fact on the list. He should also search his register for each of the lives accepted at the ordinary rate, and note the result of the search on the list. When he has completed his registration and search, he should initial the sheet, and return it to the office from which he received it. It would then bear a note against each name, to the effect either that the name of the life does not already appear in the register, or that the name appears in the register as having been previously proposed to another office or other offices with the result stated on the sheet. In this way each contributing office would have the means of comparing the information furnished by the registrar with the answers given by the proposers to the usual question as to former proposals. The office would thus realize directly the advantage of contributing to the register, and would feel that those advantages would be lessened, if not altogether lost, by any irregularity of procedure. Thus the self-interest of all the offices would be a sufficient inducement to them to contribute regularly; the question of breach of faith among the contributing offices could not arise through any irregularity to contribute, and there would be no occasion to provide against such breach of faith, by attaching to it the penalty of a fine or exclusion from further membership. For the complete success of the registration it would be necessary that the offices should agree not to allow a proposal to be withdrawn when there is a prospect of its being declined, but to have all such cases formally submitted to the directors, and either definitely accepted or rejected. The contributing offices should have a right at any time to search the register themselves, if for any reason they thought that desirable; but it should be distinctly provided that no other person than the contributing offices and their officials should, under any pretext, have access to the register; and, this being done, I do not think there would be any danger of unpleasant legal consequences from the registration of the results of the proposals made to the different offices.

It will probably be thought by some managers of offices that participation in the registration would involve an undue disclosure to their rivals, of the character of the business of their offices. I do not myself see how any bad result could follow from this; but, in order to meet the objection, it would be desirable that the register open to the inspection of the offices should not contain the name of the office declining any life contained in it. This could very easily be arranged. Each declined life should receive a number, and this number be written on the card, but the name of the office should not be written on the card. It would then be necessary to have a numerical list of the declined lives, which would state the name of the office declining; and this list should be kept strictly in the personal custody of the registrar. If then any office, by searching the register, finds that a certain life has been declined, and desires to know which office declined the life, the only way of obtaining this information would be to ask the registrar to look in his numerical list.

It would obviously be just that the expense attaching to the registration should be borne by all the contributing offices, in proportion to the number of names they respectively contribute; for the labour caused by each office would be in direct proportion to the number of names, and it may be presumed that on the average the benefit derived by each office would also be in proportion to the number of names.

It will now, I think, be clear from the description I have given of the duties and function of the registrar, that he must simply be an official employed by the contributing offices, paid by them, and responsible only to them. It would therefore be entirely impracticable to have the work in any way under the charge of such a body as the Institute of Actuaries, over which the contributing offices could not exercise effective control. It is a very laudable wish on Mr. Walford's part to extend the usefulness of the Institute, but it seems to me that his present proposal is incompatible with the proper functions of the Institute, and that the adoption of it would imperil the prosperity, if not the existence of the Institute. There is a distinction which does not seem to have been brought out by any of the speakers at the discussion on Mr. Walford's paper. The Institute of Actuaries is a scientific society, consisting of actuaries, managers, and other persons, acting in their individual capacity; but in any association for business purposes it would be necessary that the persons composing it, whether managers or actuaries, should sink their individuality and bear in mind that they are there, not to speak their own sentiments, but to represent their respective offices, and to carry into effect the instructions they have received from their Directors. The Institute of Actuaries has now for a long series of years shown a remarkable vitality, and has discharged very useful public functions, by enabling the younger men in the profession to give public proof of their abilities, and by affording a field for the impartial discussion of controverted questions of principle; and its success has probably been in great measure due to its steadily abstaining from any attempt to exercise discipline, either over the insurance companies as to their manner of conducting their business, or over its own members as to the professional advice they may give when consulted. Some attempts which were made in this direction in the earlier years of the Institute, were, I believe, not very successful in their object, but, for a time, interfered more or less with the prosperity of the Institute and its usefulness in other ways. It is to be hoped that the present generation of actuaries will scrupulously abstain from imitating these mistakes of their predecessors.

I have no sympathy with the remarks made by several of the speakers as to the competition between life offices. I believe competition to be in itself healthy for all parties, both for the public and for the offices; and it is, in my opinion, a great mistake to talk about existing competition as if it were a thing to be deplored. It is, of course, possible for competition to proceed to an unwise length; but those who are of opinion that this is now the case should not indulge in mere general complaints, but should specify the particular points they object to; and until this is done I shall continue to consider their complaints as undeserving of serious attention.

LAW INTELLIGENCE.

High Court of Justice, Chancery Division, Rolls Court, *February* 1879.

THE LONDON ASSURANCE *v.* MANSEL.

This was an action by the above-mentioned Corporation for a declaration that a contract for a life assurance on the defendant's life was void, on the ground of concealment of material facts, and that the plaintiffs were not bound to deliver the defendant his policy. In the proposal for assurance the defendant was asked the following questions:—"Has a proposal ever been made on your life at any other office or offices? If so, where? Was it accepted at the ordinary premium, or at an increased premium, or declined?" The defendant answered the questions as follows:—"Insured now in two offices for £16,000 at ordinary rates. Policies effected last year." The defendant signed the usual declaration at the foot of the proposal, declaring that the "particulars given were true, and that he agreed that the proposal and declaration should be the basis of the contract between him and the *London Assurance*". The defendant, on being examined by the medical officer of the plaintiffs, was asked, among other questions, "Whether he had been rejected by any other office", and the defendant in reply answered "No". The plaintiffs subsequently accepted the defendant's proposal, and the first year's premium was paid; but shortly afterwards discovering that the defendant's life had been proposed to, and rejected by, several other offices, tendered back the premium, which the defendant declined to receive. The defendant contended that his life was never refused by any office, and that the fact that proposals for assurances on his life had been declined, was not a material fact in a contract of life assurance.

The present proceedings were taken under the new system provided by the Judicature Act, whereby the judgment of the Court may be asked on the statements in the claim and defence without the production of evidence.

The following is the Judgment as transcribed from the short-hand notes of Messrs. Knight and Wright, 88 Chancery Lane:—

JUDGMENT.

The Master of the Rolls.—The action in this case is to set aside an agreement for assurance for life on the ground of concealment of a material fact in effecting the assurance. The first question to be decided is—what is the principle on which the Court acts in setting aside contracts of insurance. Now, as regards the general principle, I am not prepared to lay down the law as showing any difference in substance between one contract of assurance and another. Whether it is for life or fire or marine, I take it good faith is required in all cases, and though there may be certain circumstances, from the peculiar nature of marine insurance, required to be disclosed, and which do not apply to other contracts of insurance, that is rather in my opinion an illustration of the application of the principle than a distinction in principle.

But I think the law has been laid down very often, and I am going to refer to two or three statements of it which at all events are binding on me. Lord Cranworth, then Mr. Baron Rolfe and Lord Commissioner, says this about insurances in the case of Dalglish *v.* Jarvie, which is reported in the 2nd Macnaghten and Gordon Reports. It has nothing to do with insurance, but the dicta are at page 243. The case referred to the principles on which a special injunction ought to be granted *ex parte*, and Mr. Baron Rolfe says this:—"Upon one point it seems to me proper to add thus much, namely, that the application for a special injunction is very much governed by the same principles which govern insurances, matters which are said to require the utmost degree of good faith, *uberrima fides*. In cases of insurance the party is required not only to state all matters within his knowledge which he believes to be material to the question of the insurance, but all which in point of fact are so. If he conceals anything which he knows to be material, it is fraud. Besides that, if he conceals anything which may influence the rate of premium which the underwriter may require, although he does not know it would have that effect, such concealment entirely vitiates the policy."

Now it is to be observed that Lord Cranworth, then Mr. Baron Rolfe, speaks "In cases of insurance": he does not say one kind of insurance or another kind of insurance, but it is the more valuable because he is stating the law as settled as a mere illustration of the similar law which he considers to apply to applications for special injunctions when a man comes for one *ex parte*. If he conceals anything he knows to be material, it is fraud; and if he conceals anything that may influence the rate of premium, although he does not know it, it still vitiates the policy.

In another case, which again is not directly in point, turning on a contract, the case of Owens *v.* Hayworth, in the 10th Meeson and Wellsby, 147, we have a dictum of Mr. Baron Parke. It was a case of mercantile contract, not an insurance contract. Mr. Baron Parke says, at page 157, "The case of a policy of insurance does not appear to me to be analogous to the present. Those instruments are made upon an implied contract between the parties that every thing material known to the assured should be disclosed by them. That is the basis on which the contract proceeds, and it is material to see that it is not obtained by untrue representation or concealment in any respect"—that means, of course, concealment in any *material* respect.

Then, in the case to which I have already referred, which was an action on a policy of insurance, Lindemam *v.* Desborough, reported in the 8th Barnewall and Cresswell, Lord Tenterden says, at page 591, "Then it is said that the party is not bound to do more than answer the questions proposed, unless he can be charged with some fraudulent concealment. Admitting this not to fall within any of the specific questions, which is not by any means clear, still the general question put by the office requires information of every fact which any reasonable man would think material." That passage shows that the non-answering of a specific question would, in Lord Tenterden's opinion, amount to concealment, if the man knew the fact and was able to answer it. Mr. Justice Bayley says—"I think that in all cases of insurance" (now he goes on to add what Lord Cranworth did not add, but what he meant), "whether of ships, houses, or lives, the underwriter should be informed of every material circumstance within the knowledge of the assured; and that the proper question is, whether any particular circumstance was in fact material, not whether the party believed it to be so. The contrary doctrine would lead to the suppression of information, and it would often be extremely difficult to show that the party neglecting to give the information thought it material. But if it be held that all material facts must be disclosed, it will be the interest of the assured to make full and fair disclosure of all the information within their reach." Then Mr. Justice Littledale says—"I am of the same opinion. It is the duty of the assured in all cases to disclose all material facts within their knowledge. In cases of life insurance certain specific questions are proposed as to points of fact in general to all mankind, but there may be also circumstances affecting the particular individual which are not likely to be known to the assurers, which, had they been known, would no doubt have been made the subject of specific inquiries." He puts it stronger, therefore, when it is a specific inquiry.

Now I come to the facts of this case, which certainly appear to me to be very plain and clear indeed. The office of the *London Assurance* asks these questions—"Has a proposal ever been made on your life at any other office or offices? if so, where? was it accepted at the ordinary premium, or declined?" and there is an agreement at the end—"That this proposal and declaration shall be the basis of the contract between the assured and the company." Now here is the answer—"Insured now in two offices for £16,000 at ordinary rates, policies effected last year." It is to be observed that the man proposing the assurance, who knows the facts, does not answer the question. The question was—Has a proposal ever been made at any other office or offices? if so, where? He does not state—I proposed to half-a-dozen offices, which was the truth, but simply says—Insured now in two offices—which, of course, must have been intended to represent an answer, and therefore would mislead the person receiving it, who did not look at it with the greatest attention, into the belief that he was insured in two offices, and that they were the only proposals he had made. "Was it accepted at the ordinary premiums or an increased premium?" His answer is—"At ordinary rates." That is the answer to the second, but he has not answered the question "or declined". The inference, therefore, which must have been intended to be produced on the mind of the person reading the answer, was that it had not been declined. And in my opinion that is the fair meaning of the answer, and it is not to be allowed for the assured to say, "I did not answer the question"; but if it were so, it would make no difference, because, if a man purposely avoids answering a question, and thereby does not state a fact which it is his duty to communicate, that is concealment. Concealment, properly so called, means non-disclosure of a fact which it is a man's duty to disclose; and it was his duty to disclose the fact if it was a material fact.

Well, now, is it a material fact? I should say no human being acquainted with the practice of companies, or of insurance societies or underwriters, could doubt for a moment that it is a fact of great materiality—a fact upon which the offices place great reliance. They always want to know what other offices have done with respect to the lives. But in this case there could be no question as to its materiality. In the first place, we have this in the answer—"The defendant admits that proposals were made to the *Clerical, Medical and General* Life Assurance Society, The *Scottish Amicable* Life Assurance Society, and the *Law* Life Assurance Society, for an assurance on his life, such proposals were declined." There are three admitted by the answer declined in the very words of the question, and then he goes on to say, in para. 15: "The defendant is not, and never has been, of intemperate habits of life; and although proposals for assurances on the defendant's life had been made to and declined by the several offices in the Statement of Claim mentioned, the defendant's life was never rejected by an office, but was passed as a first-class life by every medical officer who examined him." Then he tells us this also, that the *English and Scottish* Life Assurance Office (which is the 4th office) passed his life as a first-class life; but they reserved the right of declining to complete the transaction at any time before the receipt of the premium. "On the 15th July the last mentioned office, having learned that the *Equity and Law* Life Assurance Company had decided not to increase their risk on the defendant's life, and would not take any part of the new risk, exercised their right of declining to complete the transaction." That is the only cause; they had heard another office had declined. There are two more offices adding them to the three admitted. In the 12th para. we have an admission by the defendant that no less than five insurance offices had declined to accept his life.

Now to suppose that any human being who knows anything about life assurance—that any decent special juryman could for a moment hesitate as to the proper answer to be given to the question when you go to the insurance office and ask for an insurance on your life, Ought you to tell them that your proposals have been declined by five other assurance offices?—is, I say, quite out of the question. There can be but one answer—that a man is bound to say, "My proposals have been declined by five other offices. I will give you the reasons, and show you that it does not affect my life," as he admits it to be by this answer, but of that the office could judge. There can be no doubt, as a proposition to be decided by a jury, that such a circumstance is material. But, in fact, I have elements here admitted on the pleadings for deciding that question, quite irrespective of the ordinary knowledge of the practice of mankind in respect of these matters, which is to be imputed to a good special juryman; because I have these two things admitted:—first of all, that the proposal which forms the basis of the contract asks a question:—Has a proposal been declined? Now, of course, if it is to form the basis of the contract, it is material, because, as was in a case in the House of Lords, of Henderson *v.*

Fitzgerald (4th House of Lords, 484), where it is part of the contract, the other side cannot say it is not material. So here we have the proposal as the basis of the contract. It is impossible for the assured to say, that the question asked is not a material question to be answered and the fact which the answer would bring out is not a material fact. Secondly, we have this—that within his own knowledge the *English and Scottish Law* Life Assurance Office, having accepted his life, which had been duly passed by their medical officer as a first-class life after examination, and merely reserving a right to decline, when they found that one other office—not five, but one—had declined the life (or rather the proposal), at once withdrew from their acceptance, and declined his proposal. So that this defendant had the strongest reasons for believing, from actual knowledge, that the fact of a proposal having been declined was a most material circumstance, and would have the greatest effect on the mind of the proposed assurers.

It seems to me, as I said before, a very plain and clear case, and the plaintiffs are consequently entitled to Judgment.

Mr. Davey.—The plaintiffs, of course, are willing to return the premium. I propose, if your Lordship thinks fit, to take an order in this form. The plaintiffs being willing and hereby offering to return the premium of £215, declare that the acceptance by the plaintiff of the life was void and of no effect, and that they are not bound to deliver the policy, and that the contract be delivered up to be cancelled. Then to tax the plaintiffs' costs, and retain them out of the premium, and pay the balance to the defendant.

The Master of the Rolls.—Yes.

On the Adjustment of Mortality Tables with reference to the Weight of the Observations. By Prof. C. F. McCay.

The law of mortality, represented by the formula $l_x = ka^x b^{c^x}$, in which l_x represents the number of the living in a stationary population at the age x, and a, b, c, and k are constants, is usually referred to as Mr. Makeham's discovery; but Prof. Bartlett, in his report on the Experience of the Mutual Life of New York for the thirty-one years from 1843 to 1873, ascribes it to Gompertz. To this distinguished actuary is certainly to be attributed the announcement in the *Philosophical Transactions* for 1825, that the decrements of human life correspond to the effects of two specified and definite forces, from which this law is a simple and immediate consequence. Not only did he announce that these decrements were due to these forces, but he distinctly developed and proved the separate effects of each. The first, irresistible in its nature, produced equal destruction at all ages. The deaths it caused were in exact proportion to the numbers exposed. It was like a stroke of lightning, or a fatal poison, or the ossification of the heart, or asphyxia, equally destructive to the young and the old, the vigorous and the weak. The other was resistible, and therefore not at once fatal, but like a battering ram striking the walls of a fortress, first weakening and then destroying. This force, acting incessantly, and including all the external and internal ills to which the body is exposed, except the irresistible causes which were alike fatal to all, was constant in its energy, but produced greater effects as the vital force decreased with age and feebleness. Mr. Gompertz shewed that the effects produced by this force would be represented by the formula Bc^x, and therefore the joint effect of both would be $A+Bc^x$. He then proved that this latter force, as it acted incessantly, led to the formula $l_x=kb^{c^x}$. This result was then compared with the best tables of mortality, and it was found to conform to them through a large period of life. In fact, he had first discovered in the tables this law of the increments of mortality, and then imposed on the facts the forces that would produce them. The steps of his investigation seem to have been, —first, the discovery of the law of geometrical progression in the rates of mortality in the several tables then known and esteemed; second, the inference that this result was the effect of a constant force like gravity; and third, the suggestion that the other force which was necessary to be added to the first to explain the phenomena, produced equal effects at all ages, in exact proportion to the numbers exposed.

Mr. Gompertz did not combine the two forces and announce the formula for the numbers of the living at all ages, as he did for the effects of the first force. This is very easily done for both, in precisely the same way as he did it for the one.

Instead of $dl_x \div l_x$ being $=Bc^x dx$ it is $=(A+Bc^x)dx$,
,, λl_x ,, $=Dc^x+C$,, $=Ax+Dc^x+C$,
,, λl_x ,, $=c^x\lambda b+\lambda k$,, $=x\lambda a+c^x\lambda b+\lambda k$,
,, l_x ,, $=kb^{c^x}$,, $=ka^x b^{c^x}$.

The influence of the former force, in the formula for the living, was not used by Gompertz; because his object was to prove the existence of the second force, and he wished a formula by which he could test its effects. The other force he merely announced, but did not compare its effects with the tables.

One of the illustrations that he gave of the second force, which he was mainly intent on establishing, may be used to illustrate both. If a receiver is exhausted by a number of air-pumps, some of which make only one stroke and then stop, while others continue their successive strokes indefinitely, the effects of the first set would be A, and of the second Bc^x, in which A represents the volume of air withdrawn by the first pumps, B the volume by the second, and c the ratio between the capacity of the receiver and B; the effect of the first being A and constant, and that of the second B in the first stroke, Bc in the next, Bc^2 in the third, and generally Bc^x.

It is thus evident that Mr. Gompertz distinctly announced the two forces that give the law used by Prof. Bartlett, and was probably aware of the formula for the numbers of the living which they would require. He did not, however, attempt to prove the existence of the first force, and it was reserved for Mr. Makeham to undertake this. From the Reports of the Registrar-General, and from the several tables of mortality, Mr. Makeham has given many confirmations of the action of the second force, but it is only in the observations of the twenty English life offices, published in 1869, that he finds any proof that the first is the only other force required to represent the facts observed.

But it is extremely improbable that so complex an organism as the human body, exposed to so many causes of destruction from within and from without, and even from the decisions of the mental element of our nature, should decay and perish under the influence of two such simple forces as those announced by Gompertz; and Mr. Woolhouse has shown that they only give approximate results for the experience of the twenty offices, not merely in the total observations, but in the separate years of insurance. While he commends the formula as very advantageous for abridging the labour of computations dependent on many lives, he intimates that it will not be safe to trust to it in the construction of tables of mortality from the observed deaths and exposures, and that it does not conform to the experience of the twenty offices in the later years of insurance, as closely as in the general observations of all the years; whereas, if it were the true law of nature, it would represent the mortality of each separate year of insurance, where the exposures are all in similar circumstances and in the precise condition required to test the accuracy of the law, better than on the mixed and irregular exposures of all the years, where the entrants at any age are in a very different condition from the average at the same age, and where the sound lives who are in the first year of insurance, bear very unequal proportions to the total exposures of each age.

But while the forces of Gompertz may not be the only element of destruction, and while at some periods of life the mortality may increase faster, and at others slower, than these two causes would require, they give a close approximation to the truth, and may be of service in the construction of tables of mortality, not by trusting them alone, but by determining the results which these two principal forces would produce, and then discussing the residual phenomena as variations from the law. Although the motion of the earth in her orbit does not conform exactly to an ellipse, the determination of the elliptic orbit which the sun would produce, enables the astronomer to determine the perturbations caused by the planets. So, likewise, in the discussion of many other natural phenomena, it is common to find the separate effects of several partial forces, and thus approximate the laws which may govern the facts. In like manner, the approximation given by the forces of Gompertz, may aid in determining the progressive increase of mortality from youth to old age.

As an example of this, it is proposed to use this law in the construction of a table for the Healthy Male and Female Lives of the twenty English life offices. In the table now in use, no greater influence is given to the large number of observations at one age, than to the small number at others; and the mortality table obtained by means of this equality, is corrected by supposing the numbers of the living to increase by differences of the first degree, instead of by the transcendental law given by the forces of Gompertz. Irregularities are thus retained, and the errors of observation are not fairly balanced and corrected. The use of the approximate law and the proper weight of each observation, may not introduce much improvement where the exposures are so numerous as they are in the H^{MF} table; but when the observations may be few, or when it is desired to construct tables for each separate year of insurance, these two principles become useful and important.

DETERMINATION OF THE CONSTANTS.

The methods used by Mr. Woolhouse and Prof. Bartlett to find the four constants k, a, b, and c, in the formula, require the use of three or four rates separated by equal intervals of time. But it is easy to find them from the total observations, so that the errors in the rates at any one age will be more or less neutralized by the excess of some balancing the deficiency of others. Let x become $x+1$ in the formula, and the quotient of the two values of

l will give the annual rate of survival $p_x = ab^{c^{x+1}-c^x}$, and then $\lambda p_x = (c-1)c^x\lambda b + \lambda a$. Now if the values of λp be obtained for each age (omitting the younger and the older ages, at which the observations may be worthy of little confidence), and divided into three equal groups, the sum of each group will contain the same multiple of λa. This can then be eliminated by subtraction, and the quotient of the two remainders will give a power of c, from which c can be obtained. Instead of the a and b, which can then be had from these equations, it will be better, after finding c, to divide the λp into two equal groups, and obtain a and b from a larger set of observations.

As an example of this, the values of λp are given in the second column of the subjoined Table 1 for the H^{MF} observations of the twenty British offices. Omitting the values under 20 and over 91, we have

$$\lambda p_{20} + \ldots\ldots + \lambda p_{43} = -\cdot 0888282 = 24\lambda a + (c-1)\lambda b\,(c^{20} + \ldots\ldots + c^{43}),$$
$$\lambda p_{44} + \ldots\ldots + \lambda p_{67} = -\cdot 2601425 = 24\lambda a + (c-1)\lambda b\,(c^{44} + \ldots\ldots + c^{67}),$$
$$\lambda p_{68} + \ldots\ldots + \lambda p_{91} = -1\cdot 7093569 = 24\lambda a + (c-1)\lambda b\,(c^{68} + \ldots\ldots + c^{91}).$$

Hence

$$-\cdot 1713143 = (c-1)\lambda b\{(c^{44} + \ldots + c^{67}) - (c^{20} + \ldots\ldots + c^{43})\},$$
$$-1\cdot 4492144 = (c-1)\lambda b\{(c^{68} + \ldots\ldots + c^{91}) - (c^{44} + \ldots\ldots + c^{67})\},$$

from which we get $\lambda c^{24} = \cdot 927$, and $\lambda c = \cdot 039$. Again,

$$\lambda p_{20} + \ldots\ldots + \lambda p_{55} = -\cdot 1715034 = 36\lambda a + (c-1)\lambda b(c^{20} + \ldots\ldots + c^{55}),$$
$$\lambda p_{56} + \ldots\ldots + \lambda p_{91} = -1\cdot 8868242 = 36\lambda a + (c-1)\lambda b(c^{56} + \ldots + c^{91}),$$

$$-1\cdot 7153208 = (c-1)\lambda b\{(c^{56} + \ldots + c^{91}) - (c^{20} + \ldots\ldots + c^{55})\} = \lambda b(c^{92} - 2c^{56} + c^{20})$$
$$= \lambda b(3872\cdot 58 - 2 \times 152\cdot 76 + 6\cdot 03) = \lambda b \times 3573\cdot 09,$$

whence $\lambda(-\lambda b) = \bar{4}\cdot 6813$, and $\lambda\{(c-1)(-\lambda b)\} = \bar{5}\cdot 654$. Also, $36\lambda a = -\cdot 10111$; and $\lambda a = -\cdot 002808 = \bar{1}\cdot 997192$.

With these values of a, b, and c, the mortality rates in the third column of Table 1 have been calculated. If these be compared with the observed or unadjusted rates, they will be found to agree with them very closely at all ages after 30. They, however, are not the best possible rates to be obtained from the formula. But by these approximate values a rate for any one age can be obtained from the observations at every other age, and the proper combination of these rates will give a corrected value for that age by means of all the observations, or by means of as many as may be desired.

ARITHMETICAL MEAN.

When two, or three, or four, or many, observations are made of the same quantity, if all are equally trustworthy, and all alike liable to positive and negative errors, and to the same amount of positive and of negative errors, the best value that can be obtained is one between the extremes, and of such a magnitude that the positive and negative errors will be equal; for otherwise, the negative, or the positive, errors would be the greater, which is contrary to the supposition. If 4, 5, 7, 8, and 9, were five such observed values, 6·6 would be the best value; for this would give negative errors to 4 and 5, of 2·6 and 1·6; and positive errors to 7, 8, and 9, of ·4, and 1·4, and 2·4, whose sum, 4·2, is the same as the sum of the negative errors 2·6 and 1·6. In all these cases, the algebraic sum of the errors is zero, and the best value is the arithmetical mean, or the sum of the observations divided by their number.

The two conditions which make this mean value the best, are that each observation is as likely to be above as below the truth, and likely to be as much above as below. In like manner, if all the probable positive errors are equal to all the probable negative errors, this combination is proper, even if the negative errors are more numerous than the positive. That is, if each possible negative error be multiplied by the chance of its occurrence, and the sum of the products equals the sum of the products of all the possible positive errors into the chances of their occurrence, the arithmetical mean may be properly taken. The general reason for the propriety of this combination may be expressed by the proposition, that the two classes of errors will be more likely to neutralize each other in this way than in any other. If the sums of the probable negative and positive errors are not equal, but can be made equal, this combination will also then become appropriate.

EQUALITY OF PROBABLE ERRORS.

When the ratio of the deaths and exposures in a life office is obtained for any given age, it may be above or below the true ratio which would be had from long-continued observations. But the possible positive errors would be far more numerous than the possible negative. If the true rate, for example, were one *per-cent,* there could be only two negative errors in 180 observations, while there might be 179 positive. If there were no deaths, or one, the observed rates, 0 ÷ 180 and 1 ÷ 180, would be smaller than the true, while 2, 3, 4, 5, &c., would all give the rate too large. But if each error be multiplied by the chance of its occurrence, the sum of the negative products will be equal to the sum of the positive, so that the probable amount of negative errors is the same as of the positive. This equality may be demonstrated. For the errors are precisely the same as they would be, if an equal-sided prism, of whose faces a are white and b black, (the whole number being $c=a+b$,) were whirled on a smooth table, and we should propose to find the ratio of the white faces to the whole number, by means of the observed number of white and of black faces on which the prism finally rested in n observations. If $a=1$ and $c=100$, and there were 180 whirls, the chance of observing no white face, or one, or two, or three, or any other number, would be exactly the same as of the same number of deaths among the 180 persons whose rate of mortality was one in a hundred; and the positive and negative errors, and the chances of their occurrence, would be identical. Now with the prism it can be shown that the probable amount of negative errors is the same as of the positive, and therefore this is true of the mortality observations.

In the prism the true ratio between the white faces and the whole number is $a \div c$, and all the possible errors are

$$\frac{0}{n}-\frac{a}{c},\ \frac{1}{n}-\frac{a}{c},\ \frac{2}{n}-\frac{a}{c},\ \frac{3}{n}-\frac{a}{c},\ \ldots\ldots\ldots\ \frac{n}{n}-\frac{a}{c}.$$

Now the chance of the prism resting on faces that are all black, or on no white face, is $b:c$ in one throw, and $b^2:c^2$ in two, and in n throws $b^n:c^n$. The chance of one white face in n throws is $nab^{n-1} \div c^n$, and of two $n \cdot \frac{n-1}{2} \cdot a^2 b^{n-2} : c^n$, and the sum of the products of the several errors into the chances of their occurrence, is,

$$\left(\frac{0}{n}-\frac{a}{c}\right)b^n \div c^n$$

$$+\left(\frac{1}{n}-\frac{a}{c}\right)nb^{n-1}a \div c^n$$

$$+\left(\frac{2}{n}-\frac{a}{c}\right)n \cdot \frac{n-1}{2} \cdot b^{n-2}a^2 \div c^n$$

$$+\left(\frac{3}{n}-\frac{a}{c}\right)n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot b^{n-3}a^3 \div c^n, \text{ \&c.}$$

But the sum of the products of the second terms within the parentheses into the factors outside, is $-\frac{a}{c}(b+a)^n : c^n$, and of the first $(b+a)^{n-1}a : c^n$, which, since $c=a+b$, are both equal and of opposite signs. Therefore, the algebraic sum of all is zero, and, therefore, the sum of the products of the positive errors into the chances of their occurrence, is the same as of the negative.

This is true, therefore, of the rates of mortality and also of the rates of survival, either of which may be represented by $a : c$.

As an illustration of this equality, let $a=1$, $b=5$, $c=6$, and $n=3$. There are in this case four possible results; there may be no white face observed, or one, or two, or three. The first gives a negative error; for the observed ratio $\frac{0}{3}$ is less than $\frac{1}{6}$, the true ratio of the white to the whole number of faces. The other three give positive errors, since $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$ all exceed $\frac{1}{6}$. The chance of the first is $5^3 \div 6^3$, and the negative product is $-\frac{1}{6}\frac{125}{216} = -\frac{125}{1296}$. The positive products are $\left(\frac{1}{3}-\frac{1}{6}\right)3 . \frac{25}{216}$, $\left(\frac{2}{3}-\frac{1}{6}\right)3 . \frac{5}{216}$, and $\left(\frac{3}{3}-\frac{1}{6}\right)\frac{1}{216}$, the sum of which, $(75+45+5) \div 1296$, equals the negative.

Amount of the probable Negative Errors.

If n is small, there may be only one negative error, which will be when no white face is observed. If n is large compared with c, there may be several. To find them generally, the several terms must be summed. To do this, it may be noted that, if the products of the first terms in the parentheses into the factors outside be multiplied by $(b+a) : c = 1$, they all become identical with the second. But though of opposite signs, so that the whole would cancel, a limited number of them will not. For two negative errors, the first terms in the parentheses being multiplied into $(b+a) : c$, will make the sum of these two

$$\left(-\frac{a}{c}b^n + \frac{b}{c}\cdot\frac{1}{n}\cdot nb^{n-1}a + \frac{a}{c}\cdot\frac{1}{n}\cdot nb^{n-1}a - \frac{a}{c}\cdot nb^{n-1}a\right) \div c^n,$$

in which the first two terms cancel. If three errors be combined, all cancel but two, which are $\left(\frac{2}{n}-1\right)\frac{a}{c}n\cdot\frac{n-1}{2}\cdot b^{n-2}a^2 \div c^n$; and so always, the sum generally being

$$\left(\frac{m}{n}-1\right)\frac{a}{c}n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdots\cdots\frac{n+1-m}{m}\cdot b^{n-m}a^m \div c^n,$$

where m is the last numerator in the first term within the parenthesis which does not give a positive error.

The negative errors are thus reduced to a single term instead of a series. If now the particular case be examined when $m:n=a:c$ and $1-m:n=b:c$, the sum of all the possible negative errors into the chances of their occurrence will become, by summing the binomial coefficients (see Lacroix, *Probabilities*, note I, no. 2°),

$$\left(\frac{m}{n}-1\right)\frac{a}{c}(2\pi)^{-\frac{1}{2}}n^{n+\frac{1}{2}}m^{-m-\frac{1}{2}}(n-m)^{m-n-\frac{1}{2}}e^{-\frac{1}{12}n}a^{m}b^{n-m}\div c^{n}=\left(\frac{ba}{2n\pi c^{2}}\right)^{\frac{1}{2}}e^{-\frac{1}{12}n}.$$

For, by substituting $-b:c$ for $m:n-1$ and $nb:c$ for $n-m$ and $na:c$ for m, the indices of a are $1-m-\frac{1}{2}+m=\frac{1}{2}$, and of b are $1+m-n-\frac{1}{2}+n-m=\frac{1}{2}$, and of n are $n+\frac{1}{2}-m-\frac{1}{2}+m-n-\frac{1}{2}=-\frac{1}{2}$, and of c are $-1-1+m+\frac{1}{2}-m+n+\frac{1}{2}-n=-1$. But e being 2·71828 the Naperian base, $e^{\frac{1}{12}}=1{\cdot}087$, and the factor $e^{-\frac{1}{12}n}$ becomes very near unity, especially if n is as large as 10 or 20; and the sum is then $\left(\frac{ba}{2\pi nc^{2}}\right)^{\frac{1}{2}}$, or $\left(\frac{qp}{2\pi n}\right)^{\frac{1}{2}}$, q being the mortality and $=1-p$. Hence it may be seen that the probable negative or positive error for the rate of survival is the same as for the rate of mortality, and that for either it varies as the square root of their product directly, and of the number of observations inversely.

This result is nearly true for all values of n which are not the same multiples of c that m is of b. When n becomes $n+1$ there is indeed no change in the error, but for every subsequent value of n until $(m+r)\div(n+s)$ again equals $a:c$, the error regularly decreases, and nearly by this law.

From this law of decrease in the errors of observed mortality rates, it follows that the error is less at the younger ages than at the older, and decreases continually at all ages as the number of observations becomes larger. How, then, can these errors be made equal?

When single observations, all equally trustworthy, are combined, the positive and negative errors neutralize each other in part, so that for 4 the probable value obtained will have only half the probable error of 1, and for 9 only one-third of this probable error. This is true, because, as has just now been shown, the probable error varies inversely as the square root of n. If, then, the rate obtained for 4 is to be combined with the rate obtained for 9, it is necessary to divide the two rates, not by $\frac{1}{2}$ and $\frac{1}{3}$ the probable errors, but by $\frac{1}{4}$ and $\frac{1}{9}$ the squares of the probable errors, to obtain the same result as if the 13 had been combined together. If a, b, c, d, &c., are 13 such observations, the best rate for 4 is $(a+b+c+d)\div 4$, and for the other 9 it is $(e+f+g+h+i+k+l+m+n) \div 9$; the first result must be divided by $\frac{1}{4}$ and the second by $\frac{1}{9}$, the squares of their probable errors, so as to obtain the best result, which is one-thirteenth of the sum of all. And, generally, as precisely the same reason applies to the rate of mortality and of survival, the rates obtained from one set of observations must be divided by the squares of the probable error to obtain such values as can be combined by addition. Therefore, the mortality rate at each age must be divided by $qp:n$ to give such values as can be combined by addition, so that their errors will be best neutralized in combining them together.

As we have found from the observations the approximate rates at every age by means of the forces of Gompertz, the mortality at each age can be found from the observations at every other; and these values being multiplied by such factors as will permit their proper combination, the rate for each age can be obtained by combining all the observations at all ages, or as many as may be desired.

Mode of Combination.

After the rate at any age has been obtained from the other ages, and the errors equalized so that they can be combined by addition, the several values will have coefficients representing the unequal influence each must have in the combination. If the several values of the rate at an age thus obtained, be q', q'', q''', &c., and Q be the best value, then $\mathrm{A}q'$, $\mathrm{B}q''$, $\mathrm{C}q'''$, &c., will represent these values, A, B, and C being the factors that will make the probable errors equal, and

$$\mathrm{Q}=(\mathrm{A}q'+\mathrm{B}q''+\mathrm{C}q'''+\text{\&c.}) : (\mathrm{A}+\mathrm{B}+\mathrm{C}+\text{\&c.}).$$

For, if A were a whole number, the result would be the same as if the observation q' were repeated as many times as there are units in A, and so of the others. And then the arithmetical mean would give the above value of Q. In like manner, if A, B, C, &c., were fractions, the multiplication of all by the common denominator would reduce them to the preceding case, where all were whole numbers. Therefore, in both cases this quotient will give the best value of q.

Formula for the best Rate.

If now D′, D″, D‴, &c., and E′, E″, E‴, &c., be the observed deaths and exposures; and q', q'', q''', &c., and p', p'', p''', &c., the approximate rates already obtained, and q_x the approximate rate and Q_x the best rate for some single age, the several values of the rate at each age are $D':E'$, $D'':E''$, $D''':E'''$, &c. From each of these observed rates, a rate for the age x may be had by multiplying by the ratio between the approximate rate at x, already obtained, and the approximate rate for each age. This will give for the rates at x, $\frac{D'q_x}{E'q'}$, $\frac{D''q_x}{E''q''}$, &c. But the rates observed are affected by probable errors, and must be divided by $p'q':E'$, &c., before they can be combined by addition. These errors are increased in finding the rates at x by the factors by which they are multiplied, and therefore these rates for x must be divided by $p'q_x:E'$, &c., or multiplied by their reciprocals, before they can be properly combined by addition: these multipliers are the A, B, C, &c., of the preceding paragraph. Therefore Q_x is

$$\left(\frac{D'q_x}{E'q'}\times\frac{E'}{p'q_x}+\frac{D''q_x}{E''q''}\times\frac{E''}{p''q_x}+\&c.\right):\left(\frac{E'}{p'q_x}+\frac{E''}{p''q_x}+\&c.\right),$$

or $$\left(\frac{D'}{p'q'}+\frac{D''}{p''q''}+\&c.\right)q_x:\left(\frac{E'}{p'}+\frac{E''}{p''}+\&c.\right).$$

By this formula, and the approximate values hitherto obtained for every age, the best values for each age could be found. But as these results would be affected with the defects in the approximate rates, it will be better to return to the groups hitherto used, and find the best values of the rate of mortality at the middle age of each, and then find new values of the constants a, b, and c, in the formula for the law of mortality.

For this purpose, a column may be formed for $D':q'p'$ at every age, and also for $E':p'$ (in which p', being nearly unity, and affecting numerator and denominator alike in the formula for the best value, may be omitted), and from these Q_x, the middle rate in each group, may be obtained.

In the example above given for the H^{MF} observations, the sum of the $D \div qp$ for the first group from 20 to 43, was found to be 689095, and of $E \div p$, 684136, which shows that the λq for 32, which is 3·9175, must be increased to

$\bar{3}{\cdot}9206307$, which gives for λp_{32}, $-{\cdot}0036327$.

In like manner,

The λQ for 56 is $\bar{2}{\cdot}3469837$, which gives for λp_{56}, $-{\cdot}0097645$,
and „ 80 „ $\bar{1}{\cdot}1431771$, „ λp_{80}, $-{\cdot}0650231$,
„ „ 38 „ $\bar{3}{\cdot}9844280$, „ λp_{38}, $-{\cdot}0042103$,
„ „ 74 „ $\bar{2}{\cdot}9226955$, „ λp_{74}, $-{\cdot}0379594$.

The 24th root of the quotient of the differences of the first three, gives $\lambda c = {\cdot}0398$. The difference of the last two, being divided by $c^{74} - c^{38}$, gives $\lambda\{(c-1)(-\lambda b)\} = \bar{5}{\cdot}5994$, and thence λa is found to be $\bar{1}{\cdot}9970833$. From these values of the constants, the rates of mortality in the fourth column were obtained.

These rates are the best that can be had by the formula founded on the forces of Gompertz, since every observation has been used, each having its proper influence in the combination, and the errors have been reduced to their minimum by the proper combination of all the observations.

If these results be compared with the observed or the adjusted rates commonly used, it will be seen that under twenty they are entirely too large and have no pretensions to accuracy, and that from twenty to thirty they are also too large, and after thirty they have more or less accordance throughout the table. They do not favour very much the proposition that the two forces of Gompertz are the sole agents in human mortality; for the errors are greatest at those ages where the large numbers observed and the small rates of mortality have had the greatest influence assigned them.

It will now, however, be possible to test this more carefully. For this purpose, we will compare the main force with the observations, and find what other force is needed to produce the effects observed. If this be constant, or only changes in such a way that its variations may be supposed to be due to errors of observation, the forces of Gompertz will be proved to be the law of the phenomena. If the changes are not of this character, they will only be an approximation to the law of nature.

Instead of comparing the first force with each single observation, it will be better to combine several before the comparison is made, so as to neutralize more or less the errors of observation. In the example, seven ages were used for this purpose.

Let $u_x = b^{(c-1)c^x}$, so that $p_x = a u_x$. These values of p must be multiplied by $\mathrm{E} : qp$, to make their probable errors equal, before they can be combined by addition. If then the sum of seven of these products of the observed rate of survival into $\mathrm{E} : qp$, be divided by the sum of seven of the $u \times \mathrm{E} \div qp$, the quotient will measure the second force which the observations require to be united with the first to obtain the observed rate of survival. In this way column (5) was formed for the values of the second force which makes u_x equal to the observed p_x.

In the example; at ten $\lambda c^x = 10 \times {\cdot}0398$, and $\lambda(\overline{c-1}\lambda b) = \bar{5}{\cdot}5994$, and $\lambda u_{10} = {\cdot}0000994$. The value of $\lambda\mathrm{E} : qp = \lambda 620 - \lambda({\cdot}006921)$

$-\lambda(1-\cdot006921)$. Hence $\lambda(E \div qp)=4\cdot9552$. Multiplying into the observed p_{10} and the u_{10}, we have 89326 and 90178. And the sum of seven of these being divided gives $a=\cdot99648$, as in the table, for the middle age, 13. And so the other numbers in this column have been found. These represent the force which must be united with the first force of Gompertz to give the observed mortalities.

These values decrease almost invariably to the age of 40, then become quite irregular, rising and falling to the age of 70; but for the next twelve years there is a large depression. At 77, a is only ·98437, making a deviation of ·00894 from the value of a in the formula. These differences are probably too large and long-continued to be due to accidental errors in the observations, especially for the years under 40 and for the decade after 70; and if so, it may be asserted that the forces of Gompertz are only an approximation.

THE BEST RATES FOR THE OBSERVATIONS.

As the successive values of a have been found for every group of seven, these multipliers of u will determine the rate for every age which the observations require, and it is only necessary to use them to obtain a satisfactory adjustment. There are, however, some of these values of a which rise and fall in several places in the series, indicating that larger groups than seven ought to have been used to correct the errors of observation, or that the arithmetical means of these quantities should be employed to increase their regularity. Using the latter method and taking the average of five of these, twice in succession, the values of a' in the sixth column were obtained. This uses fifteen observations to determine each rate, and makes the age of 17 the youngest that can be obtained from the observations; the method of differences being used for the seven years before this. With these values of a' the rates of mortality in the seventh column were obtained by combining them with the u previously found for each age from 10 to the end of life.

CRITERIA OF A GOOD ADJUSTMENT.

If the rates of mortality be multiplied by the exposures, and the number of deaths which they would give for all ages be added together from the earliest age to the oldest, these sums, compared with the corresponding sums of the observed deaths, will be a severe test of the accuracy of the adjustment. If the rates were too high or too low at any part of the table, the errors would be accumulated and continued until they were balanced by opposite errors; and if they were not balanced, they would show an excess or deficiency in the total rates. If they were balanced soon, the sum of the errors would be small, unless the errors themselves were large.

The excellence of the adjustment would therefore be shown, 1st, by the general regularity in the rates, 2nd, by the near agreement in the whole number of deaths, 3rd, by the equality of the positive and negative deviations, 4th, by the smallness of the accumulated deviations, and 5th, by the frequency of the changes in their signs from positive to negative.

To apply this test to the rates that have been obtained, the deaths are calculated for each age and placed in column (8), and their sums in the next column. Thus for the 4,116 living at twenty, the rate ·00638 gives 26 deaths, and these added to the deaths for the ages under twenty, make the total 97, which are the numbers opposite 20 in these two columns.

The excess or deficiency of these sums compared with the sum of the actual deaths, is inserted in the tenth column, and these five tests can then be easily applied.

It will be noticed that the increments in the adjusted rates are very regular: that the whole number of deaths is only one too large, being 23,848 instead of 23,847, or less than a half of a hundredth of one per-cent: that the positive accumulated errors are nearly the same as the negative, being in the ratio of four to five: that the largest deviation is $+49$, and this only two removes from a negative -36 before it, and three from a negative -28 after it: that there are only six accumulated deviations over 30, and neither of these is one-tenth of the annual deaths at the corresponding age; and that there are in all twenty-three changes in the algebraic signs of the deviations. In all these five particulars this adjustment is better than the excellent table which has been generally adopted.

For the purpose of determining whether our differences under 40 and over 70 were due to accident, an adjustment was made of the experience after the tenth year. The exposures and deaths are given in the second and third columns of the second table and the values of λp in the fourth column.

These give for the three groups of twenty-four ages, each from 21 to 92, the sums ·1086345, ·2973311, and 1·9241270, from

TABLE 1.—*Adjustment of the* H^{MF} *Observations.*

Age (x) (1)	λp_x Unadjusted. (2)	q_x First Adjustment. (3)	q_x Second Adjustment. (4)	a (5)	a' (6)	q_x Final Adjustment. (7)	Expected Deaths by Col. (7). (8)	Sum of Expected Deaths. (9)	Errors. (10)
10	(−) ·0042233	...	·006921	...	...	·00311	2	2	− 4
11	·0000000	...	·006943	...	...	·00320	2	4	− 2
12	·0010864	...	·006967	...	...	·00341	3	7	− 1
13	·0009514	...	·006993	·99648	...	·00371	3	10	0
14	·0027687	...	·007021	·99700	...	·00407	4	14	− 3
15	·0023186	...	·007053	·99620	...	·00448	6	20	− 4
16	·0008309	...	·007087	·99518	...	·00491	8	28	+ 1
17	·0015912	...	·007125	·99481	·99509	·00534	10	38	+ 4
18	·0025445	...	·007169	·99460	·99474	·00574	14	52	+ 4
19	·0034749	...	·007214	·99456	·99443	·00609	19	71	− 2
20	·0026459	·007066	·007262	·99391	·99419	·00638	26	97	− 1
21	·0029636	·007124	·007316	·99361	·99405	·00658	36	133	− 2
22	·0026693	·007188	·007376	·99402	·99397	·00671	49	182	+ 2
23	·0034296	·007258	·007442	·99400	·99391	·00684	66	248	− 8
24	·0032785	·007334	·007513	·99398	·99385	·00697	85	333	−15
25	·0024248	·007418	·007592	·99381	·99380	·00710	107	440	+ 8
26	·0031799	·007509	·007678	·99368	·99373	·00726	130	570	+ 7
27	·0030751	·007609	·007773	·99360	·99364	·00744	156	726	+15
28	·0035134	·007719	·007876	·99368	·99356	·00762	182	908	+ 5
29	·0034377	·007838	·007990	·99340	·99349	·00781	209	1117	+ 3
30	·0036687	·007967	·008114	·99339	·99343	·00799	236	1353	− 9
31	·0033368	·008112	·008250	·99328	·99338	·00818	260	1613	+ 8
32	·0038357	·008269	·008399	·99337	·99334	·00837	284	1897	− 6
33	·0036495	·008440	·008563	·99335	·99329	·00858	307	2204	+ 2
34	·0039022	·008627	·008742	·99334	·99325	·00880	330	2534	− 3
35	·0037147	·008832	·008938	·99316	·99320	·00904	351	2885	+17
36	·0040148	·009056	·009154	·99309	·99316	·00930	372	3257	+21
37	·0042335	·009301	·009389	·99308	·99312	·00958	389	3646	+16
38	·0045284	·009569	·009648	·99309	·99310	·00985	405	4051	− 6
39	·0047090	·009862	·009931	·99303	·99311	·01013	421	4472	−33
40	·0044087	·010182	·010241	·99308	·99313	·01042	434	4906	−20
41	·0046952	·010533	·010581	·99314	·99317	·01072	444	5350	−21
42	·0047871	·010916	·010953	·99324	·99321	·01105	451	5801	−13
43	·0047761	·011335	·011361	·99337	·99324	·01143	459	6260	+ 7
44	·0051241	·011793	·011808	·99326	·99324	·01187	467	6727	+13
45	·0053884	·012295	·012297	·99329	·99321	·01240	474	7201	+14
46	·0056576	·012842	·012833	·99325	·99318	·01296	483	7684	+15
47	·0062138	·013441	·013420	·99307	·99314	·01359	490	8174	− 7
48	·0061535	·014096	·014064	·99299	·99311	·01426	496	8670	0
49	·0065231	·014812	·014768	·99296	·99310	·01497	503	9173	+ 2
50	·0071847	·015595	·015540	·99312	·99312	·01572	510	9683	−20
51	·0073500	·016450	·016384	·99333	·99315	·01654	513	10196	−28
52	·0076419	·017385	·017310	·99313	·99319	·01742	517	10713	−29
53	·0078805	·018406	·018322	·99308	·99324	·01839	519	11232	+18
54	·0081854	·019522	·019431	·99329	·99327	·01947	521	11753	+39
55	·0090222	·020742	·020645	·99336	·99329	·02066	526	12279	−10
56	·0099660	·022075	·021974	·99364	·99330	·02198	527	12806	−27
57	·0100649	·023531	·023428	·99317	·99328	·02345	528	13334	−15
58	·0110408	·025121	·025019	·99301	·99326	·02506	531	13865	−16
59	·0107081	·026858	·026760	·99333	·99321	·02685	534	14399	+34
60	·0132818	·028754	·028668	·99336	·99313	·02884	535	14934	+10

TABLE 1—*(continued)*.

Age (x)	λp_x Unadjusted.	q_x First Adjustment.	q_x Second Adjustment.	a	a'	q_x Final Adjustment.	Expected Deaths by Col. (7).	Sum of Expected Deaths.	Errors.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	(−)								
61	·0134899	·030824	·030747	·99318	·99301	·03104	532	15466	+18
62	·0150457	·033084	·033025	·99304	·99285	·03347	532	15998	+ 9
63	·0164156	·035550	·035515	·99237	·99268	·03612	528	16526	− 5
64	·0172458	·038240	·038237	·99258	·99257	·03895	519	17045	− 5
65	·0189758	·041175	·041211	·99220	·99250	·04199	511	17556	−14
66	·0198026	·044375	·044460	·99199	·99250	·04524	499	18055	− 7
67	·0214344	·047863	·048008	·99258	·99252	·04876	486	18541	− 1
68	·0242044	·051665	·051882	·99303	·99251	·05264	471	19012	−15
69	·0267055	·055806	·056109	·99307	·99240	·05697	452	19464	−36
70	·0248399	·060316	·060721	·99276	·99212	·06184	435	19899	+ 8
71	·0272397	·065225	·065749	·99123	·99155	·06740	421	20320	+49
72	·0342506	·070566	·071228	·99124	·99076	·07361	409	20729	+37
73	·0366592	·076374	·077197	·99133	·98975	·08050	388	21117	+35
74	·0472332	·082685	·083696	·98944	·98871	·08794	364	21481	−28
75	·0417913	·089541	·090763	·98587	·98775	·09585	334	21815	−14
76	·0487643	·096982	·098447	·98511	·98721	·10398	306	22121	−21
77	·0487659	·105052	·10679	·98437	·98702	·11245	274	22395	− 6
78	·0599900	·113798	·11585	·98880	·98739	·12112	247	22642	−22
79	·0611185	·123268	·12568	·98700	·98804	·13030	215	22857	−24
80	·0616681	·132512	·13632	·98933	·98893	·14012	185	23042	−14
81	·0723264	·144581	·14782	·98806	·98968	·15094	159	23201	−17
82	·0709018	·156528	·16027	·99138	·99085	·16235	136	23337	− 7
83	·0884006	·169407	·17370	·99332	·99180	·17496	115	23452	−13
84	·0874435	·183271	·18817	·99210	·99276	·18862	93	23545	−10
85	·1066797	·198172	·20374	·99441	...	·20418	77	23622	−15
86	·1070813	·214162	·22046	·99131	...	·22089	60	23682	−14
87	·1084919	·231269	·23838	...	...	·23880	45	23727	−11
88	·1310832	·249600	·25755	...	...	·25796	34	23761	−11
89	·1141945	·269130	·27800	...	...	·27840	24	23785	− 7
90	·1333954	·289914	·29977	...	...	·30016	18	23803	− 5
91	·1461280	·311975	·32288	...	...	·32325	14	23817	− 3
92	·2118069	·335324	·34732	...	...	·34769	10	23827	− 4
93	·2653144	·359960	·37311	...	...	·37346	7	23834	− 5
94	·0000000	·385867	·40020	...	...	·40053	4	23838	− 1
95	·0483047	·418004	·42855	...	...	·42886	4	23842	+ 2
96	·4259687	·441328	·45808	...	...	·45838	4	23846	+ 1
97	·3010300	·470745	·48869	...	...	·48898	1	23847	+ 1
98	∞	·501155	·52027	...	...	·52053	1	23848	+ 1
99	...	...	...	...	...	·55288			

TABLE 2.—*Adjustment of the* H^{MF} *Observations after the Tenth Year of Assurance.*

Age (x)	Exposed E_x	Died D_x	λp_x Unadjusted.	q_x First Adjustment.	q_x Second Adjustment.	$1-a$	$1-a'$	q_x Final Adjustment.	Expected Deaths by Col. (9).	Errors.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
			(−)							
20	252·5	0	·0000000	·008635	·008555	...	...	·00881	2	+ 2
21	269·	3	·0048707	·008695	·008620	...	...	·00868	2	+ 1
22	277·	2	·0031471	·008760	·008690	...	...	·00861	2	+ 1
23	352·	1	·0012356	·008831	·008767	·00770	...	·00860	3	+ 3
24	406·5	4	·0042946	·008909	·008851	·00794	...	·00863	4	+ 3
25	460·	5	·0047464	·008995	·008942	·00687	...	·00869	4	+ 2
26	515·5	7	·0059377	·009088	·009042	·00745	...	·00879	4	− 1
27	619·5	4	·0028132	·009190	·009152	·00774	·00762	·00892	6	+ 1
28	747·5	4	·0023302	·009302	·009271	·00807	·00765	·00906	7	+ 4
29	920·5	10	·0047438	·009425	·009401	·00758	·00767	·00921	8	+ 2
30	1163·5	10	·0037488	·009558	·009544	·00750	·00773	·00942	11	+ 3
31	1723·	19	·0048157	·009705	·009699	·00772	·00780	·00964	17	+ 1
32	2392·	21	·0038296	·009865	·009868	·00770	·00788	·00989	24	+ 4
33	3190·	33	·0045161	·010040	·010054	·00797	·00799	·01019	33	+ 4
34	4178·5	43	·0044924	·010231	·010256	·00830	·00812	·01052	44	+ 5
35	5209·	51	·0042731	·010440	·010477	·00837	·00822	·01084	56	+10
36	6400·	76	·0051881	·010670	·010718	·00838	·00828	·01114	71	+ 5
37	7566·5	93	·0053710	·010920	·010981	·00831	·00829	·01141	86	− 2
38	8739·	107	·0053503	·011194	·011269	·00823	·00827	·01168	102	− 7
39	10089·	118	·0051094	·011494	·011583	·00842	·00820	·01193	120	− 5
40	11119·	134	·0052656	·011822	·011926	·00821	·00810	·01217	135	− 4
41	12010·	146	·0053119	·012180	·012300	·00796	·00799	·01242	149	− 1
42	12901·	175	·0059315	·012572	·012709	·00786	·00787	·01272	164	−12
43	13755·5	175	·0055606	·013009	·013155	·00757	·00776	·01306	180	− 7
44	14443·	190	·0057511	·013469	·013642	·00762	·00768	·01347	195	− 2
45	15003·5	214	·0062391	·013981	·014174	·00759	·00763	·01395	209	− 7
46	15205·	195	·0056057	·014541	·014754	·00751	·00760	·01450	220	+18
47	15522·5	246	·0069379	·015154	·015387	·00762	·00760	·01513	235	+ 7
48	15556·	244	·0068660	·015823	·016097	·00768	·00760	·01582	246	+ 9
49	15784·	271	·0075213	·016555	·016833	·00759	·00758	·01656	261	− 1
50	15776·	285	·0079174	·017355	·017657	·00760	·00755	·01735	274	−12
51	15587·	288	·0080995	·018230	·018556	·00750	·00751	·01821	284	−16
52	15297·	291	·0083414	·019185	·019536	·00752	·00743	·01911	292	−15
53	14888·	269	·0079187	·020230	·020606	·00745	·00736	·02011	299	+15
54	14600·5	317	·0095332	·021371	·021376	·00720	·00728	·02120	310	+ 8
55	14248·	325	·0100211	·022618	·023046	·00715	·00723	·02243	320	+ 3
56	13718·	338	·0108347	·023982	·024435	·00691	·00718	·02377	326	− 9
57	13208·	326	·0108538	·025469	·025949	·00737	·00716	·02527	334	− 1
58	12762·5	351	·0121114	·027094	·027601	·00719	·00715	·02691	343	− 9
59	12209·	328	·0118271	·028869	·029401	·00712	·00716	·02872	351	+14
60	11592·5	367	·0139715	·030807	·031364	·00696	·00718	·03070	356	+ 3
61	10948·5	352	·0141923	·032923	·033503	·00721	·00724	·03290	360	+11
62	10370·	367	·0156485	·035233	·035833	·00720	·00732	·03531	366	+10
63	9728·	370	·0168406	·037752	·038372	·00758	·00742	·03795	369	+ 9
64	9054·	380	·0186211	·040502	·041138	·00732	·00748	·04077	369	− 2
65	8419·5	376	·0198412	·043500	·044150	·00770	·00748	·04380	369	− 9
66	7811·5	359	·0204324	·046770	·047428	·00811	·00746	·04705	368	0
67	7191·	355	·0219872	·050333	·050996	·00766	·00741	·05057	364	+ 9
68	6642·5	374	·0251680	·054217	·054878	·00660	·00738	·05442	361	− 4
69	6064·5	381	·0281791	·058447	·059099	·00659	·00744	·05870	356	−29

TABLE 2—*(continued).*

Age (x)	Exposed E_x	Died D_x	λp_x Unadjusted.	q_x First Adjustment.	q_x Second Adjustment.	$1-a$	$1-a'$	q_x Final Adjustment.	Expected Deaths by Col. (9).	Errors.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
			(−)							
70	5483·	313	·0255277	·063055	·063698	·00709	·00767	·06351	348	+ 6
71	4961·5	294	·0265287	·068066	·068693	·00884	·00823	·06902	342	+54
72	4516·5	354	·0354478	·073519	·074090	·00845	·00902	·07523	340	+40
73	3998·	335	·0380059	·079449	·079969	·00802	·01009	·08204	328	+33
74	3509·5	366	·0478317	·085891	·086347	·01004	·01127	·08949	314	−19
75	3016·5	275	·0415150	·092888	·093263	·01463	·01246	·09747	294	0
76	2602·	281	·0496322	·100488	·10075	·01552	·01329	·1057	275	− 6
77	2187·	236	·0495915	·108717	·10887	·01606	·01384	·1142	250	+ 8
78	1843·5	247	·0624742	·117633	·11765	·01333	·01389	·1230	227	−12
79	1500·	205	·0638215	·127289	·12716	·01358	·01363	·1322	198	−19
80	1207·	163	·0630068	·137732	·13739	·01322	·01306	·1419	171	−11
81	966·	151	·0738195	·149012	·14844	·01512	·01228	·1522	147	−15
82	770·	111	·0676053	·161184	·16035	·01141	·01146	·1634	126	0
83	621·5	115	·0888617	·174299	·17317	·00858	·01066	·1758	109	− 6
84	467·	84	·0861181	·188413	·18695	·00962	·01008	·1888	88	− 2
85	360·5	78	·1058868	·203576	·20174	·00642	·00943	·2030	73	− 7
86	261·	57	·1070102	·219140	·21758	·01392	·00907	·2185	57	− 7
87	186·	42	·1111504	·237251	·23452	·00693	...	·2355	44	− 5
88	127·	34	·1353208	·255843	·25269	·01050	...	·2535	32	− 7
89	83·	20	·1197376	·275684	·27188	·00412	...	·2728	23	− 4
90	58·	15	·1299595	·296776	·29235	·00662	...	·2932	17	− 2
91	41·	12	·1503859	·319143	·31406	...	...	·3149	13	− 1
92	28·	11	·2167091	·342798	·33701	...	...	·3378	9	− 3
93	16·	8	·3010300	·367737	·36119	...	...	·3620	6	− 5
94	8·	0	·0000000	·393935	·38660	...	...	·3874	3	− 2
95	8·	0	·0000000	·421356	·41320	...	...	·4139	3	+ 1
96	7·	5	·5440680	·449330	·44094	...	...	·4416	3	− 1
97	2·	1	·3010300	·479578	·46974	...	...	·4704	1	− 1
98	1·	1	∞	·510184	·49950	...	...	·5001	1	− 1

which $\lambda c = {\cdot}039$; and for the two of thirty-six years, ·2044705 and 2·1256221. These give $\lambda\{(c-1)(-\lambda b)\} = \bar{5}{\cdot}6644$ and $\lambda a = \bar{1}{\cdot}9965119$. With these constants the first approximation for the rates of mortality in column (5) were obtained.

With these approximate rates $D \div qp$ and $E : p$, for each age were found and their sums for each of the five groups, required that

λq_{32} λq_{57} λq_{82} λq_{38} λq_{75}

must be changed from

$\bar{3}{\cdot}9941$ $\bar{2}{\cdot}4060$ $\bar{1}{\cdot}2073$ $\bar{2}{\cdot}0490$ $\bar{2}{\cdot}9680$

to $\bar{2}{\cdot}0029667$ $\bar{2}{\cdot}4054879$ $\bar{1}{\cdot}2134571$ $\bar{2}{\cdot}0518837$ $\bar{2}{\cdot}9697276$

making the corresponding values of λp

$-{\cdot}0100674$, $-{\cdot}0254383$, $-{\cdot}1634772$, $-{\cdot}049289$, $-{\cdot}425205$.

The first three give $\lambda c = \cdot 0383$, and the last two $\lambda\{(c-1)(-\lambda b)\} = \bar{4}\cdot 7196$ and $\lambda a = \bar{1}\cdot 9965744$.

With these constants the rates in the sixth column were obtained, which are the best values to be had from the formula by giving to each observation its appropriate influence.

If now the values $E \div qp$ be found for each age and then multiplied on every rate of survival and value of u, and the sum of each seven of these products be divided into each other, we get the value of a required by these seven observations as the multiplier of u; and the average of each five of these being taken twice in succession, we will have the mean values of a' of which the complement is given in the eighth column. These multiplied on the u give the rate of survival, and hence the final value of the rates of mortality, which are inserted in the ninth column. With these rates the deaths calculated from the exposures in the second column may be compared with the observed deaths, and the accuracy of the result tested by summing both from the beginning of the table. These calculated deaths and the differences between the sums of these and of the observed deaths, are to be found in the last two columns. These differences are everywhere small, and change their signs frequently. The positive and negative errors have nearly the same sum, and the whole number of deaths differs only by unity; so that the adjusted rates, having great regularity throughout the whole table, comply with all the criteria of a good adjustment.

Both these tables show that up to the age of 40 the value of a' continually decreases, and that its irregularities at other ages are probably not such as are due to the accidental errors of observation. As this represents the addition which must be made to the first force of Gompertz, it follows that the second force, though nearly constant, is subject to variation at different periods of life, and that the two forces of Gompertz are not the exact representation of the whole of the causes which destroy human life.

The very satisfactory adjustments that have been obtained for these two series of observations, show also that, though these forces cannot be trusted to construct a table of mortality as Mr. Woolhouse has intimated, they nevertheless are so near an approximation to the real forces, that they can be advantageously employed to harmonize the observed rates of mortality, so that the best rates to be had from the observations, giving to each their appropriate influence, can be obtained for the exposures and deaths of each year of insurance, and for the combined numbers for the total experience in all the years of insurance.

Of the two forces that Mr. Gompertz suggested to account for the annual decrements of human life, satisfactory proof was given by him only of the first, which produces an effect increasing in geometrical progression. For the other he presented no evidence. The only observations which have since been brought forward to establish its existence, do not furnish the required proof, but indicate that it is only approximately true, by showing that the additional force besides the first is nearly, but not exactly, constant in its effects. The formula $l_x = ka^x b^{c^x}$ as an approximation may then with propriety be ascribed to Gompertz, since he announced the forces on which it is based, and proved the correctness of the only one of its factors which may be regarded as established.

How should the Liability of a Life Office under its Policies be estimated? Being Remarks by Mr. J. A. HIGHAM, Mr. A. H. BAILEY, Mr. A. BADEN, *and* Mr. T. B. SPRAGUE.

MR. BAILEY'S paper *On the Pure Premium Method of Valuation* led to a correspondence in the *Insurance Record*, from which we propose to make some extracts.

Mr. J. A. Higham wrote that the paper "condemns the re-insurance system of valuation in some of its aspects. There is, however, a mode of applying the system to which Mr. Bailey does not advert." He then proposes that, the pure premiums being calculated by the H^M 4 per-cent table, the loading for expenses shall be a constant 5*s*. per £100 assured, and that the loading to provide for profit shall be a constant addition to the single premium; (*J.I.A.* xx, 6). Taking £3. 5*s*. as a fair rate for age 40, and deducting the 5*s*. charged for expenses, £3 remains, being ·03 per £1 assured, the value of which will be

$$\cdot 03(1 + a_{40}) = \cdot 03 \times 16\cdot 135 = \cdot 48405.$$

The net single premium at 40 being ·37943, this gives the constant addition to the single premium, ·10462, which, being spread over the whole of life, gives an addition to the annual premium for the age x, of

$$\frac{\cdot 10462}{1 + a_x} = \cdot 10462(\varpi_x + d_4) = \cdot 10462\varpi_x + \cdot 00402.$$

In this way the office annual premium is formed by adding to the net premium, 10·462 per-cent for profit, and constants of ·00402 for profit and ·0025 for expenses, or a total constant of ·00652, so that $\varpi'_x = 1\cdot10462\varpi_x + \cdot00652$. This, Mr. Higham says, will give a good working scale:—

Age 20.	30.	40.	50.	60.
£2 0 7	£2 9 11	£3 5 0	£4 11 4	£6 19 4

Mr. Higham gets the office single premium by the formula

$$A'_x = (1 + a_x)\varpi'_x = 1\cdot10462A_x + \cdot0025(1 + a_x):$$

and the value of the policy is

$$\begin{aligned} {}_nV'_x &= A'_{x+n} - (1 + a_{x+n})\varpi'_x = (1 + a_{x+n})(\varpi'_{x+n} - \varpi'_x) \\ &= 1\cdot10462(1 + a_{x+n})(\varpi_{x+n} - \varpi_x) \\ &= 1\cdot10462{}_nV_x. \end{aligned}$$

Mr. Higham says this is a system which is "everywhere consistent".

"It employs, throughout, the actual premiums and the assumed true rates of mortality and interest, and it leaves the loading untouched. If Mr. Bailey thinks that it brings out a larger reserve than is necessary, he will concede that it does afford means for forming an opinion as to the extent of the excess. The office reserve is everywhere the H^M four per-cent pure premium reserve increased 10·462 per-cent, and thereby freed from any necessity for jumping to $H^{M(5)}$.

"It places the policyholders surviving a distribution on equal terms with new entrants for future bonus.

"It makes the surplus dependent on the facts; and leaves no room for ingenuity in discovering a pure premium which will yield the surplus desired.

"It is intelligible to the assured. He can understand us when we say we hold in respect of his policy the same sum which he must put into the chest to obtain a second policy precisely like the first.

"It avoids anticipation of future profit, which the pure premium system *does not*. That system anticipates the profit for all that remains of the period for which a premium has been paid; and, more than this, it anticipates the provision for expenses during the same remaining period, and calls it profit. When the premium has been paid in one sum, the purists, on this account, abandon their system, admitting that it is wrong. They are in similar difficulty with regard to 'limited payments', to avoid being landed, when the premiums cease, with a reserve affording nothing for bonus and expenses during the remainder of life.

"As regards insurances at annual premiums for the whole of life, the pure premium system, taking one policy with another, anticipates the whole profit of the office, and dissipates the whole provision for expenses, during the six months next following upon the valuation."

If a policy was taken out at age x and has been $n+\frac{1}{2}$ years in force, the formula for the value by the pure premium method is, according to Mr. Higham,

$$\begin{aligned} & \mathrm{A}_{x+n+\frac{1}{2}}-(\tfrac{1}{2}+a_{x+n+\frac{1}{2}})\varpi_x \\ =&\tfrac{1}{2}(\mathrm{A}_{x+n}+\mathrm{A}_{x+n+1})-\tfrac{1}{2}(1+a_{x+n}+a_{x+n+1})\varpi_x \\ =&\tfrac{1}{2}(1+a_{x+n})(\varpi_{x+n}-\varpi_x)+\tfrac{1}{2}(1+a_{x+n+1})(\varpi_{x+n+1}-\varpi_x)+\tfrac{1}{2}\varpi_x. \end{aligned}$$

"If we fill this up with the gross premiums," he says, "we hold in hand half the loading of the current year, and leave intact the loading of the future. We can pay our bills till the end of the year, and lay aside the profit contemplated in the calculation of the premiums. But if we use the pure premiums, then, disregarding considerations of less importance, where is the actuary's salary to come from, until, on the assumption involved in a valuation by classes, some more loading shall be received six months hence?"

Mr. Bailey replied—"It may be readily admitted that the reassurance method of valuation yields consistent results. The objection to it is that its assumptions are at variance with facts. Take Mr. Higham's example. £3. 5*s.* per-cent is the office premium at age 40; for commercial reasons different methods of calculation have to be so adjusted as to yield about this result. This may be divided into 2·352 pure premium H^{M} 4 per-cent and ·898 surplus. So far Mr. Higham and I are agreed; beyond this I do not think that the analysis can be carried. He would subdivide the surplus into definite portions for expenses and profits; here we join issue. The pure premium can be determined, because the future rates of mortality and interest can be predicated within a reasonable margin of error. Not so the expenses; no man can form a judgment of what will be the future rate of expense. Mr. Higham says that out of a premium of £3. 5*s.* a fair allowance for expenses is 5*s.*, that is, 7·69 per-cent. Of the fairness of this allowance I offer no opinion; but, as matters of fact, I learn from a published analysis of the Board of Trade returns that the average expenditure of all the companies is at a rate almost exactly double this, and that not five per-cent of the offices are worked so cheaply. So that, if it be necessary to make the further reserve for future profits which Mr. Higham advocates, the prospect of any bonus at the next valuation must be rather small for at least 95 per-cent of the companies.

" I cannot admit that the pure premium method of valuation anticipates either the profits or the provision for expenses, for all that remains of the period for which a premium has been paid. Take the case of a valuation made on the 30th June in any year. The risks being classified, it is assumed that as many persons are born in the first half of the year as in the second, and that therefore the difference between the years of valuation and birth may be taken to be the exact age of all the lives born in any year. It is further assumed that the payment of the premiums is equally distributed throughout the year, so that their value may be taken to be the arithmetical mean between what would be the values if all the premiums were due on the 1st July, and if they were all due on the 30th June, annually. It is not pretended that these assumptions are strictly correct, but they are sufficiently near for the purpose. Like some others, as, for instance, the equal distribution of deaths throughout the year, they are adopted to bring our work within the compass of common algebraical formulas without having recourse to higher mathematics."

On these assumptions Mr. Bailey makes the formula for the policy value not (as given above)

$$A_{x+n+\frac{1}{2}}-(\tfrac{1}{2}+a_{x+n+\frac{1}{2}})\varpi_x,$$

but

$$A_{x+n}-(\tfrac{1}{2}+a_{x+n})\varpi_x.$$

"There is", he says, "no six months' anticipation. The actuary may be easy about his salary, which, if need be, can be paid day by day; the surplus on the premiums receivable on the 1st July and each subsequent day being quite sufficient for the purpose."

Mr. Bailey's formula is equivalent to

$$(1+a_{x+n})(\varpi_{x+n}-\varpi_x)+\tfrac{1}{2}\varpi_x;$$

and in this shape of it we see more readily the application of Mr. Higham's argument.

Mr. Baden wrote that, on the usual suppositions as to age, the pure premium formula becomes

$$(1)\quad A_{x+n}-(\tfrac{1}{2}\varpi_x+a_{x+n}\varpi_x)={}^{1}V\text{, suppose.}$$

If the gross premiums were valued, the formula would be

$$(2)\quad A_{x+n}-\{\tfrac{1}{2}(\varpi_x+\phi_x)+a_{x+n}(\varpi_x+\phi_x)\}={}^2V.$$

Hence $${}^1V-{}^2V=\tfrac{1}{2}\phi_x+a_{x+n}\phi_x;$$

and this is the loading reserved by the pure premium method.

He then proceeds. "Is this correct or not? Does it anticipate and throw into surplus half-a-year's unearned loading or not? The whole premium revenue is treated as an annuity payable momently, and by adding half-a-year's purchase to its value as a whole, or to the values of its component parts, ϖ_x and ϕ_x, everything that is in futurity from the moment of valuation, is taken account of, be it for risk, expenses, or profit. How, then, can it be said that half-a-year's loading is anticipated? It is true that half-a-year's premium for risk and loading alike, is reckoned as in hand, and that the whole loading of this is excluded from the reserve; but so it ought to be, since it is the contribution for profit and expenses for the half-year *preceding* the moment of valuation. The loading for the half-year next succeeding that moment, is reserved and will contribute to the expenses and profit of that period. The actuary's salary is, I think, quite safe.

"No 'purist', so far as I know, has ever maintained that the above formula (1) provides for the true valuation of any assurances but those of which the bulk of every business is composed, namely, whole life risks at uniform annual rates; nor is any actuary likely to find a difficulty or see an inconsistency in providing the due reserve of loading for other classes of assurance. In my own office, where 95 per-cent of the policies are valued on the pure premium system, the possibility of mistake in any class of assurance is guarded against by the plan adopted, that, namely of entering in the class lists the office premium and the loading of each policy, the loading in *all* cases being that for the ordinary annual premium, whilst the pure premium is not entered at all. The column of office premiums is valued, and also the column of loadings, so that, even if there be no office premium chargeable, the loading must be reserved all the same."

Mr. Higham replied that his proposed charge for expenses, of 5*s*. per £100 insured, would yield £10,000 on a business of four millions. Re-stating to some extent his argument, he said that it was not essential to his method of valuation that the office premiums should be formed in any particular way. He only assumed that the loading increases at every year of age.

"An annual premium being due and not paid, the value of the policy deduced from the gross premium [by Mr. Higham's method], exceeds the value deduced from the pure premium, by the quantity

$$(1+a_{x+n})(\phi_{x+n}-\phi_x).$$

Both systems make the same provision for claims; but whereas the one secures prospectively the loading which attaches to age at entry, the other secures prospectively the loading which attaches to present age. The future premiums as we receive them will bring ϕ_x annually. The loaded premium valuation supplements this by $\phi_{x+n}-\phi_x$, and makes the total annual provision for profit and expenses ϕ_{x+n}, being the charge which would be made for those purposes to a new entrant at age $x+n$.

"If an actuary is prepared to give at every distribution on a continued policy, the same cash bonus; which, in consequence of its diminishing purchasing power will yield at every division a diminishing reversion, then there is no objection to his using the pure premium valuation, which provides during life a uniform annual contribution to the surplus. But if it be the rule of the office to assign to every man of given present age the same reversionary addition, regardless of the age at which he entered, then the cost is in every case alike, and the actuary must make the contributions during the bonus period alike, by using the gross premium valuation, or else mulct the recent customer for the benefit of the man who entered at an earlier age.

"If the annual premium is not now due, the [policy] value is obtained by interpolating between the values after payment of the last premium and before payment of the next premium; and this brings under notice a further difference between the two systems. Midway between two dates of payment the value amounts to half-a-year's premium plus a mean between the values (premium unpaid) at last and next completed years. If the half premium be half the gross premium, we have half-a-year's loading to carry us to the end of the year. If the pure premium be used, this is wanting. We shall agree that immediately after receipt of the first annual premium on a policy, the value of the risk is one year's premium. But *which* premium? If the loaded premium be reserved, we have as yet made no profit, and we hold the loading to provide for expenses and profit during the first year. If the pure premium be reserved, then the first year's loading is already accounted profit, including what was meant for expenses, and we must keep the office open on credit till the end of the year, or pay expenses out of the fund which was meant and is sufficient for claims alone. In the case supposed, twelve months will elapse before any more loading comes to hand. Taking the policies all round, six months is the period for which the well has been pumped dry."

Mr. Higham points out that the policy value by his method becomes

$$\tfrac{1}{2}\varpi'_x + (1 + a_{x+n})(\varpi'_{x+n} - \varpi'_x),$$

when the ages are estimated as is done by Mr. Bailey. Comparing this with the pure-premium value, as given above, the difference is

$$\tfrac{1}{2}\phi_x + (1 + a_{x+n})(\phi_{x+n} - \phi_x),$$

and he says "I can see here the missing half-year's loading, and also the required provision for sustaining the reversionary bonus." Supposing a valuation to be made on 30th June, "One set of the assured have paid their loading in advance up to periods ranging from the 1st January to the 30th June next; and another set have paid in advance up to periods ranging from tomorrow (1st July) to 31st December next. Taking one with another, the company has been paid for six months' work not yet done; and Mr. Bailey's formula makes no provision to enable the company to pay its officers for doing it. He collects the business of the office into two policies of equal amount, one of which has been renewed today for a year, and the other will be renewed tomorrow for a year. The loading received this morning is not reserved, and consequently we shall have nothing but the loading receivable tomorrow on half the business wherewith to work the whole business for a year. If the actuary rebel against no pay for six months, we can offer him half-pay for twelve months.

"The answer I expect is that the loading receivable tomorrow on policy B is to pay expenses from tomorrow till the 31st December; and that the expenses from 1st January to 30th June next will be recouped from the loading receivable on the 30th June next upon policy A. But the premium may not be received in June next; and, if it be received, the loading it contains will, by the construction of the tables, be applicable to the service of the succeeding year."

"Why was any loading taken this morning upon policy A? Suppose both are new policies and portions of the same risk. The charge was not wanted to pay off a debt incurred in the preceding six months. It is an additional profit, which the calculation of the premiums spread over the whole of life, and which the valuation has at once appropriated."

Mr. Baden subsequently wrote: "That a premium revenue is distributed over the year will not, I presume, be disputed. . . . . . . The 'assumptions' are that every year's expenses are paid out of the premium revenue of the year, without regard to the theory on which the individual premiums that make up the revenue, were calculated; and further, that no condition is imposed on the valuation as to the how and when of a loading being spent, save only that it shall not be spent *before* it is in hand. And these assumptions, I take it, are in agreement with the facts."

He adds that Mr. Higham's method of valuation is not properly to be called a modification of the reinsurance system, but is rather a surrender of the principle of that system. He further says, that both the reinsurance method of valuation and Mr. Higham's "alike are characterized by the fundamental mistake of incorporating the arbitrary quantity ϕ_x with the scientific quantity ϖ_x, and treating this mechanical compound as an organic unity. By this means results are arrived at, the true bearing of which on the facts to be dealt with can only be ascertained afterwards, by resorting to some generally recognized standard of value, and measuring the results thereby. Surely the rational way of dealing with things that have to be measured, is to resort to a true standard at once; and this is what the pure premium system does. If there be, as there well may be, any circumstances which, in the judgment of the actuary, make it desirable to strengthen the reserve, let it be done in a straightforward way, and not by making believe that a false standard of value is a true one. I am rather sceptical as to the public understanding the reassurance or any other system; one thing however may fairly be expected of it, namely, that when it does understand the points at issue, it will give a decided preference to fact over fiction."

Mr. Higham rejoined: "when we know what will be the future rates of mortality and interest, and consequently what is the real value of an annuity for the remainder of life, we shall have one fact to begin with, and will look at the other quantities which enter into the calculation. Not the pure premium, but the gross premium, is receivable under the policy. As regards this part of the formula, fact enters into my method and fiction into Mr. Baden's. If he reply that the case is reversed in the other part of the formula, still we come off quits upon the present count. But I say that the office annual premium at age x is a fact, so is the office annual premium at age $x+n$, so is the value of the annuity. My method uses these three quantities alone, and claims to be founded exclusively upon fact.

"The reinsurance system is the system under which an expiring valuation-period (or partnership) reinsures its outstanding risks with a new period (or partnership) on the terms, as to a present payment, upon which the latter would undertake new business at the premiums receivable on the business taken over. * * * * The annual premium in the prospectus will show how much of an old risk we can take over without a payment down; and the single premium will show the charge to be made for the excess.

"The reinsurance system, as above described, is all that I undertake to defend; and my contribution to the subject is an endeavour to eliminate errors or inconsistencies attributed to the methods by which it has heretofore been applied.

"The old Northampton method included the reinsurance valuation, but it was based on an incorrect table of mortality and a debateable rate of interest. When office charges were introduced, based on assumed true rates of mortality and interest, they were loaded by percentage alone, and this made the reinsurance valuations unworkable. Then came all sorts of valuations; one of them involving an imaginary annuity deduced from the premiums; another, involving negative values for the policies; another, making the true standard of value anything we please. My humble suggestion is that, if we calculate the premiums aright, we may combine the use of true rates of mortality and interest, with the reality, safety, and soundness of the reinsurance valuation. If the mode of calculating the premiums were new or of my own devising, I should offer it with more diffidence; but nearly thirty years ago I quoted De Morgan as the authority for making the charge for expenses a constant addition to the annual premium (*J.I.A.* xx, 6), and I think I am right in saying that the late Griffith Davies had previously in practice made the addition for profit, perhaps for both profit and expenses, a constant addition to the single premium."

Mr. Sprague writes to us on this subject:—Having carefully considered the various arguments brought forward in this correspondence, I find that, although I agree with much that has been said by Mr. Higham on the one side, and Mr. Baden on the other, I am unable to accept the final conclusions of either of them. I quite concur with Mr. Higham in thinking that the office premium actually payable by the policyholder is a "fact", and that in comparison with this the net premium, ϖ_x, which depends upon the table of mortality and the rate of interest that may happen from time to time to be employed in the valuation, is rather deserving of being called a "fiction". Notwithstanding Mr. Baden's great attachment to the net premium, which he dignifies with the title of a "scientific quantity", it is to be observed that, when he makes his valuation, he quietly throws over his net premium. He tells us that in his class lists he enters in one column the office premium and in another column the loading of the ordinary annual premium,—the same loading being valued in all cases, as I understand, whether the premium is payable for a term of years, or has been reduced by bonus, or has been altogether extinguished. In other words, Mr. Baden values in every case the office premiums, and he deducts from the value of them a sum to provide for the expenses and profits of the future. This is, in my opinion, the proper principle upon which life office valuations should be made, and the only difference between Mr. Baden and myself is as to the proper amount of the deduction. This deduction, I maintain, should depend upon the particular circumstances of the office which is being valued, and especially on the amount of the expenditure that it habitually incurs for the purpose of obtaining new business. Mr. Higham and Mr. Baden are both agreed upon the propriety of dealing with facts rather than fictions, but there is one very hard fact which both of them disregard in their valuations, namely, the expense of the new business. If a policy for £500 is effected at the age of 40, at the rate of £3. 5*s*. per-cent, recommended by Mr. Higham, the premium will be £16. 5*s*., out of which must be paid certain expenses attaching exclusively to this policy—probably two medical fees (say), £2. 2*s*.; policy stamp, 5*s*.; commission to the agent, which is very rarely less than 10 per-cent, or £1. 12*s*. 6*d*. These together amount to £3. 19*s*. 6*d*., or (say) £4; and this sum, which must be considered as a minimum expense attaching to the new policy, amounts to 24·6 per-cent of the premium. In many cases there will be much larger expenses attaching specially to the policy, such as agent's procuration fee, over-riding commission paid to a district inspector or branch secretary, which at a very moderate estimate may together be put down at £2. 10*s*. more, thus raising the expense attaching to the individual policy to £6. 10*s*., which is 40 per-cent of the premium. This, however, is not the whole of the expense that is properly chargeable against the policy. The insurance company has incurred a considerable expense for the purpose of obtaining new business; it keeps up a more expensive office than would be necessary if there were no new business, and there is a large outlay for advertisements, printing, stationery, travelling expenses, and various other things, which are all properly chargeable against the new business; and it would be unfair, as between old and new policyholders, to charge these expenses against the premiums of the former. When proper allowance is made for these expenses, I believe that in the case of offices that do a large new business among the general public, it will be found that the expense chargeable against the new policies is about 80 per-cent of the first year's premium. This leaves in hand 20 per-cent of the premium of £16. 5*s.*,—that is, £3. 5*s.*;—and in the circumstances supposed, I hold that this is the proper reserve for the liability under the policy immediately after the premium has been paid. The probability of dying in the year, as calculated in my paper read before the Institute in November last, is ·0055, and the 4 per-cent net premium for an insurance of one year is therefore £2. 12*s.* 11*d.* for the £500, so that we infer that the abovementioned reserve is more than sufficient to cover the risk for the first year. When, therefore, Mr. Higham says that "immediately after receipt of the first annual premium on a policy, the value of the risk is one year's premium. But *which* premium?" I should reply, in the above circumstances, the net premium for a single year's insurance. Mr. Higham speaks of mulcting the recent customer for the benefit of the man who entered at an earlier age. A precisely contrary injustice is committed when a larger reserve is made for recent policies than arises from the excess of their contributions over the claims and expenses; for then a portion of the surplus that is properly divisible among the old policyholders is taken from them, in order to constitute what is often a needlessly large reserve for recent policies; and the injustice is aggravated when a further portion of the surplus arising from the old policies, is diverted from them and applied to declare a bonus on the recent policies.

If there were no special expenses attendant on the issue of a new policy, it would be perfectly right to adopt Mr. Higham's principle to hold in respect of each policy the same sum which the policyholder must put into the chest to obtain a second policy precisely like the first; but when we bear in mind that heavy special expenses attach to the new policy, which do not attach to the precisely similar one already effected, we see there is a good reason for charging a new policyholder a larger sum than we hold in hand; or, to put it in another way, there is legitimate reason for holding in hand a very much smaller sum than we should require a new policyholder to put into the chest.

I agree with Mr. Bailey and Mr. Baden that there is no necessity for making any provision whatever for expenses attaching to the policy before the next annual premium becomes payable. In general, the only expense will be the issuing of a renewal notice, and this may very well be paid out of the general funds of the company. The actuary has done his year's work in respect of a new policy, when it has been issued and the particulars have been duly noted in the office books; and if the company thinks it necessary to make a reserve for the expenses of its staff between the date of the valuation and the date when the next premium falls due, this should be done, not in estimating the value of each policy, but by means of a special entry in the balance-sheet. In this way, although, as Mr. Higham puts it, the well may be pumped dry for the next six months, a sufficient supply will be stored in the water tank for that period.

As regards the general expenses attaching to the business after the first year, my own estimate agrees very closely with Mr. Higham's. As Mr. Bailey says, the 5*s*. per hundred pounds assured amounts to 7·69 per-cent upon the premium; and the conclusions I arrived at, after an examination of the accounts of a large number of different offices, are that the expense of ordinary life insurance companies, when economically managed, amounts to about 50 per-cent of the first year's premiums and 7½ per-cent of the renewals; and that in sound companies, where the expense is larger than this, the excess is chargeable against the new premiums rather than against the renewals.

Mr. Higham does well to point out a fact that is too often forgotten, that when we are considering as to the proper method of valuing the liabilities of a life office, we should not lose sight of the principle upon which the profits are divided. As intimated by him, the method of valuation recommended by Mr. Bailey produces, other things being equal, the same cash bonus at every distribution of profits to a policy that is maintained in force; and the method is therefore inapplicable, when it is the practice of the office to give either a constant or an increasing reversionary bonus. This consideration explains why an office may properly make its valuations at 3 per-cent, although it steadily realizes more than 4 per-cent on its investments.

Mr. Baden, having read the foregoing, writes to us to explain that he calls the net premium, that is, ϖ_x, a scientific quantity, because it is a quantity which has been deduced by scientific methods from scientifically arranged data, embodying certain observed facts. He adds that he is at a loss to conceive what quantity deserves to be called scientific, if that does not; also, that his practice in his valuations is not inconsistent with his theory, as Mr. Sprague's remarks may be thought to imply; in fact, he does not throw over the net premium. The actual formula of his valuation is $A_{x+n}(S+B)-P_x(\cdot 5+a_{x+n})+\phi_x(\cdot 5+a_{x+n})$, where S is sum assured, B bonus, and P office premium. It is true that in this expression the quantity ϖ_x does not appear, but it is necessarily included in $P_x=\varpi_x+\phi_x$.

On the Application ov Lubbock's Formula ov Summation to find the Value ov Annuities and Assurances. *By* T. B. Sprague, M.A.

LUBBOCK'S Formula is one ov a very general character. In the improovd form givn by De Morgan, it may be stated as follows. If $y_0, y_1, y_2, \ldots\ldots\ldots\ldots y_m$ ar successiv equidistant values ov any function, and between each two ajacent values $n-1$ terms ar interpolated folloing the same law, then the sum of the $mn+1$ terms will be equal to

$$n(y_0+y_1+\ldots\ldots+y_m)-\frac{n-1}{2}(y_m+y_0)$$
$$-C_1(\Delta y_{m-1}-\Delta y_0)\quad -C_2(\Delta^2 y_{m-2}+\Delta^2 y_0)-C_3(\Delta^3 y_{m-3}-\Delta^3 y_0)$$
$$-C_4(\Delta^4 y_{m-4}+\Delta^4 y_0)-C_5(\Delta^5 y_{m-5}-\Delta^5 y_0)-\ldots\ldots\ldots,$$

where C_1, C_2, ... ar coeficients depending wholly upon n, the numerical values ov which, as far as C_6, ar givn in my paper on Lubbock's formula in the 18th volume ov this *Jurnl* (see p. 313). In that paper I hav givn som instances ov the aplication ov the formula to the calculation ov anuities, and I now purpos giving som further illustrations, together with a few remarks which they sugest. It may be mentiond in passing that ther is an error ov transcription in the formula as stated on page 310 ov the abov-mentiond paper, and that it shoud be corected so as to agree with the one givn abov.

If the series ov terms, $y_0, y_1, y_2, \ldots\ldots$ is such that the differences ultimatly becom zero, then the abov formula wil be exactly true. This is the case in the exampl I hav givn on page 316 in my former paper, in which I find the sum ov the 4th powers ov the numbers from 1 to 29 inclusiv; but this wil ov cours be very seldom the case in practis, and it therefore becoms ov importance to ascertain to what extent aproximat results givn by the formula can be relyd upon. In my former paper I employd the somwhat simpler form ov the formula obtaind by considering $\Delta y_m, \Delta^2 y_m, \ldots\ldots$ to be al zero, but subsequent experience has led me to the conclusion that better results wil be obtaind by taking acount ov the concluding differences, $\Delta y_{m-1}, \Delta^2 y_{m-2}, \ldots\ldots$, and I shal therefore do this in al the folloing illustrations.

The simplest and most convenient means ov ascertaining the accuracy ov the aproximat results givn by the formula wil be to aply it to calculate the values ov quantities which ar otherwise knoen. I wil therefore in the first instance aply it to calculate the value ov an anuity on a life ov 50 acording to the Carlisle 3 per-cent Tabl. For this purpos we require to kno the values ov the anuity payments (or ov the *endowments*) made at certain fixt intervals, say at the end ov the 5th, 10th, 15th, ... years. In the present case it is not necessary to calculate these values, as they ar al givn in Mr. W. T. Thomson's Actuarial Tabls. Extracting the values then from those tabls and differencing, we get the folloing results :—

$V_0 =$ 1·00000						
	− ·20095					
$V_5 =$ ·79905		+ ·01840				
	− ·18255		− ·01178			
$V_{10} =$ ·61650		+ 662		+ ·01286		
	− ·17593		+ 3108		− ·09381	
$V_{15} =$ ·44057		+ 3770		− 5095		+ ·17455
	− ·13823		− 1987		+ 8074	
$V_{20} =$ ·30234		+ 1783		+ 2979		− ·10887
	− ·12040		+ 992		− 2813	
$V_{25} =$ ·18194		+ 2775		+ 166		+ 281
	− 9265		+ 1158		− 2532	
$V_{30} =$ ·08929		+ 3933		− 2366		+ 5178
	− 5332		− 1208		+ 2646	
$V_{35} =$ ·03597		+ 2725		+ 280		− 3118
	− 2607		− 928		− 472	
$V_{40} =$ ·00990		+ 1797		− 192		+ 1193
	− 810		− 1120		+ 721	
$V_{45} =$ ·00180		+ 677		+ 529		
	− 133		− 591			
$V_{50} =$ ·00047		+ 86				
	− 47					
$V_{55} =$ ·00000						
		·20048 ÷ 5		+ 8240		
Sum = 3·47783		= ·04010		− 7653		
				+ 587 ÷ 4 = 147		

Lubbock's Formula then givs, making $n=5$,

$$\begin{aligned}
a_{50} = 5 \times 3{\cdot}47783 &- {\cdot}2({\cdot}00000 + 1{\cdot}00000) - {\cdot}4(-{\cdot}00047 + {\cdot}20095) - {\cdot}2({\cdot}00086 + {\cdot}01840) \\
&- {\cdot}1264(-{\cdot}00591 + {\cdot}01178) - {\cdot}0896({\cdot}00529 + {\cdot}04286) - {\cdot}068032({\cdot}00721 + {\cdot}09381) \\
&- .05408({\cdot}01193 + {\cdot}17455)
\end{aligned}$$

$$\begin{array}{r|ll}
= 17{\cdot}38915 - 2{\cdot}00000 & = 15{\cdot}38915 & (\text{Error} = +{\cdot}0870) \\
- \;{\cdot}08019 & ,,\; 15{\cdot}30896 & (\;,, \quad ,, \; +{\cdot}0068) \\
- \;{\cdot}00385 & ,,\; 15{\cdot}30511 & (\;,, \quad ,, \; +{\cdot}0029) \\
- \;{\cdot}00074 & ,,\; 15{\cdot}30437 & (\;,, \quad ,, \; +{\cdot}0022) \\
- \;{\cdot}00431 & ,,\; 15{\cdot}30006 & (\;,, \quad ,, \; -{\cdot}0021) \\
- \;{\cdot}00687 & ,,\; 15{\cdot}29319 & (\;,, \quad ,, \; -{\cdot}0090) \\
- \;{\cdot}01008 & ,,\; 15{\cdot}28311 & (\;,, \quad ,, \; -{\cdot}0191)
\end{array}$$

The true value ov the anuity is 15·3022. We see therefore that the values obtaind by taking aditional terms aproach nearer to the truth until we com to the fifth, that is to say, so long as the values ov the terms diminish. The fifth and sixth differences ar so large that we can scarcely expect to obtain a satisfactory result from them. Stopping, then, at the fourth differences, it is to be notist that we obtain a more satisfactory result if, insted ov the sum ov the leading and concluding differences, we substitute a fourth ov the sum ov al the eight fourth-differences, namely, ·00147. The term then becoms $-{\cdot}00132$ insted ov $-{\cdot}00431$, and the value ov the anuity-due, 15·30305, which is very near the true value, the error being ·0008 in exces.

If we stop at second differences and take one-fifth ov the ten second-differences, we hav to substitute for ·00385, $\cdot 2 \times {\cdot}04010 = {\cdot}00802$, and we get as the value ov the anuity, 15·30094. In this case the error is only ·0013 in defect.

As a second exampl, I wil calculate the same anuity value, taking intervals of 11 insted ov 5 years. Proceeding as before, we get the values—

$$\begin{array}{rlllll}
V_0 = & 1{\cdot}00000 & & & & \\
 & & -{\cdot}42150 & & & \\
V_{11} = & {\cdot}57850 & & +{\cdot}09736 & & \\
 & & -{\cdot}32414 & & +{\cdot}02584 & \\
V_{22} = & {\cdot}25436 & & +{\cdot}12320 & & +{\cdot}00096 \\
 & & -{\cdot}20094 & & +\;2680 & \\
V_{33} = & {\cdot}05342 & & +{\cdot}15000 & & -{\cdot}12834 \\
 & & -\;5094 & & -{\cdot}10154 & \\
V_{44} = & {\cdot}00248 & & +\;4846 & & \\
 & & -\;248 & & & \\
V_{55} = & {\cdot}00000 & & & & \\
\hline
\text{Sum} = & 1{\cdot}88876 & & & &
\end{array}$$

Hence

$$\mathrm{a}_{50} = 11 \times 1{\cdot}88876 - 5 - \tfrac{10}{11}(-{\cdot}00248 + {\cdot}42150) - \tfrac{5}{11}({\cdot}04846 + {\cdot}09736)$$

$$- {\cdot}287754(-{\cdot}10154 - {\cdot}02584) - {\cdot}204358(-{\cdot}12834 + {\cdot}00096)$$

$$\begin{array}{rr|lc}
= 20{\cdot}77636 - 5 & & = 15{\cdot}77636 & (\text{Error} = +{\cdot}4742) \\
- & {\cdot}38093 & = 15{\cdot}39543 & (\quad ,, \quad ,, \ +{\cdot}0932) \\
- & {\cdot}06627 & = 15{\cdot}32916 & (\quad ,, \quad ,, \ +{\cdot}0270) \\
+ & {\cdot}03665 & = 15{\cdot}36581 & (\quad ,, \quad ,, \ +{\cdot}0636) \\
+ & {\cdot}02603 & = 15{\cdot}39184 & (\quad ,, \quad ,, \ +{\cdot}0896)
\end{array}$$

Here the formula dos not giv a satisfactory result when we aply it to fourth differences. It wil be seen, however, that the third differences run very irregularly. Stopping then at the second differences, and substituting ½ ov the sum ov those differences insted ov the sum ov the first and last, we get the anuity value

$$= 15{\cdot}77636 - {\cdot}38093 - \tfrac{5}{11}({\cdot}41900)$$

$$= 15{\cdot}39543 - {\cdot}09523$$

$$= 15{\cdot}30020,$$

which is a remarkably close aproximation, considering the smal number ov terms we hav taken, the error being $-{\cdot}0020$.

I wil now take an ilustration ov a different kind, and aply the formula to find the value ov an anuity-certain for 50 years at 5 per-cent interest. Taking intervals ov 7 years, that is, making $n=7$, we get from the tabl upon page 80 ov David Jones's work, the values ov v, v^8, v^{15}, ... where $v = 1 : 1{\cdot}05$.

$$\begin{array}{rllllllll}
v = & {\cdot}952381 & & & & & & & \\
 & & -{\cdot}275542 & & & & & & \\
v^{8} = & {\cdot}676839 & & +{\cdot}079720 & & & & & \\
 & & -{\cdot}195822 & & -{\cdot}023065 & & & & \\
v^{15} = & {\cdot}481017 & & +\ 56655 & & +{\cdot}006673 & & & \\
 & & -{\cdot}139167 & & -\ 16392 & & -1929 & & \\
v^{22} = & {\cdot}341850 & & +\ 40263 & & +\ 4744 & & +554 & \\
 & & -\ 98904 & & -\ 11648 & & -1375 & & -152 \\
v^{29} = & {\cdot}242946 & & +\ 28615 & & +\ 3369 & & +402 & \\
 & & -\ 70289 & & -\ 8279 & & -\ 973 & & \\
v^{36} = & {\cdot}172657 & & +\ 20336 & & +\ 2396 & & & \\
 & & -\ 49953 & & -\ 5883 & & & & \\
v^{43} = & {\cdot}122704 & & +\ 14453 & & & & & \\
 & & -\ 35500 & & & & & & \\
v^{50} = & {\cdot}082704 & & & & & & & \\
\hline
\end{array}$$

Sum $= 3{\cdot}077598$

Hence $a_{\overline{50}|} = 7 \times 3·077598 - 3(·082704 + ·952381) - \frac{4}{7}(-·035500 + ·275542)$

$- \frac{2}{7}(·014453 + ·079720) - ·180758(-·005883 + ·023065)$

$- ·12828(·002396 + ·006673)$

$- ·097510(-·000973 + ·001929)$

$- ·077595(-·000402 + ·000554)$

$= 21·543186 - 3·118755$	$= 18·424431$
$- ·137167$	$= 18·287264$
$- ·026907$	$= 18·260357$
$- ·003106$	$= 18·257251$
$- ·001163$	$= 18·256088$
$- ·000093$	$= 18·255995$
$- ·000074$	$= 18·255921$

The value ov the anuity, as givn on page 92 ov the same work, is 18·255925.

It apears, then, that the formula givs very satisfactory results when aplyd to the calculation ov anuities. Let us see whether it succeeds equally wel with asurances, and for this purpos let us calculate the value ov A_{50} at Carlisle 3 per-cent. In this case also, Mr. Thomson's Tabls giv us the elementary values required, and save us the labor ov calculation. Proceeding as before, then, we get the folloing values :—

$V_1 = ·01302$						
	+ ·00088					
$V_6 = ·01390$		+ ·00526				
	+ 614		− ·01387			
$V_{11} = ·02004$		− 861		+ ·02253		
	− 247		+ 866		− ·02710	
$V_{16} = ·01757$		+ 5		− 457		+ ·01540
	− 242		+ 409		− 1170	
$V_{21} = ·01515$		+ 414		− 1627		+ 5008
	+ 172		− 1218		+ 3838	
$V_{26} = ·01687$		− 804		+ 2211		− 7150
	− 632		+ 993		− 3312	
$V_{31} = ·01055$		+ 189		− 1101		+ 4593
	− 443		− 108		+ 1281	
$V_{36} = ·00612$		+ 81		+ 180		− 1508
	− 362		+ 72		− 227	
$V_{41} = ·00250$		+ 153		− 47		+ 92
	− 209		+ 25		− 135	
$V_{46} = ·00041$		+ 178		− 182		
	− 31		− 157			
$V_{51} = ·00010$		+ 21				3) + 2575
	− 10					+ 858
$V_{56} = ·00000$				4) + 1230		
		5) − 98		+ 307		
Sum = ·11623		− 20				

Hence

$$\begin{aligned}
A_{50} = {} & 5\times\cdot 11623-2\times\cdot 01302-\cdot 4(-\cdot 00010-\cdot 00088)-\cdot 2(\cdot 00021+\cdot 00526) \\
& -\cdot 1264(-\cdot 00157+\cdot 01387)-\cdot 0896(-\cdot 00182+\cdot 02253) \\
& -\cdot 068032(-\cdot 00135+\cdot 02710)-\cdot 05408(\cdot 00858)
\end{aligned}$$

$$\begin{array}{r|lr}
=\cdot 58115-\cdot 02604 & =\cdot 55511 & (\text{Error}=+\cdot 00081) \\
+\cdot 00039 & =\cdot 55550 & (\quad ,, \quad ,, \; +\cdot 00120) \\
-\cdot 00109 & =\cdot 55441 & (\quad ,, \quad ,, \; +\cdot 00011) \\
-\cdot 00154 & =\cdot 55287 & (\quad ,, \quad ,, \; -\cdot 00143) \\
-\cdot 00185 & =\cdot 55102 & (\quad ,, \quad ,, \; -\cdot 00328) \\
-\cdot 00175 & =\cdot 54927 & (\quad ,, \quad ,, \; -\cdot 00503) \\
-\cdot 00046 & =\cdot 54881 & (\quad ,, \quad ,, \; -\cdot 00549)
\end{array}$$

The corect value is ·55430. We therefore get the best result by stopping at second differences; and the hiher differences run so irregularly, and increas in absolute magnitude so fast, that we cannot be at al surprized that it is useles to take acount ov them. If we substitute the fifth part ov the sum ov the second differences, —·00020, insted ov the sum ov the first and last, we get the value ·55554, which is not so accurat as the value abov found, ·55441.

The general conclusions ar that, when the terms givn by the formula increas, we must stop, at al events, after the first that is greater than the previos one; and that, when the differences run irregularly, it is useles to take acount ov those after the second. The question, however, seems to require further consideration.

ACTUARIAL NOTES.

Boole's Finite Differences.

We have received from Mr. Richard Wilding a letter, pointing out an inconsistency in Boole's *factorial* notation, in his standard work on Finite Differences.

On page 6, Boole puts

$$x(x-1)\;\ldots\ldots\;(x-m+1)=x^{(m)}.$$

From this expression, we see that

$$x^{(m-1)}=x^{(m)}\div(x-m+1).$$

Now $$x^{(2)}=x(x-1).$$

Hence

$$x^{(1)}=x(x-1)\div(x-1)=x$$

$$x^{(0)}=x\div x=1$$

$$x^{(-1)}=1\div(x+1)=\frac{1}{x+1}$$

$$x^{(-2)}=\frac{1}{x+1}\div(x+2)=\frac{1}{(x+1)(x+2)}\,.$$

And following the same law, we shall have

$$x^{(-m)}=\frac{1}{(x+1)(x+2)\,\ldots\ldots\,(x+m)}\,.$$

But Boole has, on page 7, used $x^{(-m)}$ to denote the following function:

$$\frac{1}{x(x+1)\,\ldots\ldots\,(x+m-1)}\,,$$

which should rather be denoted by $(x-1)^{(-m)}$.

A similar inconsistency occurs in Chapter iv, Article 2, where Boole puts $u_x=(ax+b)$ and $u_xu_{x-1}\ldots u_{x-m+1}=(ax+b)^{(-m)}$. It may be shown, as above, that $(ax+b)^{(m)}$ should denote

$$\frac{1}{\{a(x+1)+b\}\{a(x+2)+b\}\,\ldots\ldots\,\text{to } m \text{ terms}}\,,$$

or

$$\frac{1}{u_{x+1}u_{x+2}\ldots u_{x+m}}\,,$$

but Boole uses it to denote

$$\frac{1}{u_xu_{x+1}\ldots u_{x+m-1}}\,,$$

a quantity which it would be more correct to denote by

$$\{a(x-1)+b\}^{(-m)}.$$

Mr. Wilding further remarks that the equation

$$\Sigma(ax+b)^{(m)}=\frac{(ax+b)^{(m+1)}}{a(m+1)}+c$$

holds good with both notations.

The Market Value of an Annuity. By RICHARD WILDING.

THE usual formula for the market value of an annuity of 1 on life (x) is obtained by supposing that a lender of a unit should receive from the borrower an annuity-due of $P_x(1+j)$ to assure his capital and last year's interest, and an annuity of j—where P_x is the office premium for assuring 1 on life (x), and j is the rate of interest which the lender expects to realize, *i.e.*, the "remunerative" rate. The formula deduced is

$$\frac{1}{P_x+d}-1, \text{ where } d=\frac{j}{1+j}.$$

Now, it seems to be simpler to suppose that the capital *only* should be assured, and some other arrangement made for securing to the lender the *last* year's interest, j.

This may be done by allowing the borrower to pay every year's interest one year in advance (thus making his payments of premium and interest together), and giving him discount on these prepayments at the rate of interest, say that granted on 12 months' deposits—*i.e.*, it should be discounted at the "accumulative" rate, i. The borrower pays thus an annuity-due of $P_x+vj\left(v=\frac{1}{1+i}\right)$ for the loan of a unit, or an annuity of 1 for a loan of $\frac{1}{P_x+vj}-1$.

This may be arrived at in another way. Suppose the investor of 1 to receive, as before, an annuity-due of P_x (to replace his capital), an annuity of j, and such a present value of the interest j at the end of (x)'s year of death as would accumulate to j, at the accumulative rate i, by that time. This is evidently $j.A_x$, where A_x is calculated at rate i—*i.e.*, $A_x=\frac{1-ia_x}{1+i}$.

Thus the lender receives for his unit

$$P_x(1+a'_x)+j.a'_x+j\cdot\frac{1-ia'_x}{1+i}=1$$

(a'_x being the *market* value of an annuity of 1 on (x)), from which is easily deduced, as before,

$$a'_x=\frac{1}{P_x+vj}-1.$$

The above formula is clearly to the advantage of the buyer of the annuity—$(i<j)$.

The same supposition on which the above is based, being made, it is necessary to substitute vj for d in all Mr. Jellicoe's formulas.

INSTITUTE OF ACTUARIES.

PRELIMINARY EXAMINATION, 1879.

Examiners.—Messrs. J. Heron Duncan, J. Whitcher, F. B. Wyatt, and T. E. Young.

I.

1. Explain the use of the *Journal* in a system of book-keeping by Double Entry.

2. (α) In a Profit and Loss Account, the losses exceed the profits: on which side of the Balance Sheet should the balance of this account appear? Give your reason.
   (β) An annual premium, debited to an agent for collection, is altered by endorsement to a half-yearly one. Explain what entries should be made in consequence of the change.

3. (α) Explain concisely the difference between a Decimal and a Vulgar Fraction.
   (β) Reduce a guinea to the decimal of £4. 10*s*. 4*d*.
   (γ) Multiply 97260 by $\cdot 1\dot{6}$.
   (δ) Given as dividend thirteen millions seven hundred and seventeen thousand four hundred and twenty-one, and as divisor one hundred and eleven times the sum of one, one thousand, and one million,—Find the quotient in the quickest possible way.

4. Solve the equations:—

$$(\alpha)\quad \frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b}.$$

$$(\beta)\quad \left.\begin{aligned} x^2(x+y+z)&=108\\ 2y^2(x+y+z)&=384\\ 3z^2(x+y+z)&=900 \end{aligned}\right\}.$$

5. A clerk is sent to an office to buy some stamped forms, for which he pays £3. If they had been cheaper by 2*d*. each, he would have obtained four more than he did. What was the price a dozen?

6. The same six hundred lives are proposed for assurance at each of two offices, A and B. A accepts, but B declines a majority of them; but all that A declines are also declined by B. The number A accepts is to the number B rejects as 7 to 8; and the excess of acceptances over refusals in A is half that of refusals over acceptances in B. Find the number of lives accepted in one office only.

7. Show that the roots of the equation $x^2+mx+n=0$ are real and different, real and equal, or impossible, according as m^2 is $>$, $=$, or $< 4n$.

8. The product of the roots of the equation $kx^2+lx+m=0$ is one-third the sum of their squares: prove that $l^2=5km$.

9. The product of five numbers in Arithmetical Progression is 945, and their sum is 25. Find them.

10. Show that any term in an infinite Geometrical Progression is $>$, $=$, or $<$ the sum of all the terms that follow, according as the common ratio is $<$, $=$, or $> \frac{1}{2}$.

11. (*a*) Investigate the formulæ for Permutations and Combinations.
    (*β*) How many different weights, consisting of a whole number of pounds, can be formed by means of the six weights, 1, 2, 4, 8, 16, and 32 lbs., taking them singly, then two by two, three by three, and so on, till all are taken together?

12. Expand $\sqrt{(\sqrt[3]{a}-x)^3}$ to four terms by the Binomial Theorem, and prove that no coefficient in the whole expansion can be $> \frac{3}{8}$.

13. (*a*) Find the logarithm of $81\sqrt[3]{3}$ to the base $\sqrt[3]{3}$.
    (*β*) How would you transform a system of logarithms from the base 8 to the base 4?
    (*γ*) What advantages belong to a system of logarithms with 10 as the base?

14. (*a*) Find, without tables, to three places of decimals, the number of which $1\cdot\dot{3}$ is the common logarithm.
    (*β*) If the logarithm of 25 is $\cdot\dot{6}$, find the base of the system.
    (*γ*) Solve the equation:—

$$\left.\begin{aligned}\frac{3^x}{2^{x+y}} &= 8\\ 2^x &= 8^y\end{aligned}\right\}.$$

15. A is one of a Board of ten Directors who are about to elect, first a Chairman, and then a Deputy-Chairman for the year, whose fees will amount to £200 and £150 respectively. Find the values of A's expectation (*a*) of being elected Chairman; (*β*) of being elected Deputy.

II.

16. A is twice as old as B was when A was as old as B is now, who is three times as old as C: and B will be twice as old as C, and as old as A is now, when their united ages amount to 154 years. What are their present ages?

17. *Assuming* that $1^2+2^2+3^2+\ .\ .\ .\ +n^2=n.\overline{n+1}.\overline{2n+1}\div\underline{|3}$, find $1+3+6+10+\ .\ .\ .\ .\ .$ to n terms.

18. Sum to n terms:—

$$(a)\quad 1+r+(1+b)r^2+(1+b+b^2)r^3+\ .\ .\ .\ .\ .\ .$$

$$(\beta)\quad 1.2+3.4+5.6+7.8+9.10+\ .\ .\ .\ .\ .\ .$$

19. (*a*) The reciprocals of four numbers are in Arithmetical Progression; the first, second and fourth are in Geometrical Progression; and their sum is 100. Find them.
    (*β*) Between a and b are inserted n arithmetical means; and between the pth and qth of these last, r more; find the kth of these r means.

20. In the expansion of $\frac{1}{(1-x)^2}$, prove that if $x=\frac{n+1}{n}$ the expression will be equal to the first n terms of the expansion.

21. A and B play five games. A's skill is to B's as 2 to 3. What is the chance

(α) that A wins the first two games?

(β) that he wins two and only two?

22. Two persons are known to have passed over a piece of road in opposite directions within the time $x+y+z$, and in the times x and y respectively. Find the chance that they will meet.

23. If u_x be any function of x of n dimensions, prove that $\Delta^n u_x$ is constant: and hence show how to form a table of cubes of natural numbers expeditiously.

24. Investigate the expressions:

(α) for u_{x+n} in terms of Δu_x.

(β) for $\Delta^n u_x$ in terms of u_x and its successive values.

25. (α) Having given $\lambda 101 \; = 2{\cdot}0043214$
$\lambda 101{\cdot}5 = 2{\cdot}0064660$
$\lambda 102 \; = 2{\cdot}0086002$
$\lambda 102{\cdot}5 = 2{\cdot}0107239$
$\lambda 103 \; = 2{\cdot}0128372$
$\lambda 104 \; = 2{\cdot}0170333$,

and assuming that the 6th differences vanish, find log 103·5.

(β) Given $u_x=100000$, $\delta_1=-490$, $\delta_2=+93$, $\delta_3=-25$, $\delta_4=2$, $\delta_5=0$, find the first eight terms of the series, u_x, u_{x+1}, &c.

INTERMEDIATE EXAMINATION, 25TH APRIL, 1879.

Examiners.—Messrs. T. G. BROWNE, T. H. COOKE, C. D. HIGHAM, and H. W. MANLY.

PART I.

1. Find an expression for the probability that a person aged x will pre-decease another aged y.

2. Two offices have each £1,000,000 assured—Office A, by 100 policies of £10,000 each; B, by 1,000 of £1,000 each. Assuming all the ages equal, and the rate of mortality to be 2 per-cent per annum, give an expression for the probability in each case that the claims will amount in one year to £30,000 at least.

3. Find what accumulative Sinking Fund per-cent will repay a loan in 20 years at 5 per-cent interest; the amount to which 1 will accumulate in 20 years at that rate being 2·653.

4. The present value of an annuity of £a for n years certain is equal to the amount of an annuity of £b for n years certain.—Find the present value of £1 due at the end of n years.

5. Show that the time in which a sum of money will double itself at compound interest is approximately represented by $\cdot 69 \div i$.

6. Give a short account of the materials from which the "Peerage" Table was constructed, and state for what class of benefits it is specially suitable.

7. Define "force of mortality" and "rate of mortality", and write down the approximate relation between the two functions.

8. $a_x - a_{x.y} - \frac{1}{4}\mathrm{A}_{x.y}^{\;1}$ has been given as the formula for the value of a half-yearly reversionary annuity payable during the life of x after the death of y.—On what assumption is this solution approximately correct?

9. Explain the construction of a Table of life annuity values by the continuous method.

10. Given a table of Single and Annual Premiums for Survivorship Assurance,—Show how the rate of interest can be found.

11. Given the formula for converting an annuity payable yearly into one payable m times a-year,—Investigate a rule for finding a premium payable m times a-year from the annual premium.

12. If l_x persons each buy an endowment at an annual premium, show that the total amount which will be paid to the survivors is made up of an accumulation of (1) the premiums paid by the survivors, and (2) the premiums paid by those that die.

13. Analyze the risk undertaken by an office in granting a Ten-Year Non-Forfeitable Policy, and explain what element is left out of consideration when the ordinary ten-year premium is charged.

PART II.

14. Describe the construction of the D and N columns for Joint Lives proposed by Professor De Morgan, and applied by Dr. Farr to the English Life Table No. 3; and point out in what respect it differs from that previously used. Write down in commutation symbols the single premium for a survivorship assurance by both methods.

15. Prove that

$$ {}_n\mathrm{V}_x = 1 - (1 - {}_1\mathrm{V}_x)(1 - {}_1\mathrm{V}_{x+1}) \;.\;.\;.\;.\;.\;.\;.\; (1 - {}_1\mathrm{V}_{x+n-1}), $$

and give a verbal explanation of $\quad {}_n\mathrm{V}_x = \mathrm{A}_{x+n}\left(1 - \frac{\mathrm{P}_x}{\mathrm{P}_{x+n}}\right)$.

16. "Suppose the annual premiums to increase or decrease a certain sum every t years, and at the end of v intervals of t years each the premium to continue constant during the remainder of life, —What annual premium should be required during the first t years?" Jones gives as the answer to this question

$$ \mathrm{P} = \frac{\mathrm{M}_x \mp q(\mathrm{N}_{x+t-1} + \mathrm{N}_{x+2t-1} + \;.\;.\;.\;.\;.\;.\; \mathrm{N}_{x+vt-1})}{\mathrm{N}_{x-1}}, $$

while Chisholm, correcting him, gives

$$ \mathrm{P} = \frac{\mathrm{M}_x}{\mathrm{N}_{x-1} \pm q(\mathrm{N}_{x+t-1} + \mathrm{N}_{x+2t-1} \;.\;.\;.\;.\;.\;.\; \mathrm{N}_{x+vt-1})}. $$

State the different conditions under which both answers are correct.

17. Derive an expression for the nth presentation to a living.

18. Find a formula to express the average amount at death to which the payments of a life annuity due invested at compound interest will accumulate; and explain the difference between that value and $N_{x-1} \div M_x$.

19. Find a formula for the annual premium to assure a payment on x attaining the age of $x+n$, or at death if previous, provided y be living in either case; and show how it can be adapted for the use of the values tabulated in the Institute of Actuaries' Life Tables.

20. A person wishes to convert an endowment assurance policy into one payable at death; what premium should he be charged in future? and state the condition under which the conversion could be allowed.

21. An annuity certain of £*a* for the term of *n* years is to be enjoyed by P and his heirs during the joint existence of two lives aged x and y; and if that joint existence fail before the expiration of *m* years, the annuity is to go to Q and his heirs for the remainder of the term of *n* years. Determine the value of Q's interest.

22. Give the formula for the value of a life interest of £*a* per annum, and state for what amount the life policy should be effected. What is the value of such policy to the purchaser of the life interest after the expiration of *t* years?

23. Find the value of $A^{1}_{x:yz}$.

24. Draw up forms for

(*a*) Register of Claims.
(*b*) Valuation Class Book.

25. Suppose a person has understated his age in his proposal, and the error is not discovered until some years after the policy has been issued,—How would you propose to rectify the error? Draft an endorsement for carrying out your views.

PROCEEDINGS OF THE INSTITUTE.—SESSION 1878-9.

First Ordinary Meeting, Monday, 25 *November* 1878.

The following gentlemen were elected members of the Institute, namely—

Fellows.

Tucker, Robert Charles.
Turnbull, Andrew Hugh.

Associates.

Anderson, John.
Ashton, Frederick Musgrave.
Barnes, Joseph Howard.
Brandon, Harold.
Bridgman, Arthur Henry.
Byers, Frederick Mason Timothy.
Cooke, George.
Estcourt, Rowland Metzner.
Gill, Henry William.
Henry, Alfred.
Jones, Philip Alexander.
Marshall, William.
Palmer, James William.
Reid, William John.
Robertson, William.
Tripe, Harry.
Walton, William Gandy.
Warner, Samuel George.

A paper, by Mr. T. B. Sprague, on "The Construction and Use of a Series of Select Mortality Tables, to be employed in combination with the Institute $H^{M(5)}$ Table", was read by the Hon. Sec., Mr. G. Humphreys.

The following gentlemen took part in the discussion:—The President, Messrs. G. W. Berridge, G. Humphreys, G. King, W. Sutton, Cornelius Walford, and J. Whitcher.

Second Ordinary Meeting, 6 *January* 1879.

The President in the Chair.

The following gentlemen were elected members of the Institute, namely—

Fellow.

Cockburn, Henry.

Associates.

Berry, Berry Alfred.	Leith, Alexander Muir.
Cox, Frederick William Pope.	Stokes, Henry E.
Frankland, Frederick William.	Wall, Walter George.
Gassett, Arthur.	Williams, Henry Richmond.

The report of the adjudicators on the "Messenger Prize Essay" was read. The first prize was awarded to the Essay bearing the motto "Every Why hath a Wherefore." It was announced that the author of the Essay was Mr. G. S. Crisford.

A paper, prepared by Prof. Pell, on "The Rate of Mortality in New South Wales", with a Note on "The formation of Commutation Tables", was read by the Hon. Sec., Mr. G. W. Berridge.

In the discussion which arose on the paper, the following gentlemen took part:—The President, Messrs. A. J. Finlaison, P. Gray, Gen. Hannyngton, G. King, F. G. P. Neison, and W. Sutton.

Gen. Hannyngton showed and explained the use of a new slide rule which possesses many advantages over the rule in ordinary use.

Third Ordinary Meeting, 27 *January* 1879.

The President in the Chair.

The following gentlemen were elected members of the Institute, namely—

Associates.

Anderson, John.	Hore, William Henry.

Mr. G. S. Crisford then read the Messenger Prize Essay on "The Values that should be allowed by a Life Office for the Surrender of its Policies."

In the subsequent discussion the following gentlemen spoke:—The President, Messrs. F. A. Curtis, R. P. Hardy, J. R. Macfadyen, H. W. Manly, R. Moore, T. Y. Strachan, and Cornelius Walford.

Fourth Ordinary Meeting, 24 *February* 1879.

The President in the Chair.

The following gentlemen were elected members of the Institute, namely—

Associates.

Chappell, Henry Ailes.	Macaulay, Thomas Bassett.
Lomas, Pouis Louis.	Moon, Sidney Norman Laming.

Mr. E. Smyth then read a paper on "Tables for the Enfranchisement of Copyholds of Inheritance."

The President, Messrs. Berridge, Humphreys, Searle, and Sutton, took part in the discussion which ensued, and Mr. Smyth replied to the criticisms made.

Fifth Ordinary Meeting, 31 *March* 1879.

The President in the Chair.

The following gentlemen were elected members of the Institute, namely—

Fellow.

Besso, Marco.

Associates.

Bellamy, George Claxton. | Elliott, John David.
Tarn, Arthur Wyndham.

Mr. G. Humphreys, Hon. Sec., then read a paper prepared by Mr. T. B. Sprague, "On the Construction of a Combined Marriage and Mortality Table from Observations made as to the Rates of Marriage and Mortality among any body of Men; and on the Calculation of the Values of Annuities and Assurances that depend on the Contingency of Marriage as well as Death, and their application to determine the Rate of Premium for an Insurance against the Contingency of a Bachelor of a given age leaving Issue: illustrated by various Tables calculated from the experience of the British Peerage families."

The following gentlemen took part in the discussion on this paper:—The President, Messrs. Adler, Berridge, Bunyon, Day, Hardy, Macfadyen, Neison, Ambrose Smith, and Cornelius Walford.

Sixth Ordinary Meeting, 28 *April* 1879.

The President in the Chair.

The following gentlemen were elected members of the Institute:—

Fellow.

Harvey, Charles James.

Associates.

Boulton, Joseph. | Monilaws, William MacGregor.
Clirehugh, William George. | Roberts, Thomas B.

A paper prepared by Mr. Cornelius Walford, entitled "A Suggestion towards adding a New Feature of Usefulness to the Institute of Actuaries of Great Britain and Ireland", was read by Mr. R. P. Hardy.

In the discussion which arose on the paper, the following gentlemen took part:—The President, Messrs. Baden, Cutcliffe, Curtis, J. W. Gordon, Hardy, Higham, Humphreys, Manly, Messent, Morris, Oliver, F. W. White, and W. White.

The Thirtieth Annual General Meeting, Saturday, 7 *June* 1879.

The President (Mr. A. H. BAILEY) in the Chair.

Mr. G. HUMPHREYS (Hon. Sec.) read the notice convening the meeting, the minutes of the last ordinary meeting, the following Report of the Council, and the Statement of Accounts:—

"The Council have great pleasure in reporting on the progress of the Institute during its thirtieth session. Thirty-seven new members have been elected, two of whom are Fellows. The total number of members on the 31st March 1879 was 374, being an increase of twelve over the number at the end of the previous year.

"The total income during the year was £796. 16*s*., and the total expenditure £614. 10*s*. 9*d*. The funds of the Institute now amount to £2,803. 11*s*. 11*d*., as against £2,611. 5*s*. 10*d*. at the close of last year.

"Papers on the following subjects were read at the ordinary meetings:—

"*First*—'On the Construction and Use of a Series of Select Mortality Tables, to be employed in combination with the Institute $H^{M(5)}$ Table'—by Mr. T. B. Sprague.

"*Second*—'On the Rate of Mortality in New South Wales, with a Note on the Formation of Commutation Tables'—by Professor Pell.

"*Third*—'On the Values that should be allowed by a Life Office for the Surrender of its Policies'—by Mr. G. S. Crisford.

"*Fourth*—'On Tables for the Enfranchisement of Copyholds of Inheritance'—by Mr. E. Smyth.

"*Fifth*—'On the Construction of a Combined Marriage and Mortality Table from Observations made as to the Rates of Marriage and Mortality among any body of Men; and on the Calculation of the Values of Annuities and Assurances that depend on the Contingency of Marriage as well as Death, and their application to determine the Rate of Premium for an Insurance against the Contingency of a Bachelor of a given age leaving Issue: illustrated by various Tables calculated from the experience of the British Peerage families'—by Mr. T. B. Sprague.

"*Sixth*—'On a Suggestion towards adding a New Feature of Usefulness to the Institute of Actuaries of Great Britain and Ireland'—by Mr. Cornelius Walford.

"EXAMINATIONS.

"The following is the result of the examinations which were held on the 25th and 26th April last:—

"*First Year.*

"The examiners report that, out of thirty-four candidates that presented themselves for the first year's examination, two retired, and eight passed, in the following order of merit:—

1.—Hubert Ansell.
2.—W. M. Monilaws.
3.—J. Anderson } Æq.
R. A. Berry }
5.—A. W. Tarn.
6.—Sydney S. Allen.
7.—Herbert Lucey.
8.—W. G. Wall.

"*Intermediate Examination.*

"The examiners report that fifteen gentlemen presented themselves for this examination, and that four passed, in the following order of merit:—

1.—George Todd.
2.—H. J. Rothery.
3.—R. Wilding.
4.—W. Somerville.

"No gentleman presented himself for the final examination.

"It has been thought desirable that a complete list should be recorded in the *Journal of the Institute* of all who have passed the examinations of the Institute. Such a list will therefore be published as an appendix to this Report.

"TEXT-BOOK.

"The Council are informed by Mr. Sutton that substantial progress has been made towards the completion of the text-book. A large portion of matter is in type, nearly ready for the revision of the Text-Book Committee, and this would have been materially increased had not pressing business engagements temporarily interfered with the prosecution of the work.

"LIBRARY.

"A complete list of all the books in the library has been prepared. Some additions have been recently made to the collection, but further purchases will be desirable to make the library more generally useful. When this has been accomplished, the Council contemplate the publication of a printed catalogue for the use of the members.

"MESSENGER PRIZE.

"The competition for the Messenger Prize, on the subject of the "Surrender Values of Policies" was most satisfactory. Five essays were sent in, all of which showed a considerable amount of thought, and evidence of careful investigation. The successful competitor was Mr. G. S. Crisford."

The President moved and Mr. A. Baden seconded the adoption of the Report, and the Accounts for the year, which was agreed to unanimously.

The following gentlemen were unanimously elected as President, Vice-Presidents, Council, and Officers, for the ensuing year:—

President.

Arthur H. Bailey.

Vice-Presidents.

Andrew Baden.
Charles John Bunyon, M.A.
George Cutcliffe.
Ralph Price Hardy.

Council.

Andrew Baden.
Arthur H. Bailey.
George William Berridge.
Thomas G. C. Browne.
Charles John Bunyon, M.A.
Frank Allan Curtis.
George Cutcliffe.
Archibald Day.
David Deuchar.
John Duncan.
Alexander John Finlaison.
John Ralph Grimes.
Major-Gen. J. C. Hannyngton.
Ralph Price Hardy.
*Stewart Helder.
*Augustus Hendricks.
William Barwick Hodge.
George Humphreys, M.A.
Frank McGedy.
Henry William Manly.
James Meikle.
Edward A. Newton, M.A.
Henry William Porter, B.A.
*H. Ambrose Smith.
Thomas Bond Sprague, M.A.
William Sutton, B.A.
Spencer Campbell Thomson, B.A.
James Valentine.
John Hill Williams.
Thomas Emley Young, B.A.

Note.—Those marked * are new members.

Treasurer.

John Ralph Grimes.

Honorary Secretaries.

George William Berridge.
George Humphreys, M.A.

It was unanimously resolved that a vote of thanks be given to the President, Vice-Presidents, Council, Officers, and Examiners, for their services during the past year.

INSTITUTE OF ACTUARIES.

Income and Expenditure for the Year ending 31 *March* 1879.

Dr.

1878-9.	£ s. d.	£ s. d.
Amount of Funds on 31 March 1878—		
Messenger Legacy Fund	282 16 0	
Hardy Memorial Fund	247 19 5	
Brown Prize Fund	252 4 1	
General Fund	1,828 6 4	
		2,611 5 10
Annual Subscriptions—		
Town Fellows	247 16 0	
Country „	70 7 0	
Town Associates	308 14 0	
Country „	94 10 0	
		721 7 0
Mortality Experience Sales		4 12 6
Log. Card Sales		1 9 0
Dividends and Interest		79 8 4
		£3,418 2 8

Cr.

1878-9.		£ s. d.	£ s. d.
Rent		88 15 0	
Salaries		159 0 0	
Journal Account	205 1 0		
Less Sales	7 18 0		
		197 3 0	
Stationery		63 16 4	
Ordinary Meetings		19 11 0	
House Expenses		31 9 5	
Postage		13 0 6	
Library		12 3 4	
Miscellaneous Expenses		3 7 2	
			588 5 9
Messenger Prize			26 5 0
Amount of Funds on 31 March 1879—			
Messenger Legacy Fund		262 15 0	
Brown Prize Fund		260 1 1	
General Fund		2,280 15 10	
			2,803 11 11
			£3,418 2 8

Balance Sheet, 31 *March* 1879.

Dr.

	£ s. d.	£ s. d.
Messenger Legacy Fund (£211. 1*s*. 10*d*. Consols), cost	203 17 8	
Unappropriated Dividends	58 17 4	
		262 15 0
Brown Prize Fund (£200 Metropolitan Railway 4 per-cent Debenture Stock)	200 0 0	
Unappropriated Dividends	60 1 1	
		260 1 1
General Fund		2,280 15 10
		£2,803 11 11

Cr.

	£ s. d.	£ s. d.
Consols, £1,100 Stock		1,008 19 3
Metropolitan Railway 4 per-cent Debenture Stock, £1,200		1,225 17 3
Cash at London and Westminster Bank	560 10 10	
Less due to Petty Cash	0 3 5	
		560 7 5
Arrears of Subscriptions:		
Country Fellows	2 2 0	
„ Associates	6 6 0	
		8 8 0
		£2,803 11 11

Examined and found correct, 9 May 1879.

(Signed) U. W. Gage, Thomas J. Searle, Francis W. White, *Auditors.*

THE EXAMINATIONS OF THE INSTITUTE.

The Council of the Institute have resolved that in future no examination fee is to be returned to the unsuccessful candidates.

The Examinations will take place in April.

The following is a complete list of all who have passed the Examinations:—

1850.

Bailey, A. H.
Day, A.
Orchard, W.
Porter, H. W.
Scoones, P.
Squance, J. R.

1851.

Cheshire, E.
Child, C.
Colvin, A.
Meikle, J.
Thomson, H.

The above became entitled at once to receive Certificates of Competency, but after this date it was necessary that candidates should pass three examinations in different years before they became so entitled.

1852.

1st Year.

England.—Scott, G.
„ Galsworthy, E. H.
„ Beverly, W.
„ Haycraft, J. B.
„ Carr, T.
„ Coles, J.
„ Cutbush, E.
„ Griffiths, C.

England.—Laurence, J. P.
„ Watkins, C.
Scotland.—Huie, D. R. W.
„ Miller, T.
„ Smith, D. C.
„ Martin, C. M.
„ Wilson, J.
„ Ramsay, A. G.

1853.

1st Year.

E.—Pattison, W. P.
„ Fothergill, C. G.
„ Younger, Samuel.
„ Hatton, Robert.
„ Humphreys, Geo.

E.—Allan, J. B.
„ Robinson, D. R. C.
„ Nuth, Edward.
S.—Birkmyre, W. F.

2nd Year.

E.—Haycraft, J. B.
„ Scott, George.
„ Galsworthy, E. H.
„ Laurence, J. P.
„ Coles, Jno.
„ Watkins, Charles.

E.—Griffiths, Chas.
„ Carr, Thomas.
„ Miller, T., Jun.
S.—Huie, D. R. W.
„ Wilson, James.

1854-5.

1st Year.

Rumsey, A.
Bishop, H.
Turnbull, A. H.
Johnston, H. J.

2nd Year.

Allan, J. B.
Younger, S.
Hatton, R.
Robinson, D. R. C.
Birkmyre, W. F.

1855-6.

1st Year.

Scargill, E. T.
Racine, C. F.
Smith, J. S.
Hooton, J. G.
Berridge, G. W.
King, W.
Richmond, W.

2nd Year.

Fothergill, C. G.

3rd Year.

Haycraft, J. B.
Millar, T.
Birkmyre, W. F.
Allan, J. B.
Scott, G.
Coles, J.
Watkins, C.

1856-7.

1st Year.

Clare, S. M.
Wallis, W.
Helder, S.
Terry, J.

2nd Year.

Pattison, W. P.
Scargill, E. T.
Smith, J. S.
King, Wm.
Bishop, H.

1857-8.

1st Year.		2nd Year.	3rd Year.
Adler, M. N.	Norfolk, W. J. F.	Berridge, G. W.	Smith, J. S.
Finlaison, J., Jun.	Grant, W. B.	Terry, J.	Carr, T. Younger, S. } Eq.
Finlaison, A. J.	Walker, E. B.	Wallis, W.	
		Clare, S. M.	

1858-9.

1st Year.		2nd Year.	3rd Year.
Grimley, H. N.	Bischoff, C.	Adler, M. N.	King, W.
Mullins, W. C.	Booth, W.	Helder, S.	Clare, S. M.
Laing, C. G.	Norfolk, H. E.	Grant, W. B.	

1859-60.

1st Year.	2nd Year.		3rd Year.
Green, A. H.	Fiulaison, A. G.	Laing, C. G.	Pattison, W. P.
Makeham, W. M.	Mullins, W. C.	Bischoff, C., Jun.	Terry, J.
King, S.			

1860-1.

1st Year.		2nd Year.	3rd Year.
Saunders, C. R.	Manly, H. W.	*Nil.*	*Nil.*
Hardy, R. P.	Wilkins, C. J.		
Knowles, J. R.	Strachan, T. Y.		
Harper, F.	Henderson, J.		
Stark, J.			

1861-2.

1st Year.		2nd Year.	3rd Year.
Smith, J. N.	Searle, T. J.	Makeham, W. M.	Adler, M. N.
Bumsted, D. A.	Thomson, S. C.	Hardy, R. P.	
Taylor, F. J. C.	Christie, M. P.	Finlaison, J.	

1862-3.

1st Year.		2nd Year.	3rd Year.
Haycraft, C. F.	Smither, A.	Manly, H. W.	Finlaison, A. J.
Macgregor, D. H.	Justican, E.	Saunders, C. R.	
Macdonald, W. E. S.	Johnson, T. H.		
Waterhouse, E.	Clarke, R.		
Addiscott, F.			

1863-4.

1st Year.		2nd Year.	3rd Year.
Hare, J. M. Day, E. F. } Eq.	Hughes, W.	Bumsted, D. A.	Berridge, G. W.
	Smith, A. H.	Strachan, T. Y.	Makeham, W. M.
Curror, J.	Kincaid, S. B.		
Mountcastle, H.			

1864-5.

1st Year.	2nd Year.	3rd Year.
Vaughan, Wm.	Haycraft, C. F.	Laing, C. G.
Smith, H. S.	Wilkins, C. J.	
Crisford, G. S.	Day, E. F.	
Smyth, E.	Addiscott, F.	

1865-6.

1st Year.	2nd Year.	3rd Year.
Pierce, J.	Mountcastle, H.	Manly, H. W.
Hopkinson, J. C.		Bumsted, D. A.
Symes, R. W.		
Duncan, J.		
Nesbitt, R. H.		

1866-7.

1st Year.		2nd Year.	3rd Year.
Lambert, C. J.	Cowley, H.	Symes, R. W.	Hardy, R. P.
Sutton, W.	Kember, W.	Justican, E.	Haycraft, C. F.
Chisholm, J.			

1867-8.

1st Year.	2nd Year.	3rd Year.
Carment, D.	Vaughan, W.	Mountcastle, H.
Macdonald, Wm. Rae.	Sutton, W.	Strachan, T. Y.
Bassett, F.	Searle, T. J.	
	Smyth, E.	

1868-9.

1st Year.	2nd Year.	3rd Year.
Aldis, T. S. } Eq.	Carment, D.	Sutton, W.
Toller, T. N. }	Hughes, W.	Justican, E.
Greaves, L. H.	Bassett, F.	
Greig, J. A.	Johnson, T. H.	
Womersley, F. W.		

1869-70.

1st Year.		2nd Year.	3rd Year.
Reynolds, B.	Smith, C.	Toller, T. N, M.A.	Addiscott, F.
Waters, A. C.	Gray, W. T.	Hopkinson, J. C., B.A.	Hughes, Wm.
Todd, G.	Burne, J.		
Pringle J.	Sims, E. T.		
Park, D. F.	Adey, T. H.		
Oldham, C. H. } Eq.			
Young, T. E. }			

1870-1.

1st Year.		2nd Year.	3rd Year.
Wyatt, F. B.	Tucker, R. C.	Aldis, W. S., B.A.	Carment, D.
King, Geo.	Duncan, J. H.	Young, T. E., B.A.	
Dawson, J.	Trew, E. B.	Duncan, J.	
Higham, C. D.	Fuller, E. N.	Burne, J.	

1871-2.

1st Year.		2nd Year.	3rd Year.
Laing, F.	Andras, H. W.	Higham, C. D.	*Nil.*
Sorley, J.	Eaton, H. W. } Eq.	Crisford, G. S.	
Hunter, S.	Mills, J. }	Gray, W. T. } Eq.	
Straker, F. A.	Litchfield, E. }	Smither, A. }	
Ackland, T. G. } Eq.	McGowan, J. } Eq.	King, Geo.	
Harvey, C. J. }	Mason, C. E. }	Kember, W.	
Wilson, H. E.	Webb, V. G. }		

1872-3.

1st Year.

Cooke, T. H.
Price, E. E.
Hume, A. S.
Adlard, A. B.
Evans, Wm. }
Moore, R. M. } Eq.
Glennie, W. G.
Salter, W.
Deuchar, J. J. W. }
Simon, L. M. } Eq.

2nd Year.

Straker, F. A.
Wyatt, F. B.
Smith, Clarence
Ackland, T. G.
Greaves, L. H.
Duncan, J. H. }
Laing, F. } Eq.

3rd Year.

Duncan, J.

1873-4.

1st Year.

White, F. W.
Windett, C.
Clarke, A. D. C.
Whitcher, J. Jun.
Woods, E.
Larke, A.
Whittall, W. J. H.
Brockelbank, D. }
Ince, H. O. } Eq.
Block, R. J.

2nd Year.

Evans, W.
Cooke, T. H.
Adlard, A. B. }
Sorley, J. } Eq.
Trew, E. B.
Tucker, R. C.
Hunter, S.

3rd Year.

Higham, C. D. }
King, Geo. } Eq.

1874-5.

1st Year.

Gage, U. W.
Hardy, G. F.
King, A. T.
Boon, G. I.
Blakey, J. }
Sayce, R. C. } Eq.

2nd Year.

White, F. W.
Whitcher, J. Jun.

3rd Year.

Duncan, J. H.
Smither, A. }
Wyatt, F. B. } Eq.
Cooke, T. H.

1875-6.

1st Year.

Wilson, R., Jun.
Todd, J. M.
Pixley, F. W.

2nd Year.

Moore, R. M.
Whittall, W. J. H.
Block, R. J. }
Glennie, W. G. } Eq.
Gage, U. W.

3rd Year.

Gray, W. T.
Whitcher, J.
Ackland, T. G. }
Kember, W. }
Vaughan, W. } Eq.
Laing, F.

1876-7.

1st Year.

Wilding, R.
Robinson, A. }
Rothery, H. J. } Eq.

2nd Year.

Hardy, G. F.
Blakey, J.
Sayce, R. C.
Harvey, C. J.

3rd Year:

Sorley, J.

1877-8.

1st Year.

Graham, J.
James, G. T.
Turpin, W. G.
Somerville, W.
Clabon, C. C.
Des Vœux, H. W.
Burridge, A. F.
Carter, E. M.
Lemon, W. K.
Colquhoun, E. A.

2nd Year.

Windett, C.
Boon, G. I.
Woods, E.

3rd Year.

Straker, F. A.
Crisford, G. S.

1879.

Preliminary Examination.

Ansell, Hubert
Monilaws, W. M.
Anderson, J. }
Berry, B. A. } Eq.
Tarn, A. W.
Allen, S. S.
Lucey, Herbert
Wall, W. G.

Intermediate Examination.

Todd, George
Rothery, H. J.
Wilding, R.
Somerville, W.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

On the Rate ov Remarryge among Widoers. By T. B. SPRAGUE, M.A., *Manager ov the Scottish Equitable Life Assurance Society.*

[Read before the Institute, 24 November 1879.]

THE question ov the calculation ov premiums for the insurance agenst issue to a man at present marryd but without a family, has been on two ocasions bro't under the notis ov the Institute by Mr. Archibald Day. On 31 January 1859 he redd his paper *On the determination ov the rates ov Premium for asuring agenst Issue* (*Journal* viii, 127), in which he explaind his proces of calculating, by means ov statistics furnisht by the Registrar General's returns, the present value ov £1 payabl at the end ov the year in which a husband shal contract a second marryge. On 24 March 1862 he redd a stil more important paper, *On the statistics ov first and subsequent Marryges among the Familys of the Peerage, considerd specially with reference to the calculation ov Premiums for Asurances agenst Issue*, in which the same method was employd to obtain more accurat premiums from the more apropriat statistics furnisht by the peerage familys. The general problem is a very complex one involving several distinct contingencys. Mr. Day delt very succesfully with som ov these contingencys, but his treatment ov the subject was confessedly incomplete, and he himself speaks ov it as giving only a ruf aproximation. He omitted altogether from the calculations the circumstances that the insurance is not payabl until the deth ov the husband after contracting the second marryge, and that ther is a probability ov his second marryge being unfruitful. I hope on som future ocasion to take up the general question and carry the investigation ov it one or two steps beyond the point at which Mr. Day left it, but on the present ocasion I shal deal only with one element involvd in the calculation, namely, the probability ov the remarryge ov a widoer.

Mr. Day asumes without remark that the probability ov a widoer remarrying is the same whatever the length ov time that has elapst since the deth ov his wife. Mr. Huie, who has had ocasion to consider a similar question in his work, *On the Valuation ov Widos' Funds*, 1868, says, on p. 36, "the probability ov remarryge at any particular age may possibly depend in som mesure upon the length ov time which has elapst since the first wife dyd. At age 50, for instance, it is quite possibl that the probability ov remarryge ov a widoer whos wife has been ded for 20 years, may be greater or les than that ov another whos wife has been ded only 10 years." He is ov opinion, however, that it woud be impossibl to colect statistics suficient to enabl us to calculate the efect ov the length ov widoerhood upon the probability ov remarryge, and he concludes that "we must leav out ov the calculation the possibility that the previos duration ov widoerhood has any influence on the probability ov remarryge, and must be content to observ the law ov marryge among widoers ov al ages without reference to the date ov their wives' deth." The problem which Mr. Huie thus puts on one side is the one which I shal now consider.

Like Mr. Day I hav drawn my statistics for the investigation ov the problem from the records ov the British Peerage familys. The volume I hav uzed for the purpos is Lodge's Peerage for 1871, but for cases which that work leavs under observation on 31 December 1870, I hav consulted Burke's Peerage for 1878. My first step was to extract particulars as to every widoer (or divorst man) on a shedule ov the folloing form. When a man became a widoer a second (or third or fourth) time, he was enterd agen in the shedule.

SHEDULE 1.

Page.	Born.	Became a Widoer (or Divorst).	Age.	Remarryd.	After Years	Dyd.	Remarks. Stil alive on
3	14.10.1770	—. 3.1837	66	...	...	14. 2.1843	
4	28. 1.1784	29. 2.1812	28	8. 7.1815	3	Wid.	
„	„ „ „	26. 8.1833	49	...	...	14.12.1860	
*	*	*	*	*	*	*	
10	—. 5.1826	1. 2.1862	35	5.11.1866	4	...	31 Dec. 1877.
*	*	*	*	*	*	*	
53	6. 7.1766	10.10.1801	35	23. 6.1803	1	20.10.1839	
*	*	*	*	*	*	*	
111	8. 3.1822	9. 7.1857	35	...	...	19. 1.1859	

The next step was to bring together the particulars relating to al those who became widoers at the same age. This was don upon shedules ov the folloing form:—

SHEDULE 2.

Page.	Became a Widoer at Age	Dyd a Widoer after Years	Remains a Widoer after Years	Remarryd after Years	Dyd	Remains Marryd	Became a Widoer agen
					After further Years		
220	21	...	...	3	...	...	15
1		0	0	1			
25	22	...	...	2	...	...	6
193	„	...	...	1	...	...	45
397	„	...	...	1	9	...	...
480	„	...	...	1	28	...	...
4		0	0	4			
*	*	*	*	*	*	*	*
10	35	...	...	4	...	11	...
53	„	...	...	1	36	...	...
111	„	1	...	...	...	...	...

The numbers ov widoers (or, more strictly speaking, widoerhoods,) with regard to which the necesary particulars coud be obtaind, ar shoen in the folloing tabl. The numbers containd in Mr. Day's observations wer 945 widoers, ov whom 438 marryd, 368 dyd unmarryd, and 139 wer left existing on 31 December 1855 unmarryd. The manner in which my materials wer aranged enabld me to asertain without difficulty the number ov widoers who marryd more than twice, and I find that 34 wer marryd 3 times, and 3 wer marryd 4 times. The number ov divorses was 22.

TABL A.

Became Widoers at Age	Number.	Ov whom			Became Widoers at Age	Number.	Ov whom		
		Marryd.	Dyd.	Remaind under Observation.			Marryd.	Dyd.	Remaind under Observation.
					Bro't fwd.	626	412	139	75
21	1	1	...	...	54	17	9	5	3
22	4	4	...	...	55	14	5	6	3
23	5	4	1	...	56	20	12	5	3
24	4	2	2	...	57	28	13	10	5
25	11	10	1	...	58	15	5	5	5
26	15	13	...	2	59	20	7	8	5
27	8	7	1	...	60	9	3	3	3
28	20	16	3	1	61	15	7	8	...
29	18	15	2	1	62	17	4	7	6
30	22	16	3	3	63	14	5	4	5
31	32	20	4	8	64	13	3	8	2
32	16	11	1	4	65	21	2	14	5
33	21	17	1	3	66	15	1	14	...
34	19	13	6	...	67	19	2	13	4
35	23	21	2	...	68	12	...	9	3
36	21	16	1	4	69	11	...	9	2
37	23	15	7	1	70	16	...	13	3
38	20	16	1	3	71	9	2	7	...
39	23	15	5	3	72	11	2	7	2
40	29	18	9	2	73	9	...	7	2
41	29	19	7	3	74	8	2	4	2
42	29	13	12	4	75	9	...	8	1
43	21	12	2	7	76	10	2	7	1
44	22	13	6	3	77	5	...	5	...
45	22	12	9	1	78	1	...	1	...
46	23	16	5	2	79	4	1	1	2
47	21	12	5	4	80	7	1	5	1
48	33	14	15	4	81	...	...	...	...
49	19	11	3	5	82	4	...	4	...
50	25	15	7	3	83	4	...	4	...
51	18	11	6	1	84	1	...	1	...
52	20	7	12	1	85	2	..	2	...
53	9	7	...	2	86	1	...	1	...
Forward	626	412	139	75	TOTAL	987	500	344	143

The numbers becoming widoers at singl ages being too few for separat treatment, I combined them in quinquennial groops, putting together those who became widoers at ages 21–27, 28–32, 33–37, and so on. The numbers in these varios groops ar shoen in the folloing tabl.

Tabl B.

Became Widoers at Ages	Number.	Ov whom			Became Widoers at Ages	Number.	Ov whom		
		Marryd.	Dyd.	Remaind under Observation.			Marryd.	Dyd.	Remaind under Observation.
21–27	48	41	5	2	Bro't fwd.	705	451	165	89
28–32	108	78	13	17	58–62	76	26	31	19
33–37	107	82	17	8	63–67	82	13	53	16
38–42	130	81	34	15	68–72	59	4	45	10
43–47	109	65	27	17	73–77	41	4	31	6
48–52	115	58	43	14	78–82	16	2	11	3
53–57	88	46	26	16	83–87	8	...	8	...
Forward	705	451	165	89	TOTAL	987	500	344	143

The next step was to arange the numbers in each quinquennial groop ov ages, acording to the duration ov widoerhood. This was don by putting tik marks in a shedule ov the folloing form.

Shedule 3.

Became Widoers at Ages [ ].

Years elapst.	Remarryd.	Dyd.	Remaind Widoers.	Year ov Widoerhood.
0				1
1				2
2				3
3				4
&c.				&c.

In this way wer obtaind the figurs containd in colums (3), (4), and (5) ov the folloing tabl. Then adition from belo upwards gave the numbers in colum (2), and the numbers in colum (7) wer obtaind by subtracting from those in colum (2) one half ov the sum ov the numbers in colums (3), (4), and (5), that is, one-half ov the number in colum (6). I hav tho't it riht thus to giv the actual figurs I hav delt with, in order to put my readers in a position to form their oen opinions as to the degree ov confidence to which my results are entitld.

TABL C.

Became Widoers at 21–27.

Year ov Widoer-hood. (1)	Number entering on Year. (2)	Ov whom			Total Deduction. (6)	Years ov Life. (7)
		Marryd. (3)	Dyd. (4)	Remaind Widoers. (5)		
1	48	5	1	...	6	45
2	42	8	...	...	8	38
3	34	9	1	...	10	29
4	24	3	...	...	3	22·5
5	21	5	...	...	5	18·5
6	16	1	1	...	2	15
7	14	2	...	...	2	13
8	12	1	...	...	1	11·5
9	11	...	...	...	...	11
10	11	...	...	...	...	11
11	11	1	...	...	1	10·5
12	10	...	...	...	...	10
13	10	...	...	1	1	9·5
14	9	1	...	...	1	8·5
15	8	2	...	...	2	7
16	6	...	...	...	...	6
17	6	...	2	...	2	5
18	4	...	...	...	...	4
19	4	1	...	1	2	3
20	2	1	...	...	1	1·5
	303	40	5	2	47	279·5
21	1	...	...	...	...	1
22	1	...	...	...	...	1
23	1	...	...	...	...	1
24	1	1	...	...	1	·5
	4	1	0	0	1	3·5
TOTAL	307	41	5	2	48	283

Became Widoers at 28–32.

Year ov Widoer-hood. (1)	Number entering on Year. (2)	Ov whom			Total Deduction. (6)	Years ov Life. (7)
		Marryd. (3)	Dyd. (4)	Remaind Widoers. (5)		
1	108	5	...	...	5	105·5
2	103	15	1	...	16	95
3	87	12	3	...	15	79·5
4	72	11	...	...	11	66·5
5	61	8	...	...	8	57
6	53	11	1	...	12	47
7	41	4	...	...	4	39
8	37	1	1	2	4	35
9	33	2	...	3	5	30·5
10	28	2	1	1	4	26
11	24	...	2	1	3	22·5
12	21	1	...	1	2	20
13	19	1	1	1	3	17·5
14	16	2	...	...	2	15
15	14	...	...	...	...	14
	717	75	10	9	94	670
16	14	...	...	...	...	14
17	14	...	1	...	1	13·5
18	13	...	...	1	1	12·5
19	12	...	...	...	...	12
20	12	...	...	...	...	12
21	12	...	...	...	...	12
22	12	...	...	...	...	12
23	12	...	...	...	...	12
24	12	...	...	...	...	12
25	12	1	...	1	2	11
26 &c.	119	2	2	6	10	114
	244	3	3	8	14	237
TOTAL	961	78	13	17	108	907

Became Widoers at 33–37.

Year ov Widoer-hood. (1)	Number entering on Year. (2)	Ov whom			Total Deduction. (6)	Years ov Life. (7)
		Marryd. (3)	Dyd. (4)	Remaind Widoers. (5)		
1	107	4	...	...	4	105
2	103	14	1	...	15	95·5
3	88	6	...	...	6	85
4	82	12	...	...	12	76
5	70	11	2	...	13	63·5
6	57	6	1	...	7	53·5
7	50	6	...	...	6	47
8	44	3	...	1	4	42
9	40	5	...	...	5	37·5
10	35	2	1	1	4	33
11	31	...	3	...	3	29·5
12	28	...	1	...	1	27·5
13	27	2	...	...	2	26
14	25	1	1	...	2	24
15	23	2	...	...	2	22
	810	74	10	2	86	767
16	21	1	...	...	1	20·5
17	20	...	...	1	1	19·5
18	19	2	...	1	3	17·5
19	16	1	...	1	2	15
20	14	1	...	...	1	13·5
21	13	1	...	1	2	12
22	11	...	1	...	1	10·5
23	10	1	1	...	2	9
24	8	...	...	...	...	8
25	8	1	1	...	2	7
26 &c.	40	...	4	2	6	37
	180	8	7	6	21	169·5
TOTAL	990	82	17	8	107	936·5

	Became Widoers at 38–42.							Became Widoers at 43–47.							Became Widoers at 48–52.					
1	130	4	2	...	6	127	1	109	4	1	...	5	106·5	1	115	6	4	...	10	110
2	124	18	3	...	21	113·5	2	104	16	2	1	19	94·5	2	105	13	1	...	14	98
3	103	11	2	...	13	96·5	3	85	16	...	...	16	77	3	91	12	...	...	12	85
4	90	11	1	...	12	84	4	69	2	1	...	3	67·5	4	79	7	...	...	7	75·5
5	78	10	3	...	13	71·5	5	66	7	1	...	8	62	5	72	5	1	...	6	69
6	65	4	...	...	4	63	6	58	3	1	...	4	56	6	66	4	1	...	5	63·5
7	61	5	...	...	5	58·5	7	54	4	1	...	5	51·5	7	61	3	2	...	5	58·5
8	56	...	1	1	2	55	8	49	3	...	...	3	47·5	8	56	2	...	1	3	54·5
9	54	4	1	1	6	51	9	46	2	...	...	2	45	9	53	2	...	2	4	51
10	48	...	1	...	1	47·5	10	44	1	...	1	2	43	10	49	1	2	...	3	47·5
11	47	5	...	1	6	44	11	42	2	1	2	5	39·5	11	46	1	...	3	4	44
12	41	2	2	1	5	38·5	12	37	...	3	...	3	35·5	12	42	1	1	1	3	40·5
13	36	1	...	...	1	35·5	13	34	1	...	1	2	33							
14	35	...	...	1	1	34·5	14	32	1	1	...	2	31		835	57	12	7	76	797
15	34	...	2	...	2	33		829	62	12	5	79	789·5	13	39	...	2	1	3	37·5
	1002	75	18	5	98	953								14	36	...	2	...	2	35
							15	30	...	2	1	3	28·5	15	34	...	5	...	5	31·5
16	32	...	...	...	...	32	16	27	1	...	2	3	25·5	16	29	...	1	...	1	28·5
17	32	1	...	1	2	31	17	24	...	1	...	1	23·5	17	28	...	1	1	2	27
18	30	3	...	...	3	28·5	18	23	...	...	...	...	23	18	26	1	...	...	1	25·5
19	27	1	...	1	2	26	19	23	1	...	2	3	21·5	19	25	...	1	...	1	24·5
20	25	...	...	1	1	24·5	20	20	...	...	...	...	20	20	24	...	1	1	2	23
21	24	...	3	...	3	22·5	21	20	...	...	1	1	19·5	21	22	...	2	...	2	21
22	21	...	1	...	1	20·5	22	19	1	...	1	2	18	22	20	...	1	1	2	19
23	20	...	...	...	...	20	23	17	...	2	1	3	15·5	23	18	...	...	...	...	18
24	20	...	1	...	1	19·5	24	14	...	1	...	1	13·5	24	18	...	2	1	3	16·5
25	19	...	...	...	...	19	25	13	...	1	...	1	12·5	25	15	...	1	...	1	14·5
26 &c.	124	1	11	7	19	114·5	26 &c.	96	...	8	4	12	90	26 &c.	82	...	12	2	14	75
	374	6	16	10	32	358		326	3	15	12	30	311		416	1	31	7	39	396·5
TOTAL	1376	81	34	15	130	1311	TOTAL	1155	65	27	17	109	1100·5	TOTAL	1251	58	43	14	115	1193·5

TABL C—(*continued*).

Became Widoers at 53–57.

Year ov Widoer-hood. (1)	Number entering on Year. (2)	Ov whom. Marryd. (3)	Ov whom. Dyd. (4)	Ov whom. Remaind Widoers. (5)	Total Deduction. (6)	Years ov Life. (7)
1	88	1	1	...	2	87
2	86	19	...	...	19	76·5
3	67	9	...	...	9	62·5
4	58	8	3	1	12	52
5	46	1	...	...	1	45·5
6	45	3	...	1	4	43
7	41	...	...	...	...	41
8	41	2	4	1	7	37·5
9	34	1	...	1	2	33
10	32	1	...	1	2	31
	538	45	8	5	58	509
11	30	...	2	1	3	28·5
12	27	1	2	3	6	24
13	21	...	2	...	2	20
14	19	...	1	1	2	18
15	17	...	1	1	2	16
16	15	...	...	...	...	15
17	15	...	1	1	2	14
18	13	...	2	2	4	11
19	9	...	...	...	...	9
20	9	...	1	...	1	8·5
21	8	...	2	1	3	6·5
22	5	...	1	1	2	4
23	3	...	...	...	...	3
24	3	...	...	...	...	3
25	3	...	...	...	...	3
26 &c.	28	...	3	...	3	26·5
	225	1	18	11	30	210
TOTAL	763	46	26	16	88	719

Became Widoers at 58–62.

Year ov Widoer-hood. (1)	Number entering on Year. (2)	Ov whom. Marryd. (3)	Ov whom. Dyd. (4)	Ov whom. Remaind Widoers. (5)	Total Deduction. (6)	Years ov Life. (7)
1	76	1	...	...	1	75·5
2	75	6	1	...	7	71·5
3	68	6	...	...	6	65
4	62	6	2	...	8	58
5	54	4	1	...	5	51·5
6	49	...	1	1	2	48
7	47	1	4	...	5	44·5
8	42	1	3	2	6	39
	473	25	12	3	40	453
9	36	...	2	2	4	34
10	32	1	2	3	6	29
11	26	...	1	5	6	23
12	20	...	2	1	3	18·5
13	17	...	2	1	3	15·5
14	14	...	1	...	1	13·5
15	13	...	1	...	1	12·5
16	12	...	3	...	3	10·5
17	9	...	1	1	2	8
18	7	...	...	...	...	7
19	7	...	2	1	3	5·5
20	4	...	1	1	2	3
21	2	...	...	...	...	2
22	2	...	1	1	2	1
	201	1	19	16	36	183
TOTAL	674	26	31	19	76	636

Became Widoers at 63–67.

Year ov Widoer-hood. (1)	Number entering on Year. (2)	Ov whom. Marryd. (3)	Ov whom. Dyd. (4)	Ov whom. Remaind Widoers. (5)	Total Deduction. (6)	Years ov Life. (7)
1	82	3	1	...	4	80
2	78	5	4	...	9	73·5
3	69	1	3	...	4	67
4	65	...	3	2	5	62·5
5	60	...	8	...	8	56
6	52	3	3	...	6	49
	406	12	22	2	36	388
7	46	...	3	2	5	43·5
8	41	1	2	...	3	39·5
9	38	...	4	1	5	35·5
10	33	...	2	1	3	31·5
11	30	...	3	...	3	28·5
12	27	...	1	1	2	26
13	25	...	2	5	7	21·5
14	18	...	...	...	...	18
15	18	...	1	...	1	17·5
16	17	...	2	...	2	16
17	15	...	1	1	2	14
18	13	...	3	...	3	11·5
19	10	...	2	...	2	9
20	8	...	2	2	4	6
21	4	...	...	1	1	3·5
22	3	...	...	...	...	3
23	3	...	...	...	...	3
24	3	...	1	...	1	2·5
25	2	...	...	...	...	2
26 &c.	3	...	2	...	2	2
	357	1	31	14	46	334
TOTAL	763	13	53	16	82	722

Became Widoers at 68–72.

1	59	2	3	...	5	56·5
2	54	2	2	...	4	52
3	50	...	8	...	8	46
4	42	...	2	...	2	41
	205	4	15	0	19	195·5
5	40	...	7	1	8	36
6	32	...	1	1	2	31
7	30	...	1	...	1	29·5
8	29	...	3	3	6	26
9	23	...	1	1	2	22
10	21	...	3	2	5	18·5
11	16	...	2	...	2	15
12	14	...	2	...	2	13
13	12	...	1	...	1	11·5
14	11	...	2	1	3	9·5
15	8	...	1	1	2	7
16	6	...	1	...	1	5·5
17	5	...	1	...	1	4·5
18	4	...	2	...	2	3
19	2	...	...	...	...	2
20	2	...	1	...	1	1·5
21	1	...	1	...	1	·5
	256	0	30	10	40	236
TOTAL	461	4	45	10	59	431·5

Became Widoers at 73–77.

1	41	1	4	...	5	38·5
2	36	1	2	1	4	34
	77	2	6	1	9	72·5
3	32	1	4	1	6	29
4	26	1	3	...	4	24
5	22	...	1	...	1	21·5
6	21	...	3	1	4	19
7	17	...	2	...	2	16
8	15	...	2	...	2	14
9	13	...	1	2	3	11·5
10	10	...	4	...	4	8
11	6	...	1	...	1	5·5
12	5	...	1	...	1	4·5
13	4	...	1	...	1	3·5
14	3	...	...	1	1	2·5
15	2	...	2	...	2	1
	176	2	25	5	32	160
TOTAL	253	4	31	6	41	232·5

Became Widoers at 78–82.

1	16	2	2	...	4	14
2	12	...	3	...	3	10·5
3	9	...	1	...	1	8·5
4	8	...	1	1	2	7
5	6	...	...	1	1	5·5
6	5	...	...	1	1	4·5
7	4	...	1	...	1	3·5
8	3	...	...	...	...	3
9	3	...	1	...	1	2·5
10	2	...	...	...	...	2
11	2	...	...	...	...	2
12	2	...	1	...	1	1·5
13	1	...	...	...	...	1
14	1	...	1	...	1	·5
	58	0	9	3	12	52
TOTAL	74	2	11	3	16	66

Became Widoers at 83–86.

1	8	...	1	...	1	7·5
2	7	...	...	...	...	7
3	7	...	...	...	...	7
4	7	...	2	...	2	6
5	5	...	1	...	1	4·5
6	4	...	2	...	2	3
7	2	...	1	...	1	1·5
8	1	...	...	...	...	1
9	1	...	...	...	...	1
10	1	...	1	...	1	·5
TOTAL	43	0	8	0	8	39

Thus far we hav aranged the original facts in a fixt manner in which nothing is arbitrary. The conditions ov the problem ar altogether alterd when we pas on to graduate our results for the purpos ov determining the laws which they follo. The abov tabls contain particulars, not only ov the remarryges, but also ov the deths. For the present, however, I will consider only the marryges. The ratio with which I shal deal on the present ocasion is the ratio ov the number ov marryges to the number ov years ov life as givn in colum (7), and it may be useful to consider with som minuteness the exact nature ov this ratio. In the ordinary mortality tabl it is wel knoen that the ratio $(l_{x-1}-l_{x+1}):2l_x$ is a close aproximation to the force ov mortality at the age x. Similarly the ratio $(l_x-l_{x+1}) : l_{x+\frac{1}{2}}$ or $2\,(l_x-l_{x+1}) : (l_x+l_{x+1})$ is a close aproximation to the force ov mortality at the age $x+\frac{1}{2}$, that is to say, to the force ov mortality in the midl ov the year ov age to which the figurs relate; and this is the quantity which Dr. Farr and som others cal the *rate ov mortality* for the year ov age. Since the expression "the force ov mortality at the midl ov the year" is a somwhat cumbros one, I propose to substitute for it the fraze "central rate ov mortality" or "central deth-rate", which, as it seems to me, expresses the idea intended by Dr. Farr's, but is not, like his, apt to mislead thro' being confused with what is more commonly understood to be the rate ov mortality, namely, the probability ov dying in a year. Consistently with this, the abov-mentiond ratio ov the marryges to the years ov life, is the force ov marryge at the midl ov the year under consideration, or the central marryge rate. If we calculate this rate for individual years from the figurs in the abov tabls, ther wil be very great irregularitys in passing from one year to another, and it is the object ov graduation to remove these irregularitys while faithfully representing the general progression ov the figurs. As a first step toards this, I groop the years, choosing such combinations as wil giv the most regular results. These combinations can only be asertaind by trial, and it woud be quite unsuitabl for the purpos to adopt any regular system, such as grooping each successiv five years. The results ov my grooping ar shoen in Tabl D (page 87).

A study ov these figurs shos us that the marryge rates for each age at entry follo certain definit laws. These wil, however, be more reddily perceivd by the help ov the diagram on p. 88, where the absissa represents the age ov the widoer, and the ordinat the central marryge rate in the folloing year. Ther is a separat curv for each groop ov widoers, the first being for those who became widoers at ages 21–27, or say at 25; the second for those who became widoers at 28–32, say at 30; and so on.

TABL D.

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 21–27.			
1	45	5	·111
2	38	8	·211
3	29	9	·310
4, 5	41	8	·195
6, 7	28	3	·107
8–24	102	8	·078
Total	283	41	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 28–32.			
1	105·5	5	·047
2, 3	174·5	27	·155
4, 5	123·5	19	·154
6–8	121	16	·132
9, 10	56·5	4	·071
11–15	89	4	·045
16–43	237	3	·013
Total	907	78	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 33–37.			
1	105	4	·038
2	95·5	14	·147
3–6	278	35	·126
7, 8	89	9	·101
9, 10	70·5	7	·099
11–22	237·5	11	·046
23–41	61	2	·033
Total	936·5	82	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 38–42.			
1	127·	4	·031
2	113·5	18	·159
3, 4	180·5	22	·122
5, 6	134·5	14	·104
7–9	164·5	9	·055
10–12	130	7	·054
13	35·5	1	·028
14–18	159	4	·025
19–42	266·5	2	·008
Total	1311	81	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 43–47.			
1	106·5	4	·038
2	94·5	16	·169
3, 4	144·5	18	·125
5, 6	118	10	·085
7, 8	99	7	·071
9–11	127·5	5	·039
12–14	99·5	2	·020
15–17	77·5	1	·013
18–21	84	1	·012
22–41	149·5	1	·007
Total	1100·5	65	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 48–52.			
1	110	6	·055
2	98	13	·133
3	85	12	·141
4	75·5	7	·093
5	69	5	·072
6	63·5	4	·063
7	58·5	3	·051
8	54·5	2	·037
9, 10	98·5	3	·030
11–13	122	2	·016
14–41	359	1	·003
Total	1193·5	58	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 53–57.			
1	87	1	·011
2	76·5	19	·248
3	62·5	9	·144
4, 5	97·5	9	·092
6	43	3	·070
7, 9	111·5	3	·027
10, 11	59·5	1	·017
12–39	181·5	1	·006
Total	719	46	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 58–62.			
1	75·5	1	·013
2, 3	136·5	12	·088
4–6	157·5	10	·063
7, 8	83·5	2	·024
9–22	183	1	·005
Total	636	26	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 63–67.			
1	80	3	·038
2	73·5	5	·068
3	67	1	·015
4–7	211	3	·014
8–27	290·5	1	·003
Total	722	13	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 68–72.			
1	56·5	2	·035
2	52	2	·038
3–21	323	0	·000
Total	431·5	4	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 73–77.			
1	38·5	1	·026
2	34	1	·029
3	29	1	·034
4–15	131	1	·008
Total	232·5	4	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 78–82.			
1	14	2	·143
2–14	52	0	·000
Total	66	2	

Years ov Widoerhood.	Years ov Life.	Marryges.	Central Marryge Rate.
Ages at Entry 83–86.			
1–10	39	0	·000
Total	39	0	

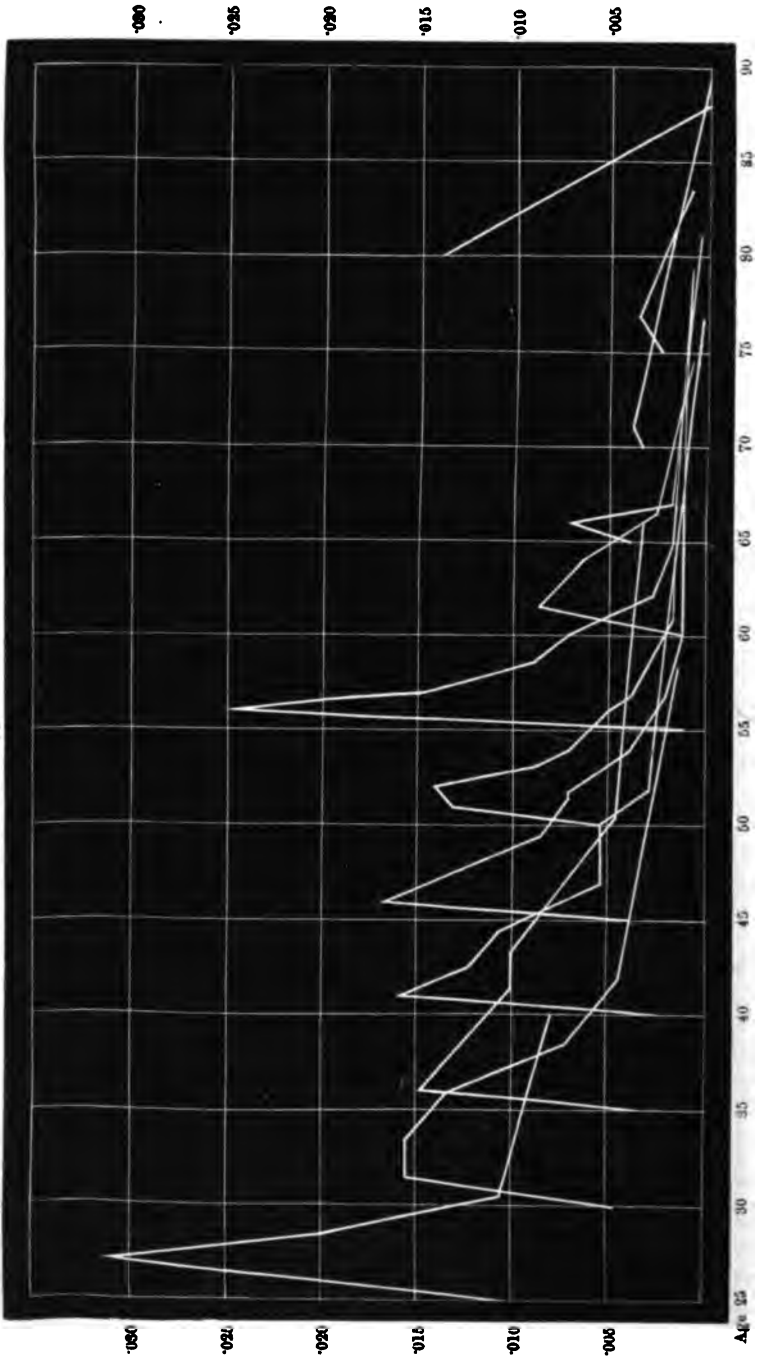


DIAGRAM 1.—*Remarryge Rate ov Widoers—Unajusted.*

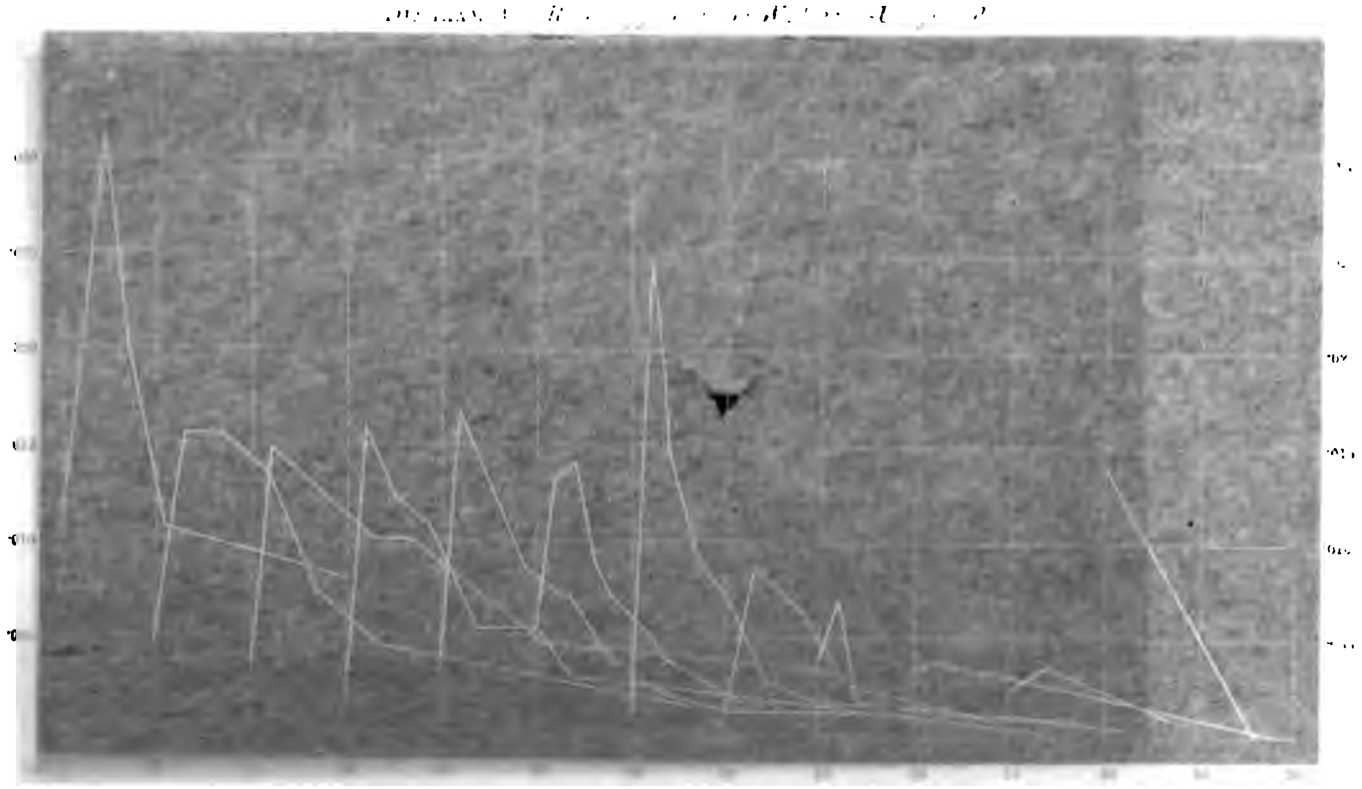
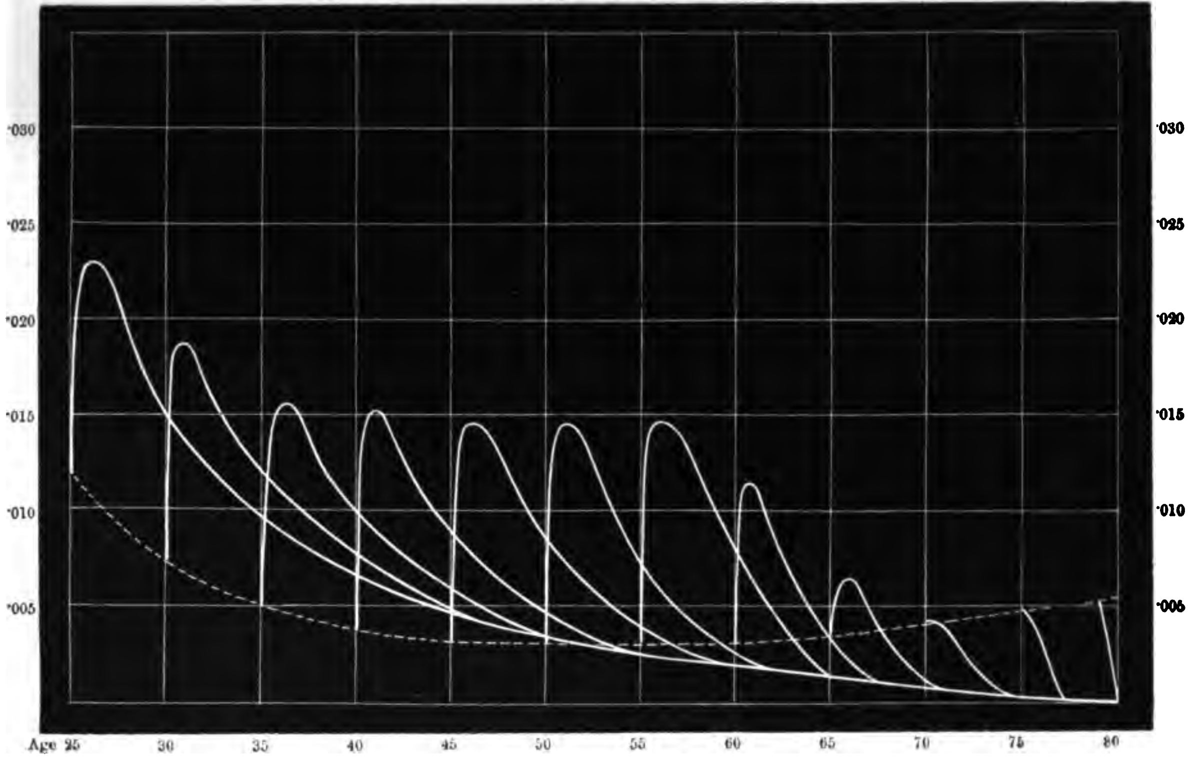


DIAGRAM 2.—*Remarryge Rate ov Widoers—Ajusted.*



The general cours ov these curvs is unmistakabl, and we see that they al more or les conform to the folloing laws. If we regard firstly the rate during the first year ov widoerhood, we see that the rate is greatest for the yungest widoers, and gradually diminishes to a minimum for those who became widoers at the ages 53–62, after which it increases. Secondly, for those who became widoers in any quinquennial groop ov ages, we see that the rate increases to a maximum in the second or third year ov widoerhood, and then gradually diminishes, ataining, after the laps ov a greater or les number ov years, what may be cald its ultimat magnitude, when it depends only on the age ov the widoer and not upon the length ov his widoerhood.

As a first step in the graduation, it wil be convenient to deal with this ultimat rate ov remarryge, and the first question to be considerd is: How many years elaps before the ultimat rate is ataind? A study ov the diagram shos that this time wil be greatest for those who becom widoers at yung ages, and wil gradually diminish as the age at the deth ov the wife (which we wil cal the *age at entry*) increases; and indicates that for ages at entry 30 and 35, the time wil be about 15 years, but for age at entry 60 only about 7 years. After varios trials, I fixt upon the folloing terms as the most suitabl.

TABL E.—*Shoing the number ov Years after which Widoers who becom such at any Age, ar subject to the Ultimat Rate ov Remarryge.*

Age at Entry.	Term.	Age at Entry.	Term.
21–27	20 years	53–57	10 years
28–32	15 "	58–62	8 "
33–37	15 "	63–67	6 "
38–42	15 "	68–72	4 "
43–47	14 "	73–77	2 "
48–52	12 "	78–82	1 "

Acordingly, for the purpos ov calculating the ultimat marryge rate, we take the facts after the laps ov 20 years among those who became widoers at ages 21–27, after the laps ov 15 years among those who became widoers at 28–32, and so on. These facts being added together with regard only to the age ataind and without regard to the age at entry, we get the figurs in colums (2) (3) (4) of Tabl F. In the Tabls C the facts thus made use of ar markt off from the others and separatly added up.

TABL F.—*Ultimat Marryge Rate.*

Age ataind.	Years ov Life.	Marryges.	Deths.	Expected Marryges.	TOTAL MARRYGES.		Age ataind.
					Actual.	Expected.	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
45	15	...	...	·7	...	·7	45
46	14·5	...	1	·6	...	1·3	46
47	13·5	...	...	·5	...	1·8	47
48	12·5	1	...	·5	1	2·3	48
49	12	...	...	·4	1	2·7	49
50	32·5	1	...	1·1	2	3·8	50
51	31·5	...	...	1·0	2	4·8	51
52	29·5	2	...	·9	4	5·7	52
53	27	1	...	·7	5	6·4	53
54	24·5	2	...	·6	7	7·0	54
55	54	1	...	1·3	8	8·3	55
56	51·5	1	1	1·2	9	9·5	56
57	47	4	1	1·0	13	10·5	57
58	43	1	...	·9	14	11·4	58
59	69	1	3	1·3	15	12·7	59
60	62·5	1	4	1·1	16	13·8	60
61	57·5	...	3	·9	16	14·7	61
62	93	1	2	1·4	17	16·1	62
63	86·5	2	5	1·2	19	17·3	63
64	79	...	5	1·0	19	18·3	64
65	102·5	...	4	1·2	19	19·5	65
66	92·5	2	5	1·0	21	20·5	66
67	80·5	1	6	·8	22	21·3	67
68	105·5	1	7	·9	23	22·2	68
69	94	1	6	·7	24	22·9	69
70	82·5	...	4	·5	24	23·4	70
71	116	...	8	·6	24	24·0	71
72	102·5	1	6	·5	25	24·5	72
73	90·5	...	8	·3	25	24·8	73
74	116·5	...	15	·3	25	25·1	74
75	100·5	...	10	·2	25	25·3	75
76	88	...	7	·1	25	25·4	76
77	104	1	12	·1	26	25·5	77
78	87·5	1	7	·0	27	25·5	78
79	76·5	...	7	·0	27	25·5	79
80	66	...	7	...	...	...	80
81	66·5	...	11	...	...	...	81
82	55·5	...	9	...	...	...	82
83	45	...	8	...	...	...	83
84	33	...	8	...	...	...	84
85	31·5	...	5	...	...	...	85
86	26	...	4	...	...	...	86
87	22	...	4	...	...	...	87
88	17·5	...	4	...	...	...	88
89	13	...	4	...	...	...	89
90	8·5	...	5	...	...	...	90
91	4·5	...	3	...	...	...	91
92	3	...	...	...	...	...	92
93	2	...	2	...	...	...	93
94	·5	...	1	...	...	...	94
TOTAL	2689·5	27	212	25·5	...	...	TOTAL

The folloing ar the ajusted results which I deduce by the grafic method.

TABL G.—*Ajusted ultimat Rate ov Remarryge among Widoers:* $(wm\mu)_{x+\frac{1}{2}}$.

Age ataind. x	Central Marryge Rate.	Age ataind. x	Central Marryge Rate.	Age ataind. x	Central Marryge Rate.	Age ataind. x	Central Marryge Rate.
45	·0463	54	·0259	63	·0139	72	·0046
46	·0431	55	·0243	64	·0127	73	·0038
47	·0402	56	·0228	65	·0116	74	·0030
48	·0376	57	·0214	66	·0105	75	·0023
49	·0354	58	·0201	67	·0094	76	·0016
50	·0333	59	·0188	68	·0084	77	·0010
51	·0318	60	·0175	69	·0074	78	·0005
52	·0294	61	·0163	70	·0064	79	·0002
53	·0276	62	·0151	71	·0055		

NOTE.—The "central marryge rate" here shoen relates to the year ov life folloing the "age ataind".

For the purpos ov shoing how far the ajusted tabl agrees with the original facts, I hav givn in colum (5) ov Tabl F the expected number ov marryges, also in colums (6) and (7) the progressiv totals ov the actual and expected marryges.

It wil be interesting to compare this marryge rate with the rates found from som other observations; and this is don in Tabl H (p. 93). I hav selected for comparison the marryge rates among the Peerage Widoers, acording to Mr. Day, the Peerage Bachelors, and the Bachelors belonging to the body ov the Writers to the Signet, Edinburgh.

The function tabulated by Mr. Day, for the Peerage Widoers (*J.I.A.* x, 188), is the one I hav denoted by $(bmr)_x$ or r_1, and the central marryge rate has been calculated from it by means ov the relation $\mu_1 = 1 \div \left(\frac{1}{r_1} - \frac{1}{2}\right)$. In my paper, *J.I.A.* xxi, 420, I hav givn the ajusted values ov $(bmq)_x$ ov q_1 for the Peerage Bachelors, and the central marryge rate has been calculated by the formula $q_1 \div \{1 - \frac{1}{2}(q_1 + q_2)\}$, the values ov q_2 being got from page 432 ov my paper. The figurs relating to the Writers to the Signet ar deduced from the tabl givn by Mr. A. Gillies Smith, C.A., on page 26 ov his *Report on the Widôs' Fund ov the Writers to Her Majesty's Signet as at Whitsunday* 1879. Starting with 10,000 bachelors alive at age 22, and asuming that the rate ov mortality among the bachelors is that ov the English Tabl No. 3,

Tabl H, *shoing the Values ov the Central Marryge Rate for e*
year ov age (or the force of marryge in the midl ov each year
age) acording to Observations made upon varios classes of Person

Age x	Peerage Widoers.		Bachelors.	
	Ultimat Rate, Sprague.	Average Rate, Day.	Peerage, Sprague.	Writers to the Signet.
22	...	·19609	·05159	·02091
23	...	·18847	·06225	·02738
24	...	·18119	·07302	·03362
25	·116	·17421	·07954	·03976
26	·231	·16753	·08116	·04587
27	·223	·16111	·08094	·05169
28	·192	·15506	·08059	·05787
29	·165	·14924	·07948	·06372
30	·149	·14367	·07838	·06945
31	·136	·13831	·07674	·06961
32	·125	·13316	·07457	·07258
33	·115	·12824	·07209	·07162
34	·109	·12350	·06864	·07322
35	·099	·11895	·06466	·07009
36	·091	·11458	·06060	·07173
37	·085	·11037	·05656	·05879
38	·078	·10923	·05253	·05831
39	·072	·10809	·04862	·04916
40	·068	·10696	·04472	·04366
41	·063	·10585	·04095	·03819
42	·058	·10475	·03719	·03063
43	·054	·09930	·03355	·02765
44	·050	·09418	·03002	·02469
45	·0463	·08924	·02661	·02312
46	·0431	·08463	·02363	·02090
47	·0402	·08025	·02116	·02081
48	·0376	·07592	·01911	·01965
49	·0354	·07182	·01738	·01940
50	·0333	·06796	·01605	·01444
51	·0313	·06431	·01482	·01442
52	·0294	·06087	·01360	·01271
53	·0276	·05686	·01238	·01144
54	·0259	·05313	·01126	·01244
55	·0243	·04965	·01014	·01355
56	·0228	·04640	·00913	·01409
57	·0214	·04338	·00822	·01470
58	·0201	·04112	·00741	·01395
59	·0188	·03897	·00659	·01239
60	·0175	·03695	·00578	·01067
61	·0163	·03503	·00518	·00956
62	·0151	·03322	·00468	·00583
63	·0139	·03058	·00407	·00436
64	·0127	·02814	·00357	·00183
65	·0116	·02591	·00306	·00096
66	·0105	·02385	·00266	...
67	·0094	·02197	·00236	...
68	·0084	·02135	·00215	...
69	·0074	·02073	·00195	...
70	·0064	·02015	·00185	...
71	·0055	·01958	·00165	...
72	·0046	·01902	·00155	...
73	·0038	·01610	·00135	...
74	·0030	·01363	·00125	...
75	·0023	·01155	·00105	...
76	·0016	·00978	·00095	...
77	·0010	·00828	·00074	...
78	·0005	·00802	·00064	...
79	·0002	·00776	·00043	...
80	...	·00752	...	...
81	...	·00728	...	...
82	...	·00704	...	...

Males, he shos how many marryd in each year ov age and how many ataind each subsequent age unmarryd, his figurs being calculated on the principl adopted by Mr. Huie, as explaind on page 418 ov my paper. In order to obtain the central marryge rate, I first ov al calculated from Mr. A. G. Smith's figurs the probability ov marrying in a year, q_1, and then got μ_1 by help ov the formula givn abov, the values ov q_2 being those got from the English Tabl No. 3, Males.

It wil be notist that, as was to be expected, the ultimat marryge rate is very greatly less than the average rate resulting from Mr. Day's calculations, but that it is stil much in excess ov the rate among the bachelors. It wil also be notist that the marryge rate among the Writers to the Signet dos not continuosly diminish as the age increases, but presents the remarkabl feature ov increasing from ·0114 at age 53 to a (relativ) maximum, ·0147, at age 57. Mr. Smith dos not giv the actual numbers ov marryges on which his tabl is based, and we therefore hav no means ov judging whether it woud hav been better to remove in the proces ov graduation this temporary increase in the marryge rate.

The next step was to graduate the marryge rate in the first year ov widoerhood. As alreddy mentiond this rate is greatest for those who becom widoers at the yungest ages, and seems to hav a minimum value for the ages at entry 55 and 60, after which ages it aparently increases. Aplying the grafic method ov ajustment, the folloing ar the figurs I get. As in other cases, the expected numbers ov marryges ar givn by the side ov the actual, in order that it may be seen how far the ajustment agrees with the original facts.

TABL I.—*Shoing the Central Marryge Rate among Widoers during the first Year ov Widoerhood:* $(wm\mu)_{[x]+\frac{1}{2}}$.

Age at Entry. x	$(wm\mu)_{[x]+\frac{1}{2}}$	MARRYGES.		Age at Entry. x	$(wm\mu)_{[x]+\frac{1}{2}}$	MARRYGES.	
		Actual.	Expected.			Actual.	Expected.
				Bro't frwd.	...	28	29·5
25	·116	5	5·2	55	·031	3	2·7
30	·073	5	7·7	60	·033	1	2·5
35	·050	4	5·3	65	·035	3	2·8
40	·038	4	4·8	70	·040	2	2·3
45	·032	4	3·4	75	·047	1	1·8
50	·028	6	3·1	80	·055	2	·8
Forward	...	28	29·5	TOTAL	...	40	42·4

It seems at first sight strange that the marryge rate shoud be greater for a widoer of 70 than for one of 50, but it is not unreasonabl to supose the latter to reason to himself, "I may probably marry agen som day, but there is no hurry"; while the former may say to himself, "If I am ever to marry agen, I must lose no time about it!"

We hav thus got for those who became widoers at the varios quinquennial ages, the starting-point for the first year ov widoerhood, and the ultimat curv to which the several curvs ar to join on after the laps ov the several terms ov years in Tabl E. In filling in by the grafic method the intermediat portions ov the curvs it is necesary to take care, not only that each curv agrees with the original facts relating to the particular age at entry, but that the different curvs ar consistent with each other. This renders the ajustment a proces ov considerabl complexity, and I found it necesary to make many trials before I coud obtain results that wer satisfactory to me. The principl ov the proces, however, is the same as in simpler cases. The diagram (1) abov givn indicates the general form ov the different curvs; and these being drawn on cros-ruled paper, the ordinats representing the marryge rate wer estimated, and the expected marryges then computed from them. The comparison ov these expected marryges with the actual indicates the corections to be made on the curvs, and these corections being made, with due regard to the avoidance ov irregularitys, the whole proces was repeated as offen as necesary. Tabl J (pp. 96, 97), and diagram (2) (page 89), sho the results eventually arived at, which seem to me to fulfil suficiently for al practical purposes the esential conditions ov a good graduation, inasmuch as they ar regular and consistent with each other, while they keep pretty close to the original facts. It wil be observd that in midl age the rate of remarryge depends principally on the duration ov widoerhood and is nearly independent ov the age.

I hope that a study ov the results I hav thus deduced from somwhat scanty materials, may lead other actuarys to try for themselvs the grafic method ov ajustment, which I believ to be generally the best that can be adopted. Those who shal stil hold the opinion that it is a ruf and unsientific method, may, I think, be fairly challenged to produce from the same unajusted facts more satisfactory results by som other method.

TABL J.—*Ajusted Marryge Rates for Quinquennial Ages at Entry:* $(wm\mu)_{[x]+n+\frac{1}{2}}$, *where* $[x]$ = *age at entry,* n = *years elapst, and* $[x]+n$ = *age ataind.*

Age ataind.	Central Marryge Rate.	Marryges. Actual.	Marryges. Expected.	Total Marryges. Actual.	Total Marryges. Expected.
Age at Entry 25.					
25	·116	5	5·2	5	5·2
26	·231	8	8·8	13	14·0
27	·223	9	6·5	22	20·5
28	·192	3	4·3	25	24·8
29	·165	5	3·1	30	27·9
30	·149	1	2·2	31	30·1
31	·136	2	1·8	33	31·9
32	·125	1	1·4	34	33·3
33	·115	...	1·3	34	34·6
34	·109	...	1·2	34	35·8
35	·099	1	1·0	35	36·8
36	·091	...	·9	35	37·7
37	·085	...	·8	35	38·5
38	·078	1	·6	36	39·1
39	·072	2	·5	38	39·6
40	·068	...	·4	38	40·0
41	·063	...	·3	38	40·3
42	·058	...	·2	38	40·5
43	·054	1	·2	39	40·7
44	·050	1	·1	40	40·8
TOTAL	...	40	40·8	...	...
Age at Entry 30.					
30	·073	5	7·7	5	7·7
31	·188	15	17·9	20	25·6
32	·170	12	13·5	32	39·1
33	·148	11	9·8	43	48·9
34	·134	8	7·6	51	56·5
35	·120	11	5·6	62	62·1
36	·109	4	4·3	66	66·4
37	·100	1	3·5	67	69·9
38	·094	2	2·9	69	72·8
39	·089	2	2·3	71	75·1
40	·080	...	1·8	71	76·9
41	·072	1	1·4	72	78·3
42	·064	1	1·1	73	79·4
43	·058	2	8	75	80·2
44	·052	...	7	75	80·9
TOTAL	...	75	80·9	...	...
Age at Entry 35.					
35	·050	4	5·3	4	5·3
36	·156	14	14·9	18	20·2
37	·149	6	12·7	24	32·9
38	·130	12	9·9	36	42·8
39	·112	11	7·1	47	49·9
40	·097	6	5·2	53	55·1
41	·088	6	4·1	59	59·2
42	·079	3	3·3	62	62·5
43	·072	5	2·7	67	65·2
44	·067	2	2·2	69	67·4
45	·060	...	1·8	69	69·2
46	·052	...	1·4	69	70·6
47	·046	2	1·2	71	71·8
48	·041	1	1·0	72	72·8
49	·036	2	·8	74	73·6
TOTAL	...	74	73·6	...	...
Age at Entry 40.					
40	·038	4	4·8	4	4·8
41	·151	18	17·1	22	21·9
42	·134	11	12·9	33	34·8
43	·115	11	9·7	44	44·5
44	·098	10	7·0	54	51·5
45	·082	4	5·2	58	56·7
46	·071	5	4·2	63	60·9
47	·064	...	3·5	63	64·4
48	·057	4	2·9	67	67·3
49	·050	...	2·4	67	69·7
50	·045	5	2·0	72	71·7
51	·039	2	1·5	74	73·2
52	·033	1	1·2	75	74·4
53	·029	...	1·0	75	75·4
54	·026	...	·9	75	76·3
TOTAL	...	75	76·3	...	...

TABL J—(*continued*).

Age ataind.	Central Marryge Rate.	MARRYGES.		TOTAL MARRYGES.	
		Actual.	Expected.	Actual.	Expected.
Age at Entry 45.					
45	·032	4	3·4	4	3·4
46	·148	16	14·0	20	17·4
47	·140	16	10·8	36	28·2
48	·122	2	8·2	38	36·4
49	·099	7	6·1	45	42·5
50	·080	3	4·5	48	47·0
51	·069	4	3·6	52	50·6
52	·061	3	2·9	55	53·5
53	·056	2	2·5	57	56·0
54	·048	1	2·1	58	58·1
55	·040	2	1·6	60	59·7
56	·033	...	1·2	60	60·9
57	·027	1	·9	61	61·8
58	·023	1	·7	62	62·5
TOTAL	...	62	62·5	...	...
Age at Entry 50.					
50	·028	6	3·1	6	3·1
51	·145	13	14·2	19	17·3
52	·135	12	11·5	31	28·8
53	·109	7	8·2	38	37·0
54	·090	5	6·2	43	43·2
55	·070	4	4·4	47	47·6
56	·055	3	3·2	50	50·8
57	·043	2	2·3	52	53·1
58	·034	2	1·7	54	54·8
59	·029	1	1·4	55	56·2
60	·025	1	1·1	56	57·3
61	·020	1	8	57	58·1
TOTAL	...	57	58·1	...	...
Age at Entry 55.					
55	·031	1	2·7	1	2·7
56	·146	19	11·2	20	13·9
57	·137	9	8·6	29	22·5
58	·121	8	6·3	37	28·8
59	·104	1	4·7	38	33·5
60	·085	3	3·7	41	37·2
61	·069	...	2·8	41	40·0
62	·051	2	1·9	43	41·9
63	·034	1	1·1	44	43·0
64	·022	1	·7	45	43·7
TOTAL	...	45	43·7	...	...
Age at Entry 60.					
60	·033	1	2·5	1	2·5
61	·117	6	8·4	7	10·9
62	·090	6	5·9	13	16·8
63	·070	6	4·1	19	20·9
64	·051	4	2·6	23	23·5
65	·033	...	1·6	23	25·1
66	·020	1	·9	24	26·0
67	·015	1	·6	25	26·6
TOTAL	...	25	26·6	...	...
Age at Entry 65.					
65	·035	3	2·8	3	2·8
66	·065	5	4·8	8	7·6
67	·050	1	3·4	9	11·0
68	·035	...	2·2	9	13·2
69	·021	...	1·2	9	14·4
70	·014	3	·7	12	15·1
TOTAL	...	12	15·1	...	...
Age at Entry 70.					
70	·040	2	2·3	2	2·3
71	·039	2	2·0	4	4·3
72	·022	...	1·0	4	5·3
73	·011	...	·5	4	5·8
TOTAL	...	4	5·8	...	...
Age at Entry 75.					
75	·047	1	1·8	1	1·8
76	·029	1	1·0	2	2·8
TOTAL	...	2	2·8	...	...
Age at Entry 80.					
80	·055	2	·8	2	·8

DISCUSSION.

The **PRESIDENT** (Mr. A. H. Bailey), in inviting remarks on the paper, said that the Council of the Institute would be pleased to receive papers from some of the younger members. Those who have passed the examinations of the Institute could not do better, in their own interest, than bring themselves before the public by continuing their studies and reading papers on practical subjects. It would be, he said, a matter of considerable interest to know how much out of the sum of ten millions of money, which is annually disbursed by the insurance companies, passes into the hands of the families of the lives insured, and how much passes to the hands of third parties. That might easily be ascertained by examining the records of claims paid; and if this were done for a few years among companies transacting ordinary business, some interesting results might be brought out. The paper is one illustration among others of how events under the control of the human will are subject to general laws almost as much as events over which we have no control. Probably the class among whom Mr. Sprague's observations were made, must bring about somewhat different results from those of the general population. If a mechanic, for instance, has the misfortune to lose his wife and is left with young children, those children must have some woman to look after them; and such a man, if he is to lead a decent life, has no other alternative than to marry again. But that does not apply to persons in a better position in life, or with larger incomes; and therefore the law of remarriage which prevails among what may be called the weekly-wage-earning class, will not prevail amongst other classes. How far the results arising from Mr. Sprague's slender materials, can be made the foundation of a law, is a matter upon which there will probably be some difference of opinion.

Mr. C. **WALFORD** said that the widows' funds in various parts of the world, from which alone we have been able to get at the law of marriage, show variations in regard to the different classes for which these funds have been established, and the particular class which Mr. Sprague has been investigating differs in its circumstances from other classes of the community. But as the actuarial questions which arise, relate very frequently to considerations of property and title and so on, the results of Mr. Sprague's investigation will probably have a wider application than at first sight appears. Mr. Sprague had lately submitted to the Royal Society of Edinburgh a paper in which the law relating to issue from remarriage has been made very clear. The law so deduced may have even a wider application than the law of remarriage itself, because in that case the operation of physical causes comes into play, and those physical causes may and probably do affect nearly all classes of mankind in much the same degree. For these reasons he hoped the paper would be reprinted in the *Journal* of the Institute. In some parts of the world a great deal of surprise is expressed that these questions of marriage and issue and analogous matters, are so much considered here; for it is not well understood how our property laws lead necessarily to their consideration.

Mr. **BERRIDGE** said that Mr. Sprague in writing the paper had had two objects in view—first to show the rate of remarriage among widowers, and next to educe an expression of opinion on the part of the members of the Institute as to the graphic method of graduation. That method was rather attacked on the occasion of Mr. Sprague's last paper being read; but it is now applied to quantities small in magnitude but irregular in character, which would be very difficult indeed to graduate by any ordinary process, and it gives regular results.

Mr. T. Y. **Strachan** remarked on the new term—central marriage rate—introduced in the paper. He preferred the use of Mr. Day's ratio, in which the denominator is $(bl)_x-\frac{1}{2}(bd)_x$, rather than Mr. Sprague's, in which the denominator is $(bl)_x-\frac{1}{2}[(bd)_x+(bm)_x]$.

Mr. **Neison** was struck with the small number of remarriages amongst the widowers. From the returns of the Registrar-General, he found that in every 100 cases of marriage there were eleven of widows and nine of widowers. But of those nine widowers two-thirds ultimately remarry; and of the eleven widows, six remain in the condition of widowhood. But in Mr. Sprague's paper, instead of two-thirds of the widowers remarrying, which would be 658, the number is only 500. There are 143 cases under observation, and that may to some extent account for the difference. In one of the large funds, in which he had thousands of cases of widows before him, the tendency to remarriage was rather in excess of the experience of the whole of England and Wales.

Mr. **George King** thought that when there are very scanty materials, and when the law of the curve is quite unknown, then the graphic method is the only one that can be adopted; and in skilful hands it is one which will prove satisfactory. But, when we have plenty of materials at our disposal, and are, from previous experience, more or less acquainted with the law, it is desirable to use some method not quite so liable to bend to the opinion of the operator. There is one great advantage in mathematical formulas, that we can all follow them out and calculate the extent of the changes they produce; but not so with the graphic method. In special cases no doubt the graphic method may be very useful, but to say (as Mr. Sprague does) that it is "generally the best", seems going a little too far.

Mr. W. **Sutton** thought that for the purpose of issue questions there must be a great deal of difference in the rate of remarriage among widowers who have children living capable of inheriting, and those who have not; and that this is a most important point. Mr. Sprague had apparently taken all the widowers he could find in the peerage, but these must differ materially as regards their social and pecuniary position. It is very desirable that all the surrounding conditions should as nearly as possible be taken into account; and he felt certain that no rate of remarriage can be of value for issue purposes until that is properly considered. He thought the graphic method of adjustment required very careful handling. It is possible that a person using it might devise one set of curves one day, and perhaps a week hence an altogether different set; and the results would depend very much upon the individual using it.

Mr. **Whitcher** had himself tried the graphic method of adjustment in dealing with certain facts deduced from limited data, but the results were such that he did not feel justified in trusting to them in any way, and he had to use mathematical formulas such as Mr. King has mentioned. After having applied these he was much strengthened in his opinion as to the graphic method. No doubt there is an immense amount of convenience in it, if any one can use it in a masterly way; but any one who trusts himself to use it should be a very expert mathematician indeed, and a man with small experience and not first-rate ability, ought to hesitate before he ventures to use it.

Mr. BERRIDGE—With regard to this graphic method, if it were merely the setting out of the first ordinates and drawing the curve from them, that would be an exceedingly rough method, and very much would depend upon individual skill in doing it; but Mr. Sprague furnishes a table showing the way in which he works and the results of his method, and puts them side by side for intervals of five years only, so that we can see how far he has deviated from the original facts. He has in his last paper brought out adjusted results in close accord with the original facts, and that closeness he set out in parallel columns—the actual and expected number according to his graduation, and in doing that he cannot go far wrong.

MR. SPRAGUE'S REPLY.

The President suggests a doubt whether results deduced from the materials I had at my disposal, can be made the foundation of a law of remarriage. I myself think that they can, and I confidently anticipate that my results, when completed, will be found very useful to actuaries, in dealing with insurances against issue in England or with disentail calculations in Scotland.

I am gratified to observe that the same objections are not now made to the graphic method of adjustment as were made when I formerly brought it under the notice of the Institute. It was then described by several speakers as rough and unscientific, but the only objections now made to it are that it must be difficult to apply, and that the success of it must altogether depend upon the skill of the computer. The only two speakers who appear to have given the method a trial, are Mr. Berridge and Mr. Whitcher, the former of whom speaks very well of the results obtained by it. Mr. Whitcher has not been successful in employing it, but from his remarks I conjecture that he must have stopped half-way in the process, or even at an earlier stage. In any case where I have used the method, I have found it necessary to make at least three or four approximations, each of which has generally been more satisfactory than the preceding one. In graduating the rate of remarriage among widowers I at first obtained a materially different set of curves from those which I have given in my paper, but they did not give satisfactory results when I submitted them to the test of comparing the expected marriages with the actual; and, as I have said, I found it necessary to make many trials before I could obtain results that were satisfactory to me. I do not believe that any other set of curves materially differing from mine would give equally satisfactory results. My final conclusion is, that notwithstanding Mr. King's protest, I believe the graphic method of adjustment is generally the best that can be adopted, and that it is equally applicable whether our materials are scanty or ample.

It may be useful to make some remarks in illustration of the principle that, when both marriage and death are at work on a body of men, the probabilities of marrying and dying within a year are not the same things as the marriage and death rates. In order to see what is the correct meaning of the phrase "probability of marrying in a year", we must go back to the fundamental definition of the theory of probabilities. If we find that, out of 10,000 bachelors of the age of 25, there are 122 who die unmarried and 741 who marry within a year, and we wish to know the probability at the beginning of the year that any one of them will marry in it; then the event we are considering is marriage within a year, and we have to examine how many possible cases there are, and how many of these are favorable to the event; and the ratio of the latter to the former gives the probability we are seeking. Under the circumstances supposed, there are 10,000 possible cases, of which 741 are favorable to the event and 9,259 unfavorable; therefore the probability of the event happening is ·0741; and the probability of its not happening is ·9259; in other words, ·0741 is the probability that any one of the bachelors (assuming them to be all similarly circumstanced) will marry within a year, and ·9259 is the probability that he will not. In precisely the same way the probability of any one of the bachelors dying unmarried within a year, is ·0122. The point may be further illustrated in the following way:—Let there be an urn containing 10,000 balls, of which 122 are black, 741 red, and the remaining 9,137 white. Then the black balls will represent the bachelors who die unmarried in the course of the year, the red balls will represent those who marry, and the white balls will represent those who are alive and unmarried at the end of the year. Now suppose that each of the 10,000 bachelors draws one ball out of the urn. What is the probability that any particular bachelor will draw a red ball? Clearly ·0741. The probability that he will draw a black ball is similarly ·0122, and the probability that he will draw a white ball, ·9137. The sum of these three fractions is unity, because it is certain that he must draw one or the other. In precisely the same way it is certain that every one of the 10,000 bachelors must either die unmarried within a year, or marry within the year, or be alive and unmarried at the end of it; and the sum of the probabilities of these three events must also be unity. Supposing, therefore, all the bachelors to be similarly circumstanced at the beginning of the year, the probability that any particular one of them will marry within the year, is ·0741. It has been suggested that in calculating the probability we must take off 61, half the dying, so that the probability is really $741 \div (10,000 - 61) = \cdot 074554$. But there is no more reason for taking off the 61 in this case, than there is for taking them off when we are calculating the chance of drawing a red ball.

The above-mentioned suggestion arises from a mistaken notion that the effect of the deaths of the bachelors upon the probability of marrying, is the same as would be the effect of their voluntary withdrawal or "secession". It is quite true that when we have a fluctuating body of persons, continually increased by entrants and reduced by secessions, we must make allowance for these in calculating the effective number of persons at risk during the year; and when we have reason to suppose that the persons seceding would, if they could be traced, be found to be subject to the same rate of mortality as those who remain under observation, we shall get the correct probability of dying in a year, by comparing the number of deaths with the effective number at risk. But, when the persons withdrawing are subject to a different death-rate from those who remain, the ratio of the dying to the effective number at risk, will not be the probability of dying in a year, but will be what I have called *the death-rate among those who do not withdraw.* We can illustrate this point by the help of the 10,000 bachelors. Suppose, for the present, that they are subject to the same death-rate after marriage as before, or (to express it more exactly) that the 741 bachelors who marry in the year are subject to the same death-rate as those who do not so marry. Then, deducting half the number of marriages, or 370·5, we have 122 deaths out of 10,000—370·5=9,629·5 persons at risk for a year; and the deaths among the 370·5 will be found by proportion 9,629·5 : 122 : : 370·5 : 4·7, so that the total number of deaths is 126·7, and the probability of dying in a year is ·01267, which is exactly the same result as we get by taking the ratio of 122 to 9,629·5. In this case the death-rate is the same thing as the probability of dying in a year. As a matter of fact, however, it is well known that the death-rate among the married is not the same as among the bachelors, but is very much less, not being more than ·004 for the year. Instead of 4·7 deaths occurring among the 370·5, the deaths would probably be only 1·5, making the total deaths 123·5. The probability therefore of any one of the 10,000 bachelors dying in a year, assuming them to be all similarly circumstanced at the outset, is ·01235. Under both suppositions, the probability of a bachelor dying *unmarried* in the year is the same, namely, ·0122. Now, after a man has married, he may die in the course of the same year; but, after he has married, he cannot die a bachelor in the same year; and since after he has died he cannot marry, it will be seen that the probability of marrying in a year is analogous to the probability of dying unmarried in the year, and not analogous to the absolute probability of dying. It follows from these considerations that the effect of the 122 deaths on the probability of marrying, is altogether different from the effect of the same number of secessions; the explanation being that the 122 who die are not withdrawn from observation so as to leave it uncertain whether they will marry or not, but they have been kept under observation until it is certain that they will not marry. As a general rule, we may say that the probability of marrying in a year is the ratio of the number of marriages, to the number of bachelors entering on the year, who all remain under observation for a year or until they either marry or die previously; that is to say, in the case supposed, the probability of marrying is ·0741.

Fire Insurance. A Theory of Statistics. By THOMAS MILLER, F.I.A.

A NUMBER of suggestions made at various times by different persons for subjecting Fire Insurance Statistics to systematic treatment were reviewed by Mr. Cornelius Walford some time since in an Article "On the scientific application of Data to the purpose of deducing Rates of Premium for Fire Insurance", which is published in the April 1878 number of the *Assurance Magazine*. The present paper is an attempt to lead the investigation into another course, and its purpose is to enunciate a THEORY OF STATISTICS in dependence on which algebraic formulæ may be constructed and arithmetical calculations made of the values of risks. The particular line it will take is that of averaging quantities, but it could be developed in other directions if that were desirable.

Proceeding, without further preface, to the consideration of the subject, we observe:—

That the readiness with which materials can be ignited by any particular agency varies—

1. In consequence of their constituents being different, as *Wool* contrasted with *Cotton*.
2. From the same kinds of substance being in different *forms*, as timber in logs, deals, shavings, sawdust, or splinters.
3. From substances existing in different *states*, as flax lying loose in repose, and flax tossed about and being cleaned in a heckling-house.

A continued supply of air is generally required for the support of combustion; and division of parts, by admitting air and the action of wind, facilitates the spread of fire. The detaching and scattering of fragments of inflammable materials by machines at work is one cause of the danger arising from their use; and artificial heat creates a similar hazard by setting free, and mixing with air, minute portions of combustible substances within fire risk of each other. It is partly owing to this, and partly to the nature of the objects, that the distillation of spirits and boiling of oils are so hazardous.

Artificial heat is, however, not necessary for the existence of such kinds of risk. When, for instance, spirituous vapours, or gases mixed in known proportions with air, or even solids, like pulverized resinous wood, or ground husks of grain, kept in suspense in sufficient quantity in the air of a confined apartment, take fire, the combustion that follows is almost as rapid as an explosion of gunpowder.

The intensity of the heat of fire depends upon the quality and quantity of the material in process of combustion at one time, and the rapidity with which it burns.

The distance to which fire has a directly igniting power depends upon the intensity of its heat, its continuance, the sparks which the fire emits, the readiness with which the materials within risk can be ignited, and the force of the wind towards or from them. Wind is generally found to be an active agent in serious conflagrations.

Obstacles may prevent the extension of fire, and the degree of resistance which they can offer to its advance depends upon their nature, dimensions, and other qualities; and upon the intensity of the action of fire upon them, and the length of its continuance.

The power of resisting fire may be multiplied by the addition of active means of repression; and the chances of extinguishing fire depend upon the sufficiency of such means, and the celerity with which they can be brought into action.

If we were to ascertain, by observation, experiment, or statistics, the ignition risk of inflammable materials within assigned limits, apart from known active causes of fire, and in a state of repose; to determine the degrees by which that risk is multiplied by active fire forces; to judge of the restraining power of passive obstacles to the spread of fire, and the chances of repressing and extinguishing fire by active means; and, if we could measure the force of loss when fire occurs, we could then calculate the rate of premium for insuring property.

The ceilings and roof of a building form divisions in it, which partially can restrain the action of fire. A building itself, with its divisions, has, again, a distinct individuality, and with its contents may be regarded as a perfect risk. We shall, therefore, consider a "RISK" to be a BUILDING AND ITS CONTENTS, on the understanding that other forms of hazard exist, the degrees of which must be estimated, in such ways as their circumstances admit of, in subordination to the general theory.

The postulates, or assumed bases of the investigation, are:—

1. That as buildings are composed of floors divided from each other, the risk of a single floor and its contents is the proper basis of calculation—*the unit of risk.*
2. That each floor of a building of more than one floor is subject to two kinds of risk; the one of fire occurring within the floor itself, the other of fire being communicated to it from floors above or below.

It must be admitted that there are different degrees of risk of fire occurring in the several floors of a building, and different probabilities of its being communicated to other floors; that the value of property in each floor is not the same; and that there are very many ways and proportions in which expectation of loss can be distributed among its different floors. Each combination of elements could be made the subject of investigation if it were necessary; but we may be permitted to enquire if a variety of questions need be answered until data are obtained; or if a building and its contents should not rather be received in relation to fire risk, as *in all respects* alike in each of its equal parts? If, for example, we make a floor the minimum of risk and calculation, then may not each floor have all the fire risk qualities attributed to it in the same degree? May we not consider the average initial risk of all the floors as the initial risk of each; and assume that each floor contains an average value of property at risk, and has an equal initial expectation of loss with each of the other floors?

Presuming that we may do so, let a represent the risk for a given period of the inflammable portions of a floor of a building and its contents in a state of repose; and b_1, b_2, &c., the various multipliers of that risk arising out of occupation, operations of any kind, heating, lighting, and other circumstances, then $a(1+b_1+b_2$, &c.) will be the total risk of fire occurring in that floor. If, now, f be the total number of floors in all the buildings of a similar class and degree of risk, then fa $(1+b_1+b_2$, &c.) will be the probable total number of fires occurring among them. If we make that number $=\mathrm{F}$, then $\frac{\mathrm{F}}{f}$ equal (as above) to $a(1+b_1+b_2$, &c.) will be the expectation of fire occurring in a single floor of the series.

Among floors of the same series there are varieties of risks; if, therefore, we take $a(1+b_1+b_2$, &c.) for each floor *of a building*, and divide the total by the number of its floors, the result will be the average risk of fire occurring in a floor *of that building*. It is desirable to distinguish this from the risk of an average floor of the series, and while the latter is known by $\frac{\mathrm{F}}{f}$, the former may be distinguished by $\frac{\mathrm{F}'}{f}$.

If, in connection with a building of more floors than one, no efficient means existed for arresting and extinguishing fire, then, if fire did not expire from want of materials within reach to feed upon, the probability of the building being destroyed would be $\frac{\mathrm{F}'}{f}$ multiplied by the number of floors it contained.

Let g be the probability that the divisions between floors, above or below, will unaided restrain the spread of fire, and h_1, h_2, &c., the degrees in which that probability is multiplied by active means of repression, then the probability that fire will be extinguished before penetrating into the next floor will be $g(1+h_1+h_2,$ &c.$)$, which expression may be represented by q. The probability, therefore, that a fire which has occurred will extend to the floor above (or to the floor below) will be $1-q$. If the means of extinguishing a fire will, whatever its extent, bear a like proportion to its magnitude,—and this is not an unreasonable supposition when the average is taken at starting,—then $(1-q)^2$, $(1-q)^3$, &c. will be the respective probabilities of its extension to the second, third, and other floors further off—above or below,—and the probability of fire occurring in any floor, and of its extension upwards or downwards, will be $\frac{F'}{f}\left(1+(1-q)+(1-q)^2, \text{ \&c.}\right)$

These observations lead directly to an estimate of the risk of fire occurring in or being communicated to the different floors of a building, and enable us to construct the following table:—

FLOORS.	1	2	3	4	5	6
Probability of fire occurring in each floor	$\frac{F'}{f}$	$\frac{F'}{f}$	$\frac{F'}{f}$	$\frac{F'}{f}$	$\frac{F'}{f}$	$\frac{F'}{f}$
Probability of Floor 1 communicating fire to the other floors .	...	$\frac{F'}{f}(1-q)$	$\frac{F'}{f}(1-q)^2$	$\frac{F'}{f}(1-q)^3$	$\frac{F'}{f}(1-q)^4$	$\frac{F'}{f}(1-q)^5$
Do. of 2 to do. .	$\frac{F'}{f}(1-q)$	...	$\frac{F'}{f}(1-q)$	$\frac{F'}{f}(1-q)^2$	$\frac{F'}{f}(1-q)^3$	$\frac{F'}{f}(1-q)^4$
Do. of 3 to do. .	$\frac{F'}{f}(1-q)^2$	$\frac{F'}{f}(1-q)$	...	$\frac{F'}{f}(1-q)$	$\frac{F'}{f}(1-q)^2$	$\frac{F'}{f}(1-q)^3$
Do. of 4 to do. .	$\frac{F'}{f}(1-q)^3$	$\frac{F'}{f}(1-q)^2$	$\frac{F'}{f}(1-q)$	...	$\frac{F'}{f}(1-q)$	$\frac{F'}{f}(1-q)^2$
Do. of 5 to do. .	$\frac{F'}{f}(1-q)^4$	$\frac{F'}{f}(1-q)^3$	$\frac{F'}{f}(1-q)^2$	$\frac{F'}{f}(1-q)$	...	$\frac{F'}{f}(1-q)$
Do. of 6 to do. .	$\frac{F'}{f}(1-q)^5$	$\frac{F'}{f}(1-q)^4$	$\frac{F'}{f}(1-q)^3$	$\frac{F'}{f}(1-q)^2$	$\frac{F'}{f}(1-q)$	...

From this table we find that, if n be the number of floors in a building, and C_n the risk that each floor has of fire being communicated to it from other floors, and consequently $\frac{F'}{f}(1+C_n)$ the combined probability, then—

The Total Probability of Fire occurring in or being communicated to the several Floors of a Building, $\frac{F'}{f}\left(1+C_n\right)$, is	Floor.	Of a Building of
$\frac{F'}{f}$	1	One floor.
$\frac{F'}{f}\left(1+1-q\right)$	Each floor	Two floors.
$\frac{F'}{f}\left\{1+(1-q)+(1-q)^2\right\}$	1	Three floors.
$\frac{F'}{f}\left\{1+2(1-q)\right\}$	2	
$\frac{F'}{f}\left\{1+(1-q)+(1-q)^2\right\}$	3	
$\frac{F'}{f}\left\{1+(1-q)+(1-q)^2+(1-q)^3\right\}$	1	Four floors.
$\frac{F'}{f}\left\{1+2(1-q)+(1-q)^2\right\}$	2	
$\frac{F'}{f}\left\{1+2(1-q)+(1-q)^2\right\}$	3	
$\frac{F'}{f}\left\{1+(1-q)+(1-q)^2+(1-q)^3\right\}$	4	
$\frac{F'}{f}\left\{1+(1-q)+(1-q)^2+(1-q)^3+(1-q)^4\right\}$	1	Five floors.
$\frac{F'}{f}\left\{1+2(1-q)+(1-q)^2+(1-q)^3\right\}$	2	
$\frac{F'}{f}\left\{1+2(1-q)+2(1-q)^2\right\}$	3	
$\frac{F'}{f}\left\{1+2(1-q)+(1-q)^2+(1-q)^3\right\}$	4	
$\frac{F'}{f}\left\{1+(1-q)+(1-q)^2+(1-q)^3+(1-q)^4\right\}$	5	
$\frac{F'}{f}\left\{1+(1-q)+(1-q)^2+(1-q)^3+(1-q)^4+(1-q)^5\right\}$	1	Six floors.
$\frac{F'}{f}\left\{1+2(1-q)+(1-q)^2+(1-q)^3+(1-q)^4\right\}$	2	
$\frac{F'}{f}\left\{1+2(1-q)+2(1-q)^2+(1-q)^3\right\}$	3	
$\frac{F'}{f}\left\{1+2(1-q)+2(1-q)^2+(1-q)^3\right\}$	4	
$\frac{F'}{f}\left\{1+2(1-q)+(1-q)^2+(1-q)^3+(1-q)^4\right\}$	5	
$\frac{F'}{f}\left\{1+(1-q)+(1-q)^2+(1-q)^3+(1-q)^4-(1-q)^5\right\}$	6	

The process may be continued, if desired, for buildings of more than six floors.

The comparative combined probabilities of fire in any floor depend, it will be seen, upon the number of floors in the building, and the means that passively exist, or can be brought into action, to resist the spread of fire. The central floors appear to be in greatest risk, but we propose to carry into all details the principle of average,—or rather of equality, in this instance,—and to consider the risk of other floors to be the same as theirs. This determination receives support from the following observations:—1. As there is greater tendency in fire to ascend than to descend, this tends to increase the risk communicated to the floors above; 2. As water used in extinguishing fires often floods the floors below, the effect—in so far as it causes loss—is similar to an addition to the fire risk from above. In future, therefore, the expression $\frac{F'}{f}(1+C_{n'})$ will represent the combined probability of fire occurring in, and of fire being communicated to the middle floor, or one of the middle floors, of a building.

Let L be the sum of all losses, per 100 insured of any denomination of money, sustained in a given period—a year for instance—in all the floors (of a series) in which fire *breaks out, and in them only*. As F represents the number of fires, $\frac{L}{F}$ will be the average loss sustained in the floors in which fires originate; and as the proportion between the numerator and the denominator will remain the same whether the losses are few or many, F in this expression will not vary as F′ does. As, further, the property insured is supposed to be distributed in equal proportions in every part of the building, the amount of loss will be similar to 1 in the floor in which fire occurs, to $1-q$, $(1-q)^2$, and so on, in the first, second, and following floors further off, so that $\frac{L}{F} \times \frac{F'}{f}(1+C_{n'})$ will be the expectation of loss if the building stands alone, or is effectually protected from the risk of other buildings; and, leaving discount out of view, the same expression will represent the NET OR UNLOADED PREMIUM that should be paid for the insurance of the property.

When the risk of fire occurring and the value of property are equal in each floor, the property insured may form part of, or be included in, one or more or all of the floors of the building, without the expectation of loss being increased or diminished. For if in one floor the whole of it incurs the risk $\frac{F'}{f}(1+C_{n'})$, and if dispersed through different floors each portion incurs a like risk $\frac{F'}{f}(+C_{n'})$, and as the losses in each floor are supposed on the average to be equal, the total risk with average loss makes the expectation of loss the same in whatever way the property is divided.

Except in special circumstances, every building adjoining, or near to another, adds to its risk; and when there are several buildings in proximity, and hazardous goods stored in them, or hazardous processes of manufacture carried on, their individual risks may be greatly increased by the risk they incur from the vicinity of the others.

Let B stand for a building including its contents; x, y, z, &c., showing the particular one, as well as indicating the number of floors in it. Let also $r_{x|y}$ mean the probability that if fire occur in B_x it will be arrested before it can extend to B_y, and $r_{y|x}$ the probability that fire occurring in B_y will be stopped before extending to B_x. Again, make $x\frac{F'_x}{f} = x$ times the probability of fire occurring in a particular floor of building X of x floors, and so on; then $x\frac{F'_x}{f}(1-r_{x|y})$ and $y\frac{F'_y}{f}(1-r_{y|x})$ will be the respective probabilities of fire occurring in B_x and extending to B_y, and of fire occurring in B_y and extending to B_x.

As $\frac{L}{F} \times \frac{F'}{f}(1+C_{n'}) =$ the premium (undiscounted) for insurance of a building X standing out of risk of all others, then to include the risk for proximity, there would have to be added to that sum $\frac{L}{F} \times \frac{F'}{f}(1+C_{n'})\,y\frac{F'_y}{f}(1-r_{y|x})$ for B's proximity, and so on: so that the total premium would be

$$\frac{L}{F} \times \frac{F'}{f}(1+C_{n'})\left\{1+y\frac{F'}{f}(1-r_{y|x})+z\frac{F'_z}{f}(1-r_{z|x}) \text{ \&c.}\right\}.$$

Even this does not show the full extent to which the premiums ought to be increased if the buildings B_y, B_z, &c., were in danger from others, as the additions to their individual risks from this cause would proportionally involve an addition to that of B_x.

A few words in continuation of the preceding remarks may be permitted on the subject of "CONFLAGRATIONS", causing the destruction of some or many buildings. We may suppose that such events can only occur under special fire-favouring circumstances. These may be drought, or scarcity of water, or prevalence of high wind, or some combination of unusual events. Whenever the means of active resistance to fire are overpowered, or rendered inoperative, the real conflagration commences. Then nothing remains to resist the flames and heat and sparks but the passive incombustibility of objects on which they can act, and the feeble efforts of individuals to improvise means for the defence of their property.

We have supposed q to be equal to $g(1+h_1+h_2,$ &c.), and if we now suppose r equal to $k(1+m_1, m_2,$ &c), in which k represents the probability of passive resistance to fire being successful, and the remainder of the expression the forces by which active measures of repression multiply that probability, then, during a conflagration, q may become equal to g only, and r to k, and the probabilities of the extension of the fire outside and inside the various buildings will be represented by $1-k$ and $1-g$ respectively.

Suppose B_u to B_z to be buildings, in risk of each other as links in a chain, and that fire has reached B_u *under ordinary circumstances*, then the probability that it will get to B_z through the intervening buildings will be

$$(1-r_{u|v})(1-r_{v|w})(1-r_{w|x})(1-r_{x|y})(1-r_{y|z})\,.$$

If the buildings are substantially built with ordinary non-combustible walls, and if water be abundant, and fire engines sufficient in number and power, the above probability is practically equal to 0.

But if we suppose that on the arrival of fire at B_u the organized power of *active* resistance to it has been overcome, then each r in that expression will have to be represented by $k=$ to the probability that the buildings themselves, unaided by active means of repression, will resist the fire attack; and each $q=$ the probability of the divisions between floors of the building successfully resisting the fire, will be represented by g; so that not only will the expectation of fire extending from one building to another be increased, but the extent of fire in buildings attacked be greater, and the risk they communicate to others be proportionally increased.

Each building may likewise be in the risk of several others on fire if there is an advancing conflagration, and the approach of fire to B_z may also be made through other lines of buildings. Under these circumstances the risk of fire upon any particular point may so accumulate as to become equal to certainty when the buildings are of ordinary construction.

When, in any place, buildings are for the most part ill-adapted by construction to resist the action of fire—that is to say, when they have unsubstantial, or combustible, or ill-protected walls or roofs, and a mass of inflammable goods is stored in them—the means of actively repressing fire may be overpowered under any combination of fire-favouring circumstances, and then, as only the passive powers of resistance to its force, outside and inside the buildings, will remain, it will continue to spread while anything is left within reach for it to feed upon, and none but the most substantial and carefully guarded buildings will escape destruction.

Fire insurance statistics may be obtained in two ways, the one by simple observation of the fire qualities of different materials under varied conditions, the other by gathering data of risks incurred and losses sustained; and each method may be mutually helpful to the other. By scientific enquiry and experiment, if necessary, the fire qualities referred to may be determined with considerable exactness. With regard to the collection of data of risks and losses, the process is a more lengthened one. All buildings insured need not, however, be brought under observation, but a sufficient number of each kind, fairly representative of the whole, should be selected. Quantities that admit of mathematical treatment by simple formulæ generally take the equated form, and it will be convenient therefore to consider each sum insured to be £100, or one hundred of some other kind of money. The losses will thus be rates per-cent exactly. Observations for a number of years will be necessary to make the statistics of real insurance value.

All data that are necessary for calculations might be collected under the following specifications:—

I.—*a.* Those that relate to the risk of *ignition* of the inflammable portion of buildings of different kinds and their contents, *when not in actual use,*—the average risk of all the floors under observation being taken as that of the risk of a single floor.

b. The degrees in which this risk in a single floor is multiplied by occupation, and operations, including the use of various kinds of machinery, heating, lighting, and other hazards, known or unknown.

These particulars suggest enquiry as to the relative risks of different materials, and the variations of them arising from differences in quality, stages of division, processes, or additional hazards to which the materials may be subjected.

II. Those that relate to the risk of fire spreading from one floor of a building to another.

a. As to comparative efficacy of passive means of restraining fire, offered by the separations between floors; and the variations in their efficiency caused by staircases, hoists, traps, and other openings; also the effects of different heights of floors.

b. The extent to which the risk is influenced by the kinds of materials stored, or in process in the buildings, the quantities of them, and the *states* in which they exist.

c. The degrees in which the probable restraining power of divisions between floors is multiplied by active means of extinguishing fire, such as the speedy application of water, even in small quantities; the accessibility, sufficiency, and constancy of the supply in larger quantities; the number, power, and efficiency of fire engines, and other active means of repressing fire.

III. Those that relate to the risk of fire spreading from one building to another.

a. The effect of distance, alone, between risks, and how the size and nature of risks influence the spread of fire.

b. The effects, at various distances, of the construction and thickness of external walls; also of division walls separating adjoining risks; and of the heights to which the latter are carried above the roofs.

c. The effects of fire-proofing of roofs, and the protection of windows and external openings.

d. The extent to which the efficiency of passive obstacles, such as these, to the spread of fire, will be multiplied by active means of extinguishing fire.

IV. Those that relate to the average percentage of loss—on "building and contents"—*in those floors* ONLY *in which fire originates.*

Direct statistics of the facts could not and need not be collected, but the information wanted might be obtained by the following process:—

1. Classify the buildings according to the number of floors.
2. Ascertain the number of fires in each class, and the rate of loss per 100 insured, on the whole "building and contents", in each case.

Then if n be the number of floors in each building of the series, the total amount of the losses in each class will be $L_n(1+C_n)$; and if the number of losses in a class be F_n, we shall have $\frac{L_n}{F_n}(1+C_n)$ as the average loss—*i.e.*, the average amount of all the losses—in buildings of n floors. If we call this A_n, we shall obtain

$$\frac{L_1}{F_1}(1+C_1)=\frac{L_1}{F_1}= \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad A_1$$

$$\frac{L_2}{F_2}(1+C_2)=\frac{L_2}{F_2}(1+1-q)= \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad A_2$$

$$\frac{L_3}{F_3}(1+C_3)=\frac{L_3}{F_3}(1+2(1-q))= \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad A_3$$

$$\frac{L_4}{F_4}(1+C_4)=\frac{L_4}{F_4}(1+2(1-q)+(1-q)^2)= \quad . \quad . \quad . \quad A_4$$

$$\frac{L_5}{F_5}(1+C_5)=\frac{L_5}{F_5}(1+2(1-q)+2(1-q)^2)= \quad . \quad . \quad A_5$$

$$\frac{L_6}{F_6}(1+C_6)=\frac{L_6}{F_6}(1+2(1-q)+2(1-q)^2+(1-q)^3)=A_6.$$

Whence $\frac{L_1}{F_1}=A_1$; $\frac{L_2}{F_2}=\frac{A_2}{1+1-q}$; $\frac{L^3}{F_3}=\frac{A_3}{1+2(1-q)}$;

$$\frac{L_4}{F_4}=\frac{A_4}{1+2(1-q)+(1-q)^2};\quad \frac{L_5}{F_5}=\frac{A_5}{1+2(1-q)+2(1-q)^2};$$

$$\frac{L_6}{F_6}=\frac{A_6}{1+2(1-q)+2(1-q)^2+(1-q)^3}.$$

But the quantities represented by $\frac{L_n}{F_n}$ are by the hypothesis equal to each other; wherefore, by equations selected from the above materials, we can deduce values for $\frac{L_n}{F_n}$ and q. It need scarcely be said that we must expect a variety of values of the two quantities, and consequently that some form of reconcilement will be necessary to harmonize the results.

THE THEORY OF STATISTICS, as developed on the lines laid down, may be concisely expressed by the formula $\frac{L}{F} \times \frac{F'}{f}(1+C_{n'}) =$ the net rate of premium not discounted. In STRICT THEORY, the values of all elementary fire probabilities should be found and recorded; but in practice this would no more be necessary than are records of the minutiæ of hazards in life assurance which make up the probabilities of lives failing from one year to another. Although each material should have its recognized place, and the probabilities of fire occurring through its means be accepted as settled, yet, as in household furniture, one degree of risk may be considered as common to many different kinds of materials. The degrees of probability of fire occurring in *one floor* of any kind of risk when not in active use might be found, and these probabilities and the multiples of them might be recorded. With such data, and the aid of a table showing the values of $\frac{L}{F}(1+C_{n'})$ for buildings of different numbers of floors, rates of premium for any class of risk could be CALCULATED, and TARIFFS OF RATES be easily constructed. A normal area of building should be assumed, but if the areas in any particular kind of risk did not materially differ, the differences might be overlooked; if not, the rate of premium for the normal risk would have to be multiplied by 1 + the increased *proportion* of the area; or by 1 − the decreased *proportion*, if the area were less.

There would be certain facilities of calculation in connection with this system. Thus, if water were scarce in a locality, or the fire engines defective in number and power, deductions from the values of q and r—or rather of the multiples in them represented by b and h—might be made in the computations of premium, of all policies, and thus distinctions be effected between risks in different quarters while still preserving the plan of calculation. If, again, buildings in a locality were crowded together, and among them hazardous goods were stored or trades carried on, so that the extension risk could not be accurately estimated, the insurance companies might, according to their judgment, impose an equitably calculated extension rate on every insurance on property in the district. It should be noticed that in some instances the risk of a building, even a hazardous one, from surrounding buildings, is really greater than the risk it incurs from fire originating within itself.

As an example of rating risks of the same class, but of different heights, the following may be given:—

$\frac{L}{F} \times \frac{F'}{f} \times (1+C_{n'})=$ the rate undiscounted.

$\frac{L}{F}$, the average rate of *loss* in those floors, only, of the series in which fire occurs. *Let this be* ·75 *per unit.*

$\frac{F'}{f}=a\left\{\begin{matrix}1+b_1+\dots b_m \\ \&c. + \&c., \&c.\end{matrix}\right\}\div n$—that is, the risk of ignition in a single floor of a building with its contents not in active use, multiplied by the *average* hazards arising from occupation, processes, heating, lighting, &c., and unknown risks.

Let $a=·025$ *per* £100 *insured.*

Let $1+b_1+\dots b_m=12$ in floor 1; 12 in floor 2; 9 in floor 3; 15 in floor 4; 7 in floor 5; and 5 in floor 6. Then the average will be:—

				For a building of
In a building of one floor,	12	which, multiplied by $a=·025$, gives	$·300=\frac{F'}{f}$	One floor
„ two floors,	12		·300 „	Two floors
„ three „	11		·275 „	Three „
„ four „	12		·300 „	Four „
„ five „	11		·275 „	Five „
„ six „	10		·250 „	Six „

$1-q=·60$; consequently—

		In a building of
$1+C_{n'}=1$	1·000	One floor
„ $1+1-q$	1·600	Two floors
„ $1+2(1-q)$	2·200	Three „
„ $1+2(1-q)+(1-q)^2$	2·560	Four „
„ $1+2(1-q)+2(1-q)^2$ . . .	2·920	Five „
„ $1+2(1-q)+2(1-q)^2+(1-q)^3$	3·136	Six „

The rates of premium are therefore—

$\frac{L}{F}$ ×	$\frac{F'}{f}$ ×	$(1+C_{n'})$				For a building of
·75	·300	1·000=·225	say	4/6	per-cent	One floor
„	·300	1·600=·360	„	7/3	„	Two floors
„	·275	2·200=·4538	„	9/	„	Three „
„	·300	2·560=·576	„	11/6	„	Four „
„	·275	2·920=·6022	„	12/	„	Five „
„	·250	3·136=·5880	„	12/	„	Six „

The nature of the calculations is shown by the above figures, but it should be clearly understood that the quantities given have been selected for the purposes of illustration, and are not real values. If we suppose $q=1$ and $1-q=0$—when fire could not penetrate from one floor to another in consequence of the building being really fireproof—the rate for each floor would be that of an independent risk of one floor. If, again, q were equal to 0, then $1-q$ would be equal to 1, and the rate of premium would be that for one floor multiplied by the number of floors in the building.

The results brought out will themselves suggest comments on the theory and its development in the line that has been followed. When, for instance, it is observed that the rate for the building of six floors is slightly *less* than the rate for the building of five floors, it might be hastily inferred that the calculation cannot be correct or the theory must be wrong, as each additional floor *adds* to the risk of all the others. One part, however, of the hypothesis is that, on the average, the property on which the £100 is insured is spread in equal proportions over every part of the risk,—and another, that the sum insured always bears the same proportion to the value of the property insured, whatever number of floors there may be in the building. These considerations show that when £100 is the sum insured, the expected amount of loss in any floor becomes reduced as the number of floors is increased. Thus, in a building of two floors, *one-half* of the expectation will be the share of each, while, in a building of six floors, the share of each will only be *one-sixth.* When, therefore, the number of floors is increased, and the risk of fire occurring in the ones added is less than in those below, a corresponding reduction in the average expectation of loss takes place, and so a smaller rate of premium *for insurance over the whole building* may be required. In practice we must expect to find insurances divided into different items, some of which may cover property in one or more floors exclusively; and perhaps in the higher floors the value of property may be comparatively small. If, in such cases, we were to consider the risk incurred to be similar to that of a building of the smaller number of floors, with the hazard added of fire being communicated from the floors above, we might arrive at a reasonable appreciation of the risk; and in similar cases of divergence from the average, other methods of adjusting interests could be found.

We cannot, however, proceed further with the consideration of questions of this kind, however useful their examination might be, as they merely relate to points of detail, and do not affect the general principles of the enquiry.

On the Probability that there will hereafter be Issue of a Marriage hitherto Childless. *By* THOMAS BOND SPRAGUE, M.A., F.R.S.E.

[Read before the Institute, 29 Dec. 1879.]

IT is sometimes necessary in disentail proceedings to estimate the probability that a husband, who has been married for several years without having any children, will hereafter have issue, either by his present or by a future wife. The probability of issue by a future wife is a subject comparatively well understood, and it is a recognized part of the business of some insurance companies to grant insurances against the contingency of such issue being left surviving at the death of a married man who has no child. The probability of issue by the present wife has been very little considered, and it is this probability that I propose now to discuss. There are very few statistics that will help us in the matter; in fact, I am only aware of two works in which any such statistics are given. Dr. J. Matthews Duncan gives the following table, in his work *Fecundity, Fertility, and Sterility* (p. 132, 2nd Ed.):—

Table showing the Interval between Marriage and the Birth of a First Child in Wives married at Different Ages.

		MOTHER'S AGE AT MARRIAGE.							Total.
		15–19	20–24	25–29	30–34	35–39	40–44	45–49	
YEARS MARRIED.	Less	94	325	126	44	15	4	...	608
	1	409	1,259	533	135	49	3	2	2,390
	2	83	202	88	45	17	2	...	437
	3	25	50	35	12	10	1	...	133
	4	8	31	13	8	1	...	...	61
	5	13	10	3	3	3	...	...	32
	6	5	14	6	1	1	...	...	27
	7	5	3	1	3	...	...	...	12
	8	1	3	1	...	...	...	...	5
	9	2	3	...	...	...	...	...	5
	10	...	1	...	...	...	...	...	1
	11	...	1	2	...	...	...	...	3
	12	2	1	1	...	...	...	...	4
	13	1	1	...	...	...	...	...	2
	14	...	...	...	...	...	...	...	...
	15	1	...	...	...	...	...	...	1
	16	...	...	...	...	...	...	...	...
	17	...	...	...	...	...	...	...	...
	18	...	1	...	...	...	...	...	1
	TOTAL	649	1,905	809	251	96	10	2	3,722
Average Interval between Marriage and Birth of First Child.	Year	1·516	1·329	1·350	1·510	1·594	1·400	1·000	1·385
		or	or	or	or	or	or	or	or
	Months	18·2	15·9	16·2	18·1	19·1	16·8	12·0	16·6

Mr. Charles Ansell, Jun., in his *Statistics of Families in the Upper and Professional Classes* (page 87) gives a table from which columns (1) to (4) of the table on the opposite page are extracted.

These two sets of statistics are far from consistent with each other, especially as regards the births in the first and second years after marriage. According to Dr. Duncan's statistics, for one first child that is born during the first year of marriage, four are born during the second; whereas, according to Mr. Ansell's statistics, the number of such children born in the first year is greater than in the second year in the ratio of about 3 to 2. The total number of fruitful marriages included in Mr. Ansell's observations is 6,035, and the total number included in Dr. Duncan's is 3,722. In order to show more distinctly the difference between the two sets of statistics, I have added three columns to the last table. Column (5) shows how many of the 6,035 marriages would have become fruitful in each

Interval between Marriage of Spinsters and the Birth of Children.

Year after Marriage.	Number of Children born in each year after the Marriage of the Parents.			Duncan's Totals for all Ages $\times \frac{6,035}{3,722}$	$\Sigma(2)$	$\Sigma(5)$
	1st Children.	2nd Children.	3rd Children.		Ansell.	Duncan.
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1st	3,159	...	...	986	6,035	6,035
2nd	2,163	675	...	3,875	2,876	5,049
3rd	421	2,904	163	708	713	1,174
4th	137	1,103	1,560	215	292	466
5th	69	403	1,491	99	155	251
6th	26	161	730	52	86	152
7th	21	80	353	44	60	100
8th	11	46	179	19	39	56
9th	7	37	95	8	28	37
10th	7	21	60	8	21	29
11th	5	15	39	2	14	21
12th	4	10	23	5	9	19
13th	3	5	23	7	5	14
14th	2	6	14	3	2	7
15th	...	3	6	...	...	4
16th	...	...	4	2	...	4
17th	...	2	5	...	...	2
18th	...	...	3	...	...	2
19th	...	...	...	2	...	2
20th	...	2	...	...	...	...
21st	...	2	...	...	...	...
22nd	...	...	1	...	...	...
TOTAL	6,035	5,475	4,749	6,035	...	...

year if they had followed the same law as the 3,722 considered by Dr. Duncan; and columns (6) and (7) show how many of them become fruitful at some future time after the lapse of any number of years, according to the two sets of observations. I have at present no means of saying which of the two sets of statistics is the more likely to be correct; and it is with considerable diffidence that I hazard a conjecture as to a possible explanation of the discrepancy as regards the first year of marriage. It is, however, to be noticed that the persons observed belonged to different grades of society. Mr. Ansell's statistics are obtained from returns voluntarily furnished to him by members of families in the upper and professional classes, whereas Dr. Duncan's are obtained from the registers of births of 16,593 legitimate children born in Edinburgh and Glasgow in the year 1855, who would therefore, for the most part, belong to the lower classes of society. It seems to me possible that in the latter class, where the wife very commonly shares with the husband the labor of providing for the household, marriage may, on the average, become fruitful at a later date than among the higher class of society.

In both the above tables mention is made only of the mother, but they may with equal propriety be read as relating to the duration of marriages, and as giving the number of marriages which became fruitful in the successive years of their subsistence. Regarded in this way, it will be noticed that they give no information as to the number of marriages which are entirely unfruitful, that is to say, the number of wives who have no children; but this is an element indispensable for the solution of the question we are considering, and no progress can be made in the discussion of it until we determine, with as much accuracy as is practicable, what proportion of the marriages originally entered into are permanently unfruitful.

In my paper, read before the Royal Society of Edinburgh last session, I obtained ·284 as the probability that a marriage entered into by a man of the age of 40 will be unfruitful. For younger ages the probability will, I believe, depend to a very small extent upon the husband's age, and the above-mentioned result would point to a probability of about ·25 for the ages at which most marriages are contracted. This proportion, however, is higher than the one given by Dr. Duncan in his chapter "On the Sterility of Marriages in the Population", page 182 of his work. He quotes from Dr. Stark's *Census of Scotland,* 1861. "We " have at present no means of ascertaining what proportion of the " marriages proves unfruitful. * * * * Taking two of the " largest registration districts of Glasgow, it was found that of " 14,523 married persons [The sequel shows that *married couples* " is intended here.] living together, 11,718 had children living " with them, while 2,805 had no children living. This would " yield a proportion of 80·686 per-cent with children and 19·314 " per-cent without children, or without the decimals that in every " 100 married couples 81 had children while 19 had none"; and from the English Registrar-General's report (apparently of the same census) "A great number of married people have no " children living; and it was shown in the previous report, from " a limited but perhaps a sufficient number of facts, that about 28 " in 100 married pairs had no children residing with them on the " census night. From other observations it may be estimated, " however, that not more than 20 in 100 families are childless, and " consequently that about 80 in one hundred have children living."

I have therefore thought it advisable to make my calculations on the two alternative suppositions (1) that 25 per-cent (2) that only 20 per-cent of the marriages are unfruitful.

Consider first the marriages in Dr. Duncan's table in which the mother's age at marriage was 20–24. These are 1,905 in all, and became fruitful at various intervals up to 18 years after marriage. Assuming that 25 per-cent of the total marriages are unfruitful, there will correspond to these 1,905 marriages 635 altogether unfruitful; so that out of a total of 2,540 marriages, 1,905 would be fruitful and 635 unfruitful. We shall have the means of answering completely the question I have proposed for consideration, if we trace these 2,540 marriages for the first nineteen years of their existence, after which time, according to the table, none of them will become fruitful. In order to do this, we must assume an age for the husband. The average age at marriage varies greatly according to the class of society considered. According to Mr. Meikle's observations, most marriages of Scottish advocates take place at the age of 34, but among clergymen of the Church of Scotland the greatest number take place at the age of 25. Observations on the peerage families of Great Britain and Ireland indicate 26 as the age at which most marriages of men take place. Comparing these, I assume 28 as the husband's age at marriage, and take 22 as the wife's, being the mean of the five ages 20–24. Then Dr. Duncan's figures give us the number of marriages that become fruitful in each of the nineteen years, and we have to calculate the number of unfruitful marriages that subsist at the beginning of each of those years. I assume that the husbands and wives are subject to the rate of mortality indicated by Dr. Farr's English Table No. 3; and as he has not given probabilities of dying in his work, it may be useful to state that they are given in the article "Annuities" in the last edition of the *Encyclopedia Britannica*. In order to avoid decimals, I multiply Dr. Duncan's figures by 10, and the problem therefore is to trace the progress of 25,400 marriages entered into by husbands of the age of 28 and wives of the age of 22. Of these marriages, according to Dr. Duncan's figures, 3,250 become fruitful in the course of the first year, 12,590 in the second, 2,020 in the third, and so on, as shown in column (5) of the following table. The number of subsisting unfruitful marriages will be from time to time reduced by the death of the husband or the wife, and I calculate, by a method which it is probably unnecessary to explain on the present occasion, that in the first year 214 unfruitful marriages are terminated by the death of the husband and 213 by the death of the wife; in the second year 90 by the death of the husband and 141 by the death of the wife; and so on, as shown in columns (6) and (7). We get the total decrement in any year from all causes, by adding together the figures in columns (5), (6), and (7). This is given in column (8), and by subtracting it from the corresponding number in column (4) we get the number of unfruitful marriages that still subsist at the beginning of the next year. Then column (9), which is obtained by summing the numbers in (5), shows the number of these unfruitful marriages which, according to Dr. Duncan's statistics, will at some future time become fruitful. The probability that a marriage which has subsisted for a given number of years, will hereafter become fruitful, will therefore be found by dividing the number in column (9) by that in column (4). Thus, after the lapse of 7 years there are 5,215 of the subsisting marriages still unfruitful; and 140 of these will become fruitful at some

Years elapsed.	AGES.		Number of Unfruitful Marriages subsisting at the beginning of the Year.	Of which			Total Decrement. (5)+(6)+(7)	Σ(5)
				Became Fruitful during the Year.	Were dissolved by the Death of			
	Husband.	Wife.			Husband.	Wife.		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
0	28	22	25,400	3,250	214	213	3,677	19,050
1	29	23	21,723	12,590	90	141	12,821	15,800
2	30	24	8,902	2,020	69	74	2,163	3,210
3	31	25	6,739	500	64	62	626	1,190
4	32	26	6,113	310	61	58	429	690
5	33	27	5,684	100	60	56	216	380
6	34	28	5,468	140	58	55	253	280
7	35	29	5,215	30	58	54	142	140
8	36	30	5,073	30	58	53	141	110
9	37	31	4,932	30	58	53	141	80
10	38	32	4,791	10	58	52	120	50
11	39	33	4,671	10	58	52	120	40
12	40	34	4,551	10	59	51	120	30
13	41	35	4,431	10	59	51	120	20
14	42	36	4,311	...	59	50	109	10
15	43	37	4,202	...	60	50	110	10
16	44	38	4,092	...	60	50	110	10
17	45	39	3,982	...	61	49	110	10
18	46	40	3,872	10	61	49	120	10
19	47	41	3,752	...	...	...	...	...
			TOTAL	19,050	1,325	1,273	21,648	...

future date; so that the probability that there will hereafter be issue born of a marriage between a husband of 28 and a wife of 22, which after the expiry of 7 years is still unfruitful, is 140 ÷ 5,215, or ·0268.

If we suppose that only 20 per-cent of the marriages originally entered into are ultimately unfruitful, the probabilities we are considering are considerably increased. The following table shows the probabilities calculated from Dr. Duncan's statistics on the two suppositions. Column (4) shows the probabilities calculated from Mr. Ansell's statistics on the supposition that 20 per-cent of the total marriages are unfruitful; and in column (5) I have added an adjusted probability which it seems to me may most safely be adopted in the present state of our knowledge.

Table showing the Probability, according to Various Assumptions, that there will hereafter be Issue of a Marriage which is still Unfruitful after the lapse of a given number of Years; the Ages at Marriage being 28 *and* 22.

Years elapsed.	The Proportion of Childless Marriages being			Adjusted Probability.
	25 per-cent.	20 per-cent.	20 per-cent.	
	Duncan.		Ansell.	
(1)	(2)	(3)	(4)	(5)
0	·7500	·8000	·8000	...
1	·7273	·7836	·6705	...
2	·3606	·4355	·3436	...
3	·1766	·2273	·1802	·180
4	·1129	·1487	·1066	·110
5	·0669	·0896	·0634	·063
6	·0512	·0691	·0461	·046
7	·0268	·0365	·0311	·032
8	·0217	·0296	·0230	·022
9	·0162	·0222	·0178	·015
10	·0104	·0143	·0122	·0115
11	·0086	·0118	·0081	·0086
12	·0066	·0091	·0046	·0063
13	·0045	·0062	·0019	·0046
14	·0023	·0032	...	·0036
15	·0024	·0033	...	·0028
16	·0024	·0034	...	·0021
17	·0025	·0035	...	·0015
18	·0026	·0036	...	·0010
19	...	...	...	·0006
20	...	...	...	·0003
21	...	...	...	·0000

The results of the two sets of statistics do not agree so closely as could be desired, and it is to be observed that one most important element in the calculation, namely, the proportion of marriages which are ultimately unfruitful, is to a great extent a matter of conjecture, about which considerable uncertainty exists. In illustration of this point, it may be useful to give the following additional quotations from Dr. Duncan's work. He says, on page 190, "Of women between the ages of fifteen and forty-four in-" clusive, there were married 4,372; among wives of the same " ages, 3,710 had first children, leaving 662 marriages sterile, " or 1 in 6·6. In other words, 15 per-cent [662 ÷ 4,372 = ·1514] " of all the marriages between fifteen and forty-four years of " age, as they occur in our population, are sterile." Dr. Duncan obtained the numbers in this passage from the registers of Edinburgh and Glasgow for 1855, and he points out that "although the first births in 1855 will not all pertain to the " women married in that year, it may be assumed that, if the " marriages be nearly the same in number for a few contiguous " years, the first births in one year will give the fertility very " accurately of any of the contiguous years." It appears to me that this is a reasonable supposition to proceed upon, and that the comparison is a fair one. It gives, however, a much smaller estimate of the proportion of unfruitful marriages than the previous ones; and, as the class of persons upon whom the observations were made is very different from the class with which we have to do in disentail calculations, I do not think the conclusion is suitable for our purpose.

Dr. Duncan quotes from Sir James Y. Simpson the following passage—"I had a census taken of two villages of considerable " size—viz., Grangemouth in Stirlingshire, and Bathgate in West " Lothian—the one consisting principally of a seafaring population, " and the other of persons engaged in agriculture and manufacture. " The following form the results in these two places:—Of 210 " [perhaps this should be 209] marriages in Grangemouth, 182 had " offspring; 27 had none; or about 1 marriage in 10 was without " issue [12·9 per-cent]. Of the 27 unproductive marriages, all " the subjects had lived in wedlock upwards of five years, and in " all, the female had been married that period before she reached " the age of 45. Again, of 402 marriages in Bathgate, 365 had " offspring; 37 had none; or about 1 marriage in 11 was unproduc-" tive [9·2 per-cent]. There were at the same time living in the " village 122 relicts of marriages, and of these 102 were mothers; " 20 were not mothers; or about 1 in 6 had no family [16·4 per-" cent]. In all, of 467 wives and widows, 410 had offspring; 57 had

" none, or about 1 marriage in 8 was unproductive. [This should " evidently be 524, of whom 467 had offspring, and 57, or 10·9 " per-cent, had none]. Of these last 57, six had not been five " years married, and there were other six above the age of 45 " when married. If we subtract these 12, we have, of 455 [512] " marriages, 410 [467] productive; 45 unproductive; or 1 in " $10\frac{1}{9}$th [8·8 per-cent] without issue." The method here pursued is not so satisfactory as Dr. Duncan's. Nothing is said as to the marriages in Grangemouth which had been dissolved by the death of husband or wife; and in Bathgate, although we are told that there were 122 widows living, nothing is said as to the widowers. There may have been a considerable number of wives who had died without children, and therefore the conclusion " about 1 marriage in 8 was unproductive" does not legitimately follow from the statement " of 467 wives and widows, 410 had offspring; 57 had none."

Sir J. Y. Simpson, as quoted by Dr. Duncan, further says —" I have analyzed, with some care and trouble, the history " of 503 marriages, detailed by Sharpe, in his work on the " *British Peerage* for 1833. Among British peers there were 401 " marriages with issue; 102 without issue; or, of 503 existing " marriages among British peers in 1833, 74 were without issue " after a period of 5 years. Of those who had not yet lived in the " married state for 5 years, 28 were still without family; and in " Burke's *Peerage* for 1842 there still remained among these 28 " marriages 7 without issue, making 81 as the total number of " unproductive marriages among the original 503; or the propor- " tion of the unproductive to the productive marriages among this " number is, as nearly as possible, 1 in $6\frac{2}{7}$ [16·1 per-cent]. In " the above calculation I have excluded 8 unproductive marriages, " in which the age of the husband at the date of marriage exceeded " 56. These 8, however, ought to be deducted from the original " sum of total marriages that were included; or, in other words, " the 503 should be reduced to 495, and then the whole result " would stand thus:—Among 495 marriages in the British " Peerage, 81 were unproductive, or 1 in $6\frac{1}{9}$ [16·4 per-cent] were " without any family." The comparison here made does not seem to be altogether satisfactory. Apparently the 503 marriages were all subsisting in the year 1833; and, if so, the omission of the unfruitful marriages which had been dissolved by the death of the husband or wife, will more or less vitiate the conclusion. I believe, however, that statistics extracted from the records of the

British Peerage families will afford the best means of determining what proportion of the marriages contracted remain permanently unfruitful. I am at the present time engaged in making further extracts of this kind, which I trust will throw much light on this point, as well as on the other points that have been considered in the present paper.

DISCUSSION.

The **PRESIDENT** (Mr. A. H. Bailey) doubted whether, in any calculations that they had to make for what Mr. Sprague called "disentail proceedings", they would venture to make any estimates of value based upon the probability of issue to a marriage with a lady still within the child-bearing period. So far as his experience went, he had never met with any one who had been bold enough to do so. As regarded the paper itself, he was struck, as in so many medical discussions, with the inconsistency of the different statistics; and one wished to know more than Mr. Sprague had been able to give. It was not quite clear from what sources Dr. Duncan's statistics were derived; and to be told that the proportion of unfruitful marriages is in one place 25 per-cent, and in another 15 per-cent, is not altogether satisfactory. The portion of the paper which pleased him best consisted of the last two sentences, in which Mr. Sprague says he is engaged on further researches into the British Peerage Families. A registry compiled with reasonable accuracy, such as the Peerage, is for our purpose a much more satisfactory means of solving questions of this character than when, as in Mr. Ansell's case, circulars are addressed to a variety of persons, and only those who care to reply send answers. In the Peerage we have the whole of the facts, favorable or unfavorable, as the case may be.

Mr. **BERRIDGE** said he noticed, in reading the paper, that the large estimates of unfruitful marriages were taken from censuses which record the number of married pairs who had no children residing with them, but it does not by any means follow that those were unfruitful marriages. Then Mr. Sprague started with a certain number of marriages contracted, which might be dissolved by death before they became fruitful. All these records leave out of the calculation the number that have died and not become fruitful, so that a second element of inaccuracy is introduced.

Mr. C. **WALFORD** was very familiar with Dr. Duncan's book, and also with Mr. Ansell's; but he had always found, in the case of the former, a difficulty in locating the data upon which the tables were based. The tables were indeed very interesting and ingenious, and the facts almost unique. Mr. Ansell's book also was full of interest, and we know the means by which his returns were obtained. The fact that they were contributed by people of intelligence, who could have no object in misrepresenting the facts, gives a *primâ facie* evidence of accuracy. Looking generally into the question of the probability of marriages becoming fruitful, many circumstances arise, some of which are not discussed in Mr. Sprague's paper. Going into the question some years ago, he found a general concensus of opinion that one woman in ten is barren, and some of the facts in Mr. Sprague's paper run in the same direction. While the proportion of women incapable of producing children is one in ten, that of men is supposed to be 1 in 200, but this cannot be clearly ascertained. With regard to the protracted period at which the first birth occurs, he had in several instances known it to be 10, 12, and 14, and only in one case 18 years. [Mr. **WHITCHER**—That is the only case on record.] Women sometimes become prolific, after years of marriage, from change of residence. In America he had known instances in which married couples had resided for 8 or 10 years in the Eastern States without having children, and then had gone to the West and become prolific immediately; and therefore, in any calculations regarding the question of first births after a series of years, he thought one important factor would be the probability of travelling or changing residence. He believed the time would come when they would have the means—and that partly in consequence of the attention drawn to the subject by this paper—of being able to calculate the actual probability of a married lady becoming prolific at any given period after marriage, and such observations might be of great service in actuarial pursuits. He had at the present time a complicated case before him with regard to property, in which three cases of married couples are involved, and two of them have not had any children, and the third only a still-born child. He was making some estimate of the probability of there being a family in each of these cases, and he found it a very difficult thing to do. He mentioned the circumstance merely to show that in practice there is a real necessity for estimating those probabilities, and he hoped and believed that this paper would set them thinking, and prove to be useful, and that the further enquiries upon which Mr. Sprague was engaged would result in something more definite. He could not help thinking that still-births enter as a factor into this question, and there is a difficulty in that no registration of them is required in this country. In France and other European countries still-births are registered.

Mr. **FINLAISON** asked Mr. Berridge to explain in what manner he considered the deaths of persons under observation would be likely to cause inaccuracy in Mr. Sprague's results? It seemed to him that the death of a married person during the currency of one of the periods of observation, would not be the cause of error; for the fact of death would be a determination in itself of the question under examination. It was a different case from that of the withdrawal of people under observation in the currency of a period in a mortality investigation; for in this instance death might occur in the current year of observation after the withdrawal took place, but a marriage could clearly not be fruitful, save in the case of a posthumous child, after the death of one of the couple.

Mr. **BERRIDGE** said that a certain number of marriages are contracted, of which, say, 80 per-cent will be fruitful and 20 per-cent unfruitful; and they had to compare at any subsequent period of time the number of subsisting unfruitful marriages with the number which will thereafter become fruitful. Consequently they had to take out from the total number of marriages, not only those which had become fruitful prior to that point of time, but also those dissolved by death and unfruitful prior to that point of time as well; and, therefore, in seeking for the percentage which remained unfruitful, if they took a census return which dealt only with existing marriages, and got out the percentage between the numbers fruitful and unfruitful, irrespective of those that had died, they would get a quantity not quite of the same nature as that factor which they desired to work into their calculation.

Mr. FINLAISON said that if they started with 10,000 married people, and wished to find the proportion of fruitful marriages out of this number, the subsequent deaths should not influence the calculation; but if they took their facts from any one edition of the Peerage, and traced all the married people who had had children, and those to whom no children were recorded, it appeared to him that they might not start from a correct number of original marriages, because the deaths that had taken place would have removed the cases of marriage that had subsisted for a short time only, while there would be left in the book an undue proportion of those where the husband and wife had lived long.

MR. SPRAGUE'S REPLY.

The discussion on my paper makes me think that English actuaries may be glad of a short explanation as to the nature of Scottish disentail proceedings. Under the old Scottish law entails were perpetual, and a property once entailed remained subject to the fetters of the entail so long as there existed any descendant of any one of the persons mentioned in the deed of entail. Successive Acts of Parliament have, however, given facilities for disentailing, and at the present time any old entail can be got rid of if the proprietor in possession of the estate, and the person entitled to succeed on his death, agree to execute the necessary deeds. It would occupy too much of your space to describe fully the conditions on which this is done, and it will suffice to say they will generally have to pay over to the two persons next entitled to succeed, (who are technically called heirs of entail) the estimated values of their contingent interests in the estate; and, when an application to disentail comes before the law courts, the judge makes a remit to some actuary to calculate the values of such interests. The actuary has then to estimate, to the best of his judgment and ability, all the various contingencies of death, marriage, and birth of issue, that are involved in the question; and I can assure Mr. Bailey that, as a matter of fact, estimates have to be made in disentail proceedings of the probability of issue to a marriage with a lady still within the child-bearing period, and that the sums actually paid in accordance with such estimates sometimes amount to thousands of pounds. In addition to this, I see no reason why, as information on the subject is accumulated and becomes more trustworthy, insurances should not be hereafter as readily effected against the birth of issue to a married lady still within the child- g period, as against the birth of issue to a man whose present past child-bearing. My paper had therefore the practical object of assisting actuaries in the calculations they make by order of the courts of law, or may hereafter make in the usual course of their business.

Dr. Duncan has fully described, in the first chapter of his work, the source from which he got his statistics. He says, on the first page of his work, "In 1855, when the systematic registration of births in Scotland was established, the schedule in use exacted from the public a variety of interesting details in connection with each return—a circumstance which gives to the registers for that year an extraordinary value. For, in consequence, I believe, of numerous complaints regarding the irksome labour of filling up the document, it was discontinued, and a much less comprehensive schedule has been in use ever since. It is from the registers of births for 1855 that I have extracted almost all the data which have yielded the results I am now about to communicate. Similar data cannot be found in the subsequent registers, nor, so far as I know, in any other registers whatsoever." If Dr. Duncan's work is not contained in the library of the Institute, I think it should be purchased and placed there without delay.

It is quite true, as pointed out by Mr. Berridge, that there are two elements of inaccuracy in the estimate deduced from census returns of the proportion of the marriages that are unfruitful; and for this reason I do not attach much weight to such estimates. It will be observed, however, that the errors arising from the two causes will be in opposite directions. If married couples are reckoned as childless because they have no children residing with them on the census night, this will have a tendency to give us too large an estimate; but the fact that the unfruitful marriages dissolved by death are wholly lost sight of, will have a tendency to make our estimate too low.

Mr. Finlaison seems to think that the marriages which have subsisted for a short time only, and have been dissolved by the death of husband or wife, are not mentioned in a Peerage. If this were so, it would be true that there would be left in the book an undue proportion of cases where husband and wife had lived long; but it is not the case, for the Peerage generally gives particulars, not only of the living members of each family, but also of the birth, marriage, and death of those members of the family who have died.

Actuarial Notes. By the late R. WILDING.

1.—*Smart's Interest Tables.*

THE numerous objections that have been made to the half-yearly numbers in Smart's *Interest Tables*—and especially the remark at p. 113, vol. xx, of the *Journal,* to the effect that it is difficult to say what certain of them represent—lead me to conclude that what they may be taken to indicate has never been pointed out.

The five tables given by Smart may be understood to have been calculated on the supposition that interest is convertible *momently;* and it may be remarked that the objection to this as not being in accordance with practice, tells with equal force against the received "continuous method".

In these tables, the continuous rate δ (the "force of discount") is (as usual, see *Journal* xv, 125) equal to

$$\log\left(1+\frac{rate\ per\text{-}cent}{100}\right).$$

Then, Smart's "First Table" gives the values of $e^{n\delta}$, the amount of a unit increased continuously for a time n, n being either integral or fractional.

The "Second Table" gives the reciprocal of this, or the sum which would amount to a unit in time n, interest continuous.

The "Third Table" gives the values of

$$\frac{e^{n\delta}-1}{e^{\delta}-1}=e^{n\delta}\cdot\frac{1}{e^{\delta}-1}-\frac{1}{e^{\delta}-1},$$

which is the sum that will produce in the unit of time the same interest as the unit of money will produce in time n; or, the sum which will give off a unit of interest in the unit of time, improved continuously for time n, *less* the sum itself which has given off a unit of interest in a unit of time: that is, the *amount* of a continuous annuity, for time n, of 1.

The "Fourth Table" gives the values of

$$\frac{1-e^{-n\delta}}{e^{\delta}-1}=\frac{1}{e^{\delta}-1}-e^{-n\delta}\cdot\frac{1}{e^{\delta}-1},$$

or, the *discount* for time n on the sum which will produce a unit of interest in the unit of time; which is equal to the sum which will produce a unit of interest in the unit of time *less* the present value of same sum due at time n: that is, the present value of a continuous annuity of 1 for time n.

The "Fifth Table" gives the reciprocals of the numbers of the "Fourth Table"; or, the interest produced in the unit of time by a sum whose discount for time n is *unity*.

In Smart's Tables, the numbers are given to integral values of n and halves only; but, as interpreted above, the functions of n have a *real* meaning when n is any fractional number—for instance, the cases where $n=\frac{1}{2}$ become perfectly intelligible, on the above hypothesis.

2.—*Formulas for Policy-Values in terms of Premiums.*

The formula

$${}_{n}V^{1}_{x\overline{t}|}=\frac{P^{1}_{x\overline{t}|}-P^{1}_{x\overline{n}|}}{P_{x\overset{1}{\overline{n}|}}},$$

may be found by the "retrospective" method of valuation, thus—

Premiums accumulated (n years)

$$=P^{1}_{x\overline{t}|}\cdot\frac{1+{}_{|n-1}a_x}{v^n{}_np_x}\cdot=\frac{P^{1}_{x\overline{t}|}}{P_{x\overset{1}{\overline{n}|}}}\quad\ldots\ldots\quad(A)$$

Claims accumulated for same time

$$=\frac{v(1+{}_{|n-1}a_x)-{}_{|n}a_x}{v^n\cdot{}_np_x}$$

$$=\left(v-\frac{{}_{|n}a_x}{1+{}_{|n-1}a_x}\right)\cdot\frac{1+{}_{|n-1}a_x}{v^n\cdot{}_np_x}$$

$$=P^{1}_{x\overline{n}|}\cdot\frac{1}{P_{x\overset{1}{\overline{n}|}}}\quad\ldots\ldots\ldots\ldots\quad(B)$$

$$\therefore\quad{}_{n}V^{1}_{x\overline{t}|}=(A)-(B)=\frac{P^{1}_{x\overline{t}|}-P^{1}_{x\overline{n}|}}{P_{x\overset{1}{\overline{n}|}}}.$$

When n exceeds t, the value will be negative, and shows the amount due *to* the assurance company under those circumstances; the formula then giving the amount which an accumulation for n years of a premium $P^{1}_{x\overline{t}|}$, which is only adequate to meet the accumulated claims for t years, falls short of meeting the accumulated claims for n years.

Also, $${}_{n}V_{x\overset{1}{\overline{t}|}}=\frac{P_{x\overset{1}{\overline{t}|}}}{P_{x\overset{1}{\overline{n}|}}}\quad\text{(Samot, }\textit{J.I.A.}\text{ xx, 345)};$$

$$\therefore\quad{}_{n}V_{x\overline{t}|}=\frac{P_{x\overline{t}|}-P^{1}_{x\overline{n}|}}{P_{x\overset{1}{\overline{n}|}}},\ \text{for }{}_{n}V_{x\overline{t}|}={}_{n}V^{1}_{x\overline{t}|}+{}_{n}V_{x\overset{1}{\overline{t}|}}.$$

When t is large, this becomes

$${}_{n}V_{x}=\frac{P_x-P^{1}_{x\overline{n}|}}{P_{x\overset{1}{\overline{n}|}}}.$$

Since $P_{x\overline{k}|}=P^{1}_{x\overline{k}|}+P_{x\overset{1}{\overline{k}|}}$, for all values of k, various other forms may be given to the above.

⁂ The above notes (the latter of which Mr. Wilding described to us as probably more curious than useful) were received by us several months ago, but their publication has been unavoidably delayed until the present time. We are sure that all who read them will share our regret at the author's premature death.—ED. *J.I.A.*

What is a Policy legally worth to the Insured?

[From the *Insurance Monitor*, of New York.]

NEGATIVE 'VALUES' RECOGNIZED BY THE SUPREME COURT OF NEW YORK.

THE recent decision of Justice Larned concerning the distribution of the funds of the insolvent New York State Life, contains a remarkable deliverance concerning the proper method of valuing the interest of the holders of current policies, which results in producing the negative values which have been such a bone of contention among the English experts. He says:—

By the contract of insurance the company agree to pay a certain sum of money at death, provided the other party pays to the company certain sums annually. It is plain, therefore, that the value of the contract to the assured is the present value of the moneys which he is to receive at death, less the present value of the moneys which he is to pay. This present value of the moneys which he is to pay is ascertainable on the principles governing the calculation of annuities.

For instance, if the assured, being fifty years old, has a policy payable at death for $2,000, provided, however, he pays the company therefor $100 per year, then the present value of the contract is the present value of a paid-up policy of $2,000 at fifty years old, less the present value, at fifty years old, of an annuity of $100. This matter has been a good deal discussed. It has been urged that, instead of deducting from the present value of a policy as it would be if paid up, the present value of the future premiums to be paid, there should be deducted only the present value of a certain part of those future premiums usually termed net. The argument for this calculation is based first on the fact that by the statute the superintendent, in investigating the solvency of a company, is to calculate its liabilities on that basis. (Laws 1868, chap. 623, amending sec. 20.) The answer to this is, that it is very well for the statute to provide a mode of computation in testing the matter of solvency which will show the net amount to be actually realized by the company. Because out of the whole amount received by the company it will pay expenses of agents, etc., and therefore it will not retain on hand, to keep itself solvent, the whole of what it receives. It will only retain what it receives less its expenses, etc., called by a very appropriate word, the loading. And the latter clause of the section shows that this is the view taken of the matter. That permits the superintendent, in thus estimating solvency, to calculate, in his discretion, by gross premiums, but says that in that case he is to deduct one sixth for future expenses and contingencies. Solvency of the company may depend on the one side of the account upon assets in hand and upon future premiums to be received, (less expenses paid,) and on the other side of the account upon future liabilities on the policies to be paid; and therefore the value of the policies to the company may not unwisely be computed on the net basis. But here the question is as to the value of the policies to the holders, and the holders must pay the whole premium called for by their policies.

Now, the fact that one sixth (or more or less) of the sums which the insured is obliged to pay for premiums is used up by the company in expenses, does not make his contract any more valuable to him. He is obliged to pay so much, and what he has to pay diminishes just so much the value of what he is to receive. Second—The argument is further based on the fact that in some cases (not in the present company,) a policy valued in the manner which I have decided upon, has no value whatever. This is an unfortunate result. But it is the truth that generally, and I believe always, when a policy is first issued the money which the company agree to pay, estimated at its then present value, is less than the premiums which the insured is to pay, estimated at their then present value. So that, at its first inception, before any payment, there is no pecuniary value, and it may be—I think it usually is—that even after the expiration of a year or two, the amount which the insured is to receive, estimated at its present value, is less than the amount which he is to pay, estimated at its present value. Indeed, the company expects the insured to pay as much as he will receive back, taking the average of life, and all the expenses, salaries and profits, of the company besides. Until, then, the insured has been so long in the company that the future premiums which he will have to pay are less than the amount insured, reducing both to a present value, there is not, when a new premium becomes payable, any pecuniary value in the policy. The view which I have adopted may be strengthened by this further consideration: It is stated that different companies have charged different rates of premium—even that the same company has charged different rates for policies of the same kind issued for the same amount to persons of the same age. Suppose, then, that a company has issued to each of two persons of the same age a running life policy for $2,000. One was to pay an annual premium of $80 and the other an annual premium of $100. Now, it is plain that when we come to inquire as to the value of these policies to the holders, the former is worth more than the latter, but if the policies were to be valued on the net premiums they would be of equal value.

It has been suggested that the value should be estimated by inquiring for how much another company would insure the party, assuming him to be in sound health. The objection to this mode of computing value is that the contract is one purely for the payment and receipt of money, not for the sale and manufacture of goods. The value of a mere money contract is a matter of calculation on some basis or other, not a matter depending on external proof. This is especially so when the question is on what value should a defunct company pay its obligations not yet matured.

CORRESPONDENCE.

RETIRANTS IN THE EIGHTH YEAR.

To the Editor of the Journal of the Institute of Actuaries.

DEAR SIR,—I would draw the attention of your readers to a rather interesting point. I refer to the withdrawals from life companies in the eighth year of insurance, or as it is called in England the seventh year. The matter is mentioned by Mr. King on p. 392, vol. xix of the Institute *Journal.* I extract a table from his paper:—

Discontinuance per-cent.

Year of Assurance.	17 OFFICES' EXPERIENCE.	20 OFFICES' EXPERIENCE.		
	Ages at Entry 40 to 50.	Ages at Entry		
		20 to 35.	40 to 50.	55 to 65.
0	·95	3·04	2·06	1·64
1	9·01	7·47	5·98	6·75
2	5·74	5·35	4·29	4·17
3	4·87	4·60	3·42	2·92
4	4·15	3·58	2·84	2·48
5	3·64	2·95	2·57	2·50
6	3·30	2·64	2·11	2·20
7	7·89	3·61	3·53	4·29
8	1·73	2·03	1·59	1·29
9	1·75	1·57	1·53	1·53

What is the cause of this rise at the seventh, or more properly the eighth, year of insurance? Mr. King has shown conclusively that term policies cannot be it. He says:—"Some other cause must have been at work, but I have looked in vain through the Blue-books to detect it." I believe that the seven-years half-credit system is at the bottom of the matter. Under this supposition, we can easily account for both the large number of lapses in the eighth year and the fact that in recent tables the peculiarity is not so noticeable. The explanation would appear perfect, but for the appearance of the anomaly in the experience of the Mutual Life of New York. This company never issued half-credit policies, so far as I am aware. I quote their figures:—

Year of Insurance.	Ratios of Retirants to Exposures.
1	·054
2	·100
3	·068
4	·048
5	·038
6	·033
7	·026
8	·038
9	·020
10	·022

The rise at the eighth year here is almost precisely the same as in the twenty offices—the ratios at the seventh, eighth, and ninth years in the total experience of both being respectively ·026 : ·038 : ·020 and ·025 : ·038 : ·019.

Professor Bartlett, of the Mutual Life, wrote me that he could give no satisfactory explanation. If, as I have suggested, we assume that the half-credit system is the root of the matter as regards the English statistics, what are we to think of the American ones? We have the option of two conclusions. We have here either a very remarkable coincidence or the indication of the working of some law. Which is it? We incline to the former opinion. Perhaps some of your readers will give us their views.

Yours respectfully,
T. B. MACAULAY.

Montreal,
164 *St. James' Street,*
May 8th 1879.

ON THE RELATION BETWEEN THE VALUE OF A POLICY AND THE RATE OF INTEREST.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In vol. xvii of the *Journal of the Institute*, both Mr. Macfadyen and Mr. Sutton prove satisfactorily that, under certain limitations, the value of a policy by a higher rate of interest is less than the value by a lower rate. As the following demonstration is possibly more simple than what these gentlemen give, it may not be uninteresting to some of your readers. In the cases referred to, the proof proceeds from the formula

$$1-\frac{1+a_{x+1}}{1+a_x},$$

but without reference to the duration of the policy. The value may be also expressed by

$$\frac{A_{x+n}-A_x}{1-A_x}.$$

Now, assuming the dash to indicate values relating to a higher rate of interest, we have

$${}_nV_x > {}_nV'_x$$

when

$$\frac{A_{x+n}-A_n}{1-A_x} > \frac{A'_{x+n}-A'_x}{1-A'_x};$$

that is, when

$$\frac{A_{x+n}-A_x}{A'_{x+n}-A'_x} > \frac{1-A_x}{1-A'_x}.$$

But

$$A_{x+n}-A_x = v\left(\frac{d_{x+n}}{l_{x+n}}-\frac{d_x}{l_x}\right)+v^2\left(\frac{d_{x+n+1}}{l_{x+n}}-\frac{d_{x+1}}{l_x}\right)+\ldots.$$

and

$$A'_{x+n}-A'_x = v'\left(\frac{d_{x+n}}{l_{x+n}}-\frac{d_x}{l_x}\right)+v'^2\left(\frac{d_{x+n+1}}{l_{x+n}}-\frac{d_{x+1}}{l_x}\right)+\ldots.$$

therefore, as $v'<v$, we have each term of the series for $A_{x+n}-A_x$ greater than the corresponding term of the series for $A'_{x+n}-A'_x$. But in the event of some of the terms being negative, which might easily be shown to be the case according to various mortality tables, we could not state which series is the greater. If the value of $A_{x+n}-A_x$ be written

$$v(q_{x+n}-q_x)+v^2(p_{x+n}q_{x+n+1}-p_xq_{x+1})+\ldots\ldots$$

it becomes apparent that, when the mortality increases with the age,

$$A_{x+n}-A_x>A'_{x+n}-A'_x,$$

and consequently $$\frac{A_{x+n}-A_x}{A'_{x+n}-A'_x}>\text{unity}.$$

But as $$A_x>A'_x,\ \frac{1-A_x}{1-A'_x}<\text{unity},\ \therefore\ {}_nV_x>{}_nV'_x.$$

In both of the demonstrations above referred to, the proof is founded on the assumption that $a_x>a_{x+1}$, and $>$ the annuity at all succeeding ages. Mr. Sprague, in vol. xxi, p. 94, referring to the theorem, investigates under what conditions $a_x<a_{x+1}$, and there proves that when such is the case we must have

$$vq_x>\varpi_{x+1}.$$

That is, the premium for a single year greater than the whole-life premium at the next higher age.

I am, Sir,
Your obedient servant,
WM. G. WALTON.

Scottish Amicable Life Society,
Glasgow, 16 *May* 1879.

MR. GRAY'S METHODS OF CONSTRUCTING LIFE TABLES.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—The example appended to my letter in your last number contained some errors (for which I am to blame), and it was not, as I now find, sufficiently explained.

Believing, as I do, that the process which I then sought to elucidate deserves more attention than it has hitherto gained, I should feel obliged by your permitting me to give a further illustration of it. For this purpose I will use the Institute H^M Table, which is accessible to all readers of the *Journal.*

The problem is to construct $\log N_x$, the log of D_x being given. This is Problem XXII, page 124 of Gray's *Tables and Formulæ.* Mr. Gray showed that the work brought out $\log a_x$ and $\log(1+a_x)$ as well as $\log N_x$; but he did not specially notice the remarkable fact that, excepting the datum $\log D_x$, not a figure besides appears in the process, and he did not mention any further uses of these quantities.

On inspection of the following example it will be seen that there are two working columns, the respective contents of which will be best shown by a typical representation, thus:—

$\log N_x$	
$\log D_x$	$\log a_x$
$\log (1+a_x)$	a_x
$\log N_{x-1})$	
$\log D_{x-1}$	$\log a_{x-1}$
$\log (1+a_{x-1})$	a_{x-1}
$\log N_{x-2}$	

Two complete steps are here exhibited.

The first portion of the work is the insertion of the initial value of $\log N_x$, which at the oldest age is the same as $\log D_{x+1}$, and of the successive values of $\log D_x$ in their proper places, in reversed order, on paper ruled in the form above shown.

The subsequent working consists of, first, the subtraction of $\log D_x$ from $\log N_x$, setting the remainder (which is $\log a_x$), in the adjoining column in line with $\log D_x$; second, the entering of the table of $\log (1+x)$ with this remainder, setting the result (which is $\log \overline{1+a_x}$), under $\log D_x$; and third, the addition of this result to $\log D_x$. The sum is $\log N_{x-1}$, and so on.

Of the 27 values of $\log N_x$ here formed with *six* figure logarithms, seven values differ by ± 1 from the corresponding values on page 28 of the Institute volume, which were found with *seven* figure logarithms. The annuities are all true in the last figure.

If $\log a_x$ only were wanted, $\log vp_x$ would take the place of $\log D_x$ and the work would be shorter, the several quantities being additive. But the extra trouble in finding $\log N_x$ is well repaid. Each value on the working sheet may be turned to good account in other processes. For example, $\log N_{x+n} - \log D_x$ gives the logarithm of a deferred annuity; and since the logarithms of a_x and $(1+a_x)$ are both in hand, the very useful result $\varpi_x = v - \frac{a_x}{1+a_x}$ can be found with peculiar facility.

I am indeed of opinion that Gray's Problem XXII is a memorable contribution to the practical valuation of life contingencies, and I think that Mr. Gray did not sufficiently appreciate his own incomparable process.

I am, Sir,
Yours faithfully,

London, 10 *November* 1879. J. HANNYNGTON.

Calculation of log N_x, H^M *Table, at 4 per-cent.*

96	$\bar{1}$·302009		87	409532		78	545008	
	0·054996 ·070630 2	$\bar{1}$·247013 (·1766)		042236 522335 67	367296 (2·3297)		934467 705735 33	610541 (4·0789)
95	·125628		86	564638		77	640235	
	512167 149398 18	613461 (·4106)		166982 543825 40	397656 (2·4984)		004403 726168 26	635832 (4·3235)
94	661583		85	710847		76	730597	
	836617 222246 26	824966 (·6683)		286326 563204 15	424521 (2·6578)		070280 746170 14	660317 (4·5742)
93	058889		84	849545		75	816464	
	087072 287159 8	971817 (·9372)		399660 581732 63	449885 (2·8176)		132281 765795 69	684183 (4·8326)
92	374239		83	981455		74	898145	
	292071 344017 37	082168 (1·2083)		505991 600770 48	475464 (2·9886)		190843 785104 2	707302 (5·0969)
91	636125		82	106809		73	975949	
	471982 390785 25	164143 (1·4593)		604654 620927 42	502155 (3·1780)		245441 804583 7	730508 (5·3766)
90	862792		81	225623		72	050031	
	631352 431963 25	231440 (1·7039)		696394 641693 22	529229 (3·3824)		296305 824225 22	753726 (5·6719)
89	063340		80	338109		71	120552	
	775180 468544 40	288160 (1·9416)		781284 663147 20	556825 (3·6043)		343945 843769 6	776607 (5·9787)
88	243764		79	444451		70	187720	
	910999 498489 44	332765 (2·1516)		860328 684662 18	584123 (3·8382)		388865 862856 48	798855 (6·2930)
87	409532		78	545008		69	251769	

CAN A LAW OF MORTALITY BE REPRESENTED IN A MATHEMATICAL FORM?

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In submitting for consideration the above question, I venture to express the opinion that it is delusive to imagine, when we have altered a mortality series in accordance with the formulas of Mr Gompertz or Mr Makeham, that we are thereby bending our facts into the expression of a law of mortality. I admire the ingenuity with which those formulas have been constructed, and the closeness with which their results approximate to the original facts; but I remain convinced, that the assumption of a law of mortality having been discovered,—using the word "law" in its scientific sense*,—so as to be expressible in mathematical relations, or, in other words, admitting of quantitative expression, is fallacious. And the following brief observations are offered in justification of this view:—

I. The fact that the laws or uniformities of nature discoverable within the sphere of a physical science can be expressed in a mathematical formula, is evidence that the particular science has attained its final shape,—has, in a word, become exact. Now, the expression of a law of mortality in mathematical language, really implies, though it may not be explicitly stated, that observations of mortality and deductions therefrom have reached the position of a science—and not merely of a science, but of a science in its completed form. It is equivalent to placing mortality deductions on a level, say, with astronomy, or at least of assuming that this is possible to be done. When this real implication of the formulas of Gompertz and Makeham is clearly seen, it will be admitted, I think, that the assumed position is untenable; for no one, so far as I am aware, has ever contended that statistics of the death-rate are entitled to the designation of an exact science.

II. But I go further, and venture to affirm that, from the nature of the data with which we have to deal, the difficulty is really an impossibility. It is only in the purely material sciences that the expression of their laws, or uniformities, in a mathematical shape can be made. In branches of knowledge which involve not merely the outward world, but also human nature, with their inter-actions,—such as political economy and sociology,—no such attempt is possible: the most distinguished cultivators of those departments—for example, Cairnes, Mill, and Spencer—admit that quantitative expression is unattainable; that, to quote Mr Cairnes's remark in relation to political economy (*The Logical Method of Political Economy*), their principles "are not susceptible of arithmetical or mathematical expression." Now, it is clear that mortality belongs to departments of investigation of this latter kind: its factors consist, not simply of outward physical circumstances, but also of principles of human nature, even morality and social habits, the progress even of medical science, the laws of inheritance and evolution, and the like. And, consequently, from the nature of the case, whatever partial uniformities of operation we may discover of the force, death, we cannot express the action in a general mathematical formula.

---

* The word Law is, generally, very inaccurately used, and even cultivated men frequently speak of laws opposing and neutralizing each other. The precise scientific use of the term may be thus illustrated:—In the physical sciences, Gravity is a *Force* exerted by particles of matter upon each other, whose *Law*, or uniform mode of action, is, that it varies directly as the mass and inversely as the square of the distance. Similarly employed in respect of mortality, we should represent Death as a *Force* operating upon all creatures, and its uniform mode of action (if such a mode could be discovered) as the *Law* of Mortality.

Not merely, therefore, do I believe that no law of mortality has been discovered which is capable of being exhibited in a mathematical shape, but I also think, from the nature of the subject, that it is futile to expect that such a result will ever be obtained.

In modifying series, accordingly, I am content, in our ignorance, whenever I desire to introduce an element into facts which they do not naturally present, to adopt some distributive process, like the excellent formula of Mr Woolhouse (which is *not* the mathematical expression of an assumed mortality law), or the old method of differences. To employ either of the formulas I have referred to, on the ground of being a representation of a law of mortality, which the facts, when altered, are then imagined to reveal, would be, I consider, to assume a non-existent rule, and to make an unjustified interpretation.

It is, I admit, difficult at first to accept this view; for all our terms connected with the alteration of mortality tables involve the preconceived notion that a law exists, and is known: and the association of ideas is consequently strong. For example, the word "adjustment", from its derivation, carries with it the assumption that adjusting a series of numbers is really fitting them into accordance with a law; and the term "graduation", too,—meaning a progression by regular degrees,—involves again the affirmation of a standard or law.

Yours faithfully,
T. E. YOUNG.

Commercial Union Assurance Co.,
Cornhill, London, E.C.
Dec. 1879.

DEATH RATES AMONG INFANTS IN SCOTLAND.

To the Editor of the Journal of the Institute of Actuaries.

DEAR SIR,—I have been much interested by Professor Pell's communication in the *Journal of the Institute of Actuaries* for January 1879, on "The Rates of Mortality in New South Wales", and more especially by the very clear account which he gives at pages 262-5 of the method which he prefers for deducing, by means of registered births and deaths, and without reference to the notoriously incorrect enumerations of persons of each age furnished by census, the probabilities of living for a year at each of the ages 0, 1, 2, 3, and 4.

The births and deaths that take place in Scotland are as perfectly registered as those of any country; and I have accordingly thought it worth while to prepare the accompanying tables, by means of which the numbers and probabilities furnished by recent experience in this country during a series of years may be compared with the similar details for New South Wales which Professor Pell has given.

My tables D and E are strictly in the form of those at pages 263-4 of his paper; and, in the calculation of the values of q_x, are treated without adjustment, and exactly according to my understanding of his original formulas.

It is perhaps doubtful whether the annual number of deaths *at unspecified ages* is in New South Wales so considerable as to justify any addition to the deaths recorded for each of the first five years of life. And positive assurance is lacking that any such number of births escape registration in the colony as would render an addition to the annual official statement of births desirable. It may, however, be gathered from Professor Pell's paper that he deems no adjustment of his Tables D and E either necessary or practicable; and, most certainly, to make any such adjustments on the Scottish Tables would involve an illusory show of accuracy, and be an abuse of arithmetic.

The record of births in the Scottish Table D is, I believe, as accurate as a good Registration Act can make it. In the Scottish Table E there will be found, both in its Male and in its Female portion, additional columns, showing how many deaths *at unspecified ages* occurred in all Scotland during each of sixteen years. Hence sticklers for accuracy have it in their power to repeat my calculation, after distributing among the first five years of life such rateable proportion of the deaths occurring at unspecified ages as they may think proper.

I have only to add that, although the values attributed to q_x and to p_x are, in *my* statement, carried to seven decimal places, I am perfectly aware that *five* places would have been amply sufficient, and that the only purpose really served by the higher decimal places is to attest the honesty of the arithmetical work.

When working with the aid of the Arithmometer it is sometimes easier to retain than to discard superfluous figures. I make no pretension to extreme accuracy in stating the values of q_x and p_x.

Yours very truly,
WILLIAM ROBERTSON, M.D.

28 *Albany St., Edinburgh,*
December 1879.

TABLE D.—*Births registered in Scotland during* 15 *years*, 1859-73.

Years.		Males.	Females.	Years.	
1	1859	54,628	51,915	1859	1
2	1860	54,409	51,220	1860	2
3	1861	54,606	52,403	1861	3
4	1862	55,257	51,812	1862	4
5	1863	56,226	53,115	1863	5
6	1864	57,374	54,959	1864	6
7	1865	58,220	54,850	1865	7
8	1866	58,360	55,307	1866	8
9	1867	58,517	55,527	1867	9
10	1868	59,222	56,292	1868	10
11	1869	58,321	55,033	1869	11
12	1870	58,959	56,431	1870	12
13	1871	60,029	56,099	1871	13
14	1872	61,293	57,472	1872	14
15	1873	61,467	58,233	1873	15

TABLE E.—*Deaths under Five Years of Age registered in Scotland during* 16 *Years, namely*, 1859-74.

		MALES.						FEMALES.							
Years.		0–1	1–2	2–3	3–4	4–5	Additional at Unspecified Ages.	0–1	1–2	2–3	3–4	4–5	Additional at Unspecified Ages.	Years.	
1	1859	6,461	2,843	1,508	976	700	84	5,062	2,714	1,503	1,005	723	51	1859	1
2	1860	7,366	3,252	1,544	1,007	711	77	6,047	3,028	1,473	1,024	741	46	1860	2
3	1861	6,653	2,859	1,396	871	611	67	5,216	2,683	1,347	865	600	32	1861	3
4	1862	6,982	3,210	1,567	932	649	52	5,574	3,005	1,587	1,029	662	41	1862	4
5	1863	7,275	3,835	2,017	1,374	880	67	5,850	3,550	1,958	1,377	935	42	1863	5
6	1864	7,786	3,355	1,761	1,283	898	38	6,394	3,176	1,705	1,195	822	50	1864	6
7	1865	7,808	3,150	1,432	960	699	58	6,291	2,995	1,369	945	679	36	1865	7
8	1866	7,673	3,069	1,474	967	640	36	6,242	2,895	1,394	891	697	38	1866	8
9	1867	7,544	3,002	1,468	902	620	41	5,977	2,728	1,350	900	628	37	1867	9
10	1868	7,527	3,158	1,549	1,036	779	38	6,073	2,971	1,472	1,048	742	24	1868	10
11	1869	8,120	3,477	1,774	1,214	862	39	6,534	3,236	1,699	1,216	909	30	1869	11
12	1870	7,894	3,029	1,508	1,053	784	26	6,272	2,783	1,500	1,056	784	20	1870	12
13	1871	8,352	3,160	1,549	1,022	712	38	6,782	3,067	1,551	1,011	729	20	1871	13
14	1872	8,180	3,073	1,468	975	677	29	6,540	2,924	1,417	941	767	24	1872	14
15	1873	8,322	3,320	1,450	951	695	27	6,628	2,981	1,415	902	677	16	1873	15
16	1874	8,692	3,388	1,831	1,297	982	43	6,755	3,014	1,710	1,236	918	18	1874	16

Calculation, by Pell's Method, of q *for the first Five Years of Life in Scotland.*

	Males.	Females.
B_0	866,888	820,668
B_1	805,421	762,435
B_2	744,128	704,963
B_3	684,099	648,864
B_4	625,140	592,433
χ_0	115,060	92,329
χ_1	45,017	42,015
χ_2	20,631	19,946
χ_3	12,852	12,615
χ_4	8,297	8,361
q_0	·1327265	·1125047
q_1	·0644462	·0620919
q_2	·0341702	·0339909
q_3	·0239732	·0241783
q_4	·0173523	·0179863

NOTE.—The above calculations were made by the Arithmometer, without any attempt to shorten the arithmetical process by means of logarithms.

ORIGINAL TABLES.

ANNUAL AND SINGLE ENDOWMENT ASSURANCE PREMIUMS —INSTITUTE H^M 3 PER-CENT.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—Waiting the long-promised continuation of Mr. Ralph Hardy's valuable tables founded on the Institute data, I think it would prove of value to the profession if each Member were to contribute to your columns any result likely to facilitate the work of those using these tables.

With that view, I trouble you with the enclosed, being the annual and single premiums for an Endowment Assurance of £100, at 3 per-cent, for a range of ages and terms which should embrace all that is required in practice.

Besides permitting the formation of tables for any number of years' duration, and facilitating the subsequent valuation of the policies, these afford the means of obtaining by inspection the want provided for in my contribution to the *Journal*, vol. x, p. 120, where an arbitrary "extra" may have been proposed by reason of the deterioration of a life.

The arrangement of the Table is, to save space, in the form with which we are familiar; the annual premium being in the modern type and the single premium in the antique.

I am, Sir,
Your most obedient servant,
H. AMBROSE SMITH.

Annual and Single Premiums for Endowment Assurances of 100, *payable in* n *years.*—HM 3 *per-cent.*

Age at Entry at side; "*n*" (the number of years' duration) at top and bottom of each column.

	5	6	7	8	9	10	11	12	
15	18·427	15·168	12·842	11·107	9·760	8·685	7·809	7·080	15
16	18·450	15·193	12·872	11·136	9·789	8·714	7·836	7·106	16
17	18·483	15·226	12·905	11·168	9·818	8·741	7·863	7·133	17
18	18·519	15·259	12·935	11·196	9·846	8·767	7·887	7·155	18
19	18·542	15·282	12·957	11·216	9·864	8·784	7·904	7·172	19
20	18·561	15·299	12·970	11·228	9·875	8·796	7·915	7·182	20
21	18·565	15·302	12·976	11·234	9·881	8·801	7·921	7·188	21
22	18·565	15·302	12·976	11·234	9·882	8·803	7·922	7·193	22
23	18·565	15·302	12·976	11·234	9·884	8·805	7·926	7·196	23
24	18·565	15·302	12·978	11·238	9·887	8·810	7·931	7·202	24
25	18·565	15·305	12·983	11·244	9·895	8·818	7·939	7·210	25
26	18·575	15·316	12·990	11·254	9·903	8·827	7·950	7·220	26
27	18·584	15·326	13·003	11·264	9·914	8·839	7·962	7·234	27
28	18·593	15·336	13·013	11·276	9·927	8·851	7·974	7·246	28
29	18·607	15·349	13·023	11·286	9·939	8·863	7·987	7·258	29
30	18·616	15·359	13·036	11·298	9·948	8·874	7·998	7·271	30
31	18·621	15·366	13·044	11·308	9·960	8·887	8·010	7·284	31
32	18·630	15·375	13·054	11·318	9·972	8·900	8·023	7·296	32
33	18·639	15·385	13·066	11·330	9·986	8·912	8·036	7·310	33
34	18·654	15·399	13·079	11·345	9·998	8·926	8·051	7·324	34
35	18·668	15·412	13·092	11·359	10·014	8·940	8·065	7·341	35
36	18·681	15·426	13·108	11·373	10·029	8·956	8·083	7·357	36
37	18·691	15·439	13·121	11·388	10·043	8·971	8·099	7·375	37
38	18·704	15·453	13·134	11·401	10·058	8·986	8·115	7·393	38
39	18·714	15·463	13·147	11·414	10·072	9·003	8·133	7·412	39
40	18·723	15·473	13·160	11·429	10·088	9·022	8·154	7·433	40
41	18·738	15·487	13·175	11·447	10·108	9·043	8·176	7·458	41
42	18·751	15·507	13·195	11·470	10·132	9·070	8·204	7·486	42
43	18·770	15·528	13·222	11·497	10·161	9·099	8·235	7·519	43
44	18·795	15·555	13·251	11·527	10·194	9·133	8·269	7·557	44
45	18·827	15·589	13·282	11·561	10·230	9·168	8·309	7·596	45
46	18·855	15·619	13·318	11·597	10·266	9·209	8·349	7·639	46
47	18·888	15·654	13·353	11·635	10·306	9·250	8·392	7·685	47
48	18·922	15·688	13·391	11·673	10·346	9·292	8·438	7·733	48
49	18·955	15·723	13·427	11·714	10·389	9·339	8·487	7·785	49
50	18·988	15·761	13·468	11·757	10·437	9·389	8·541	7·841	50
51	19·022	15·800	13·513	11·806	10·488	9·445	8·600	7·905	51
52	19·065	15·849	13·565	11·860	10·549	9·508	8·667	7·976	52
53	19·120	15·902	13·622	11·924	10·615	9·579	8·742	8·057	53
54	19·173	15·962	13·688	11·993	10·689	9·658	8·827	8·145	54
55	19·231	16·031	13·757	12·071	10·771	9·746	8·921	8·244	55
56	19·300	16·103	13·838	12·154	10·862	9·843	9·022	8·352	56
57	19·368	16·183	13·926	12·250	10·963	9·948	9·136	8·470	57
58	19·454	16·274	14·023	12·355	11·075	10·067	9·259	8·600	58
59	19·544	16·373	14·132	12·472	11·198	10·197	9·395	8·742	59
60	19·646	16·486	14·253	12·598	11·332	10·339	9·544	8·898	60
	5	6	7	8	9	10	11	12	

Annual and Single Premiums, &c.—(continued).

	13	14	15	16	17	18	19	20	
15	6·466	5·941	5·488	5·098	4·746	4·439	4·166	3·922	15
16	6·491	5·966	5·512	5·118	4·770	4·463	4·189	3·944	16
17	6·516	5·990	5·537	5·140	4·793	4·485	4·211	3·967	17
18	6·538	6·012	5·557	5·161	4·813	4·506	4·231	3·987	18
19	6·555	6·029	5·573	5·177	4·829	4·521	4·248	4·003	19
20	6·566	6·039	5·585	5·187	4·840	4·532	4·259	4·013	20
21	6·572	6·046	5·591	5·195	4·847	4·541	4·267	4·022	21
22	6·576	6·050	5·596	5·200	4·853	4·546	4·274	4·030	22
23	6·580	6·054	5·601	5·206	4·859	4·554	4·281	4·038	23
24	6·587	6·062	5·609	5·214	4·868	4·562	4·290	4·046	24
25	6·595	6·070	5·619	5·224	4·879	4·573	4·301	4·058	25
26	6·606	6·082	5·630	5·236	4·891	4·585	4·314	4·072	26
27	6·619	6·094	5·643	5·250	4·905	4·600	4·329	4·086	27
28	6·633	6·109	5·658	5·264	4·919	4·614	4·343	4·101	28
29	6·645	6·122	5·670	5·277	4·933	4·629	4·358	4·117	29
30	6·659	6·135	5·685	5·292	4·947	4·644	4·374	4·133	30
31	6·670	6·148	5·698	5·306	4·963	4·658	4·390	4·150	31
32	6·684	6·162	5·712	5·321	4·978	4·675	4·407	4·168	32
33	6·698	6·176	5·726	5·336	4·995	4·693	4·426	4·187	33
34	6·714	6·193	5·744	5·354	5·013	4·712	4·446	4·208	34
35	6·730	6·210	5·762	5·373	5·033	4·734	4·468	4·231	35
36	6·747	6·228	5·782	5·394	5·055	4·755	4·491	4·256	36
37	6·766	6·248	5·802	5·415	5·077	4·780	4·516	4·282	37
38	6·786	6·268	5·824	5·438	5·102	4·804	4·543	4·310	38
39	6·806	6·291	5·847	5·463	5·128	4·832	4·571	4·340	39
40	6·829	6·315	5·873	5·490	5·154	4·861	4·602	4·373	40
41	6·855	6·342	5·902	5·520	5·187	4·896	4·638	4·410	41
42	6·884	6·373	5·935	5·555	5·224	4·934	4·678	4·458	42
43	6·919	6·410	5·972	5·594	5·265	4·977	4·723	4·499	43
44	6·957	6·449	6·014	5·638	5·310	5·025	4·773	4·552	44
45	6·999	6·493	6·060	5·686	5·361	5·077	4·828	4·610	45
46	7·045	6·541	6·110	5·738	5·415	5·134	4·888	4·672	46
47	7·092	6·590	6·162	5·792	5·473	5·195	4·952	4·739	47
48	7·143	6·644	6·218	5·852	5·537	5·261	5·022	4·811	48
49	7·199	6·703	6·280	5·917	5·604	5·333	5·097	4·892	49
50	7·258	6·766	6·347	5·989	5·680	5·413	5·180	4·978	50
51	7·325	6·838	6·423	6·068	5·764	5·500	5·271	5·073	51
52	7·400	6·917	6·508	6·157	5·856	5·597	5·372	5·178	52
53	7·485	7·007	6·601	6·256	5·959	5·704	5·485	5·295	53
54	7·580	7·105	6·705	6·363	6·071	5·821	5·607	5·423	54
55	7·683	7·214	6·818	6·481	6·195	5·951	5·741	5·564	55
56	7·797	7·332	6·941	6·611	6·330	6·092	5·890	20	
57	7·921	7·463	7·077	6·754	6·479	6·248	19		
58	8·057	7·606	7·226	6·910	6·643	18		59	
59	8·206	7·761	7·391	7·082	17		60		
				16				31·134	16
60	8·369	7·938	7·572				30·421	30·566	15
	13	14	15				60	59	

Annual and Single Premiums, &c.—(continued).

	21	22	23	24	25	26	27	28	
15	3·708	3·504	3·323	3·159	3·010	2·872	2·747	2·631	15
16	3·725	3·526	3·345	3·181	3·032	2·895	2·769	2·653	16
17	3·747	3·548	3·367	3·204	3·054	2·916	2·791	2·675	17
18	3·766	3·568	3·388	3·223	3·073	2·937	2·811	2·695	18
19	3·783	3·584	3·404	3·239	3·089	2·954	2·828	2·712	19
20	3·794	3·596	3·416	3·252	3·102	2·965	2·840	2·726	20
21	3·803	3·605	3·426	3·262	3·112	2·976	2·851	2·736	21
22	3·810	3·613	3·434	3·270	3·121	2·986	2·861	2·747	22
23	3·818	3·621	3·442	3·279	3·131	2·996	2·871	2·758	23
24	3·828	3·631	3·453	3·290	3·142	3·007	2·884	2·770	24
25	3·839	3·644	3·465	3·303	3·156	3·022	2·899	2·787	25
26	3·853	3·658	3·479	3·318	3·172	3·038	2·916	2·803	26
27	3·869	3·673	3·496	3·335	3·188	3·056	2·934	2·823	27
28	3·885	3·689	3·513	3·353	3·207	3·074	2·954	2·843	28
29	3·900	3·706	3·531	3·371	3·226	3·093	2·974	2·863	29
30	3·917	3·724	3·549	3·390	3·245	3·114	2·996	2·886	30
31	3·935	3·742	3·567	3·410	3·266	3·135	3·017	2·910	31
32	3·954	3·761	3·587	3·431	3·288	3·159	3·040	2·933	32
33	3·974	3·782	3·610	3·453	3·312	3·183	3·067	2·961	33
34	3·995	3·805	3·634	3·478	3·338	3·211	3·095	2·991	34
35	4·019	3·830	3·660	3·506	3·367	3·240	3·126	3·023	35
36	4·045	3·857	3·688	3·535	3·398	3·273	3·160	3·058	36
37	4·073	3·886	3·718	3·566	3·431	3·307	3·196	3·096	37
38	4·102	3·916	3·750	3·600	3·466	3·344	3·236	3·137	38
39	4·134	3·950	3·785	3·638	3·504	3·385	3·277	3·181	39
40	4·169	3·986	3·823	3·677	3·547	3·430	3·325	3·230	40
41	4·208	4·027	3·866	3·723	3·598	3·479	3·377	3·284	41
42	4·252	4·074	3·914	3·773	3·647	3·535	3·434	3·344	42
43	4·301	4·125	3·969	3·830	3·706	3·596	3·498	3·411	43
44	4·356	4·183	4·029	3·892	3·771	3·663	3·568	3·483	44
45	4·416	4·246	4·094	3·960	3·842	3·738	3·645	3·563	45
46	4·482	4·314	4·166	4·035	3·919	3·816	3·728	3·649	46
47	4·552	4·387	4·242	4·114	4·002	3·903	3·817	3·742	47
48	4·628	4·467	4·325	4·200	4·091	3·996	3·913	28	
49	4·710	4·553	4·415	4·294	4·189	4·097	27		
50	4·801	4·647	4·512	4·396	4·295	26			
51	4·900	4·750	4·620	4·508	25				
52	5·010	4·866	4·740	24					
53	5·132	4·992	23						
54	5·265	22							
	21								

	58	57	56	55	54	53	52	51	
								36·255	24
							35·616	35·812	23
						35·007	35·194	35·404	22
					34·414	34·592	34·796	35·019	21
				33·813	33·985	34·181	34·399	34·638	20
			33·199	33·365	33·554	33·761	33·994	34·247	19
		32·541	32·703	32·884	33·088	33·310	33·557	33·825	18
17	31·848	32·004	32·179	32·375	32·593	32·829	33·088	33·371	17
16	31·285	31·457	31·647	31·857	32·084	32·337	32·608	32·905	16
15	30·732	30·916	31·117	31·341	31·583	31·848	32·133	32·444	15
	58	57	56	55	54	53	52	51	

Annual and Single Premiums, &c.—(continued).

	29	30	31	32	33	34	35	36	
15	2·524	2·424	2·333	2·248	2·169	2·096	2·029	1·965	15
16	2·546	2·447	2·356	2·271	2·193	2·119	2·052	1·988	16
17	2·568	2·470	2·378	2·294	2·215	2·142	2·075	2·012	17
18	2·588	2·490	2·399	2·314	2·236	2·163	2·096	2·034	18
19	2·605	2·508	2·416	2·332	2·254	2·182	2·115	2·053	19
20	2·619	2·521	2·430	2·347	2·269	2·198	2·131	2·070	20
21	2·631	2·533	2·443	2·360	2·283	2·212	2·145	2·085	21
22	2·641	2·544	2·455	2·373	2·295	2·225	2·160	2·099	22
23	2·653	2·557	2·468	2·386	2·310	2·240	2·175	2·115	23
24	2·667	2·571	2·482	2·401	2·326	2·256	2·193	2·133	24
25	2·682	2·587	2·500	2·419	2·344	2·276	2·212	2·154	25
26	2·700	2·606	2·519	2·439	2·365	2·296	2·235	2·177	26
27	2·720	2·627	2·540	2·461	2·388	2·321	2·259	2·203	27
28	2·742	2·648	2·562	2·484	2·412	2·346	2·285	2·229	28
29	2·763	2·671	2·586	2·508	2·437	2·372	2·313	2·258	29
30	2·786	2·694	2·610	2·534	2·465	2·400	2·342	2·288	30
31	2·810	2·719	2·637	2·562	2·493	2·430	2·373	2·321	31
32	2·836	2·747	2·666	2·592	2·524	2·463	2·406	2·356	32
33	2·864	2·777	2·697	2·624	2·558	2·498	2·444	2·394	33
34	2·896	2·810	2·731	2·660	2·595	2·536	2·483	2·436	34
35	2·930	2·845	2·767	2·698	2·635	2·578	2·526	2·480	35
36	2·966	2·883	2·807	2·739	2·678	2·623	2·574	2·529	36
37	3·005	2·924	2·851	2·784	2·726	2·672	2·624	2·581	37
38	3·048	2·969	2·897	2·833	2·775	2·723	2·678	2·638	38
39	3·095	3·017	2·948	2·886	2·829	2·780	2·736	2·698	39
40	3·146	3·070	3·003	2·943	2·889	2·843	2·801	36	
41	3·202	3·129	3·063	3·005	2·954	2·910	35		
42	3·264	3·193	3·130	3·075	3·026	34			
43	3·334	3·264	3·204	3·152	33				
44	3·409	3·343	3·285	32					
45	3·492	3·429	31						
46	3·581	30							
	29								

	50	49	48	47	46	45	44	43	
								42·100	32
							41·320	41·579	31
						40·557	40·807	41·087	30
					39·805	40·043	40·314	40·615	29
				39·062	39·295	39·554	39·845	40·166	28
			38·334	38·558	38·809	39·088	39·397	39·735	27
		37·618	37·833	38·075	38·346	38·643	38·969	39·327	26
25	36·925	37·131	37·364	37·626	37·912	38·229	38·573	38·948	25
24	36·456	36·680	36·930	37·210	37·512	37·845	38·206	38·599	24
23	36·027	36·269	36·537	36·831	37·152	37·501	37·877	38·288	23
22	35·637	35·893	36·178	36·488	36·822	37·187	37·583	38·004	22
21	35·267	35·541	35·838	36·164	36·514	36·896	37·303	37·742	21
20	34·900	35·188	35·503	35·841	36·208	36·601	37·026	37·478	20
19	34·524	34·828	35·156	35·509	35·890	36·298	36·735	37·204	19
18	34·117	34·434	34·774	35·142	35·538	35·961	36·411	36·893	18
17	33·677	34·009	34·364	34·745	35·153	35·590	36·053	36·548	17
16	33·226	33·569	33·939	34·335	34·757	35·206	35·683	36·188	16
15	32·776	33·135	33·517	33·927	34·362	34·825	35·311	35·829	15
	50	49	48	47	46	45	44	43	

Annual and Single Premiums, &c.—(continued).

	37	38	39	40	41	42	43	44	.
15	1·906	1·851	1·800	1·752	1·707	1·665	1·627	1·590	15
16	1·930	1·875	1·823	1·776	1·732	1·691	1·652	1·616	16
17	1·954	1·899	1·848	1·801	1·757	1·716	1·678	1·642	17
18	1·975	1·922	1·871	1·824	1·781	1·741	1·703	1·668	18
19	1·996	1·942	1·892	1·846	1·802	1·763	1·726	1·692	19
20	2·013	1·960	1·910	1·864	1·821	1·782	1·746	1·713	20
21	2·028	1·975	1·927	1·882	1·840	1·801	1·766	1·733	21
22	2·043	1·992	1·943	1·899	1·858	1·820	1·786	1·754	22
23	2·060	2·009	1·962	1·918	1·878	1·842	1·807	1·776	23
24	2·079	2·029	1·982	1·939	1·900	1·864	1·831	1·801	24
25	2·099	2·051	2·005	1·963	1·925	1·891	1·858	1·829	25
26	2·124	2·076	2·031	1·991	1·953	1·920	1·888	1·860	26
27	2·151	2·103	2·059	2·020	1·984	1·950	1·921	1·894	27
28	2·178	2·132	2·090	2·052	2·016	1·985	1·955	1·929	28
29	2·208	2·163	2·122	2·085	2·051	2·020	1·992	1·968	29
30	2·240	2·195	2·156	2·119	2·087	2·058	2·032	2·008	30
31	2·274	2·230	2·192	2·157	2·126	2·098	2·073	2·052	31
32	2·310	2·269	2·231	2·198	2·168	2·141	2·117		
33	2·350	2·310	2·274	2·242	2·214	2·188			
34	2·392	2·354	2·320	2·289	2·262				
35	2·439	2·403	2·370	2·341					
36	2·490	2·454	2·423						
37	2·544	2·510							
38	2·602								

	42	41	40	39	38	37	36	35	
								49·017	40
							48·091	48·441	39
						47·182	47·521	47·902	38
					46·294	46·618	46·984	47·392	37
				45·421	45·732	46·084	46·475	46·909	36
			44·561	44·864	45·202	45·577	45·994	46·451	35
		43·722	44·010	44·337	44·701	45·100	45·540	46·023	34
33	42·901	43·181	43·495	43·842	44·229	44·651	45·114	45·621	33
32	42·371	42·671	43·009	43·379	43·786	44·235	44·781	45·246	32
31	41·870	42·194	42·551	42·945	43·373	43·842	44·349	44·896	31
30	41·398	41·742	42·120	42·534	42·985	43·475	44·002	44·570	30
29	40·946	41·314	41·712	42·143	42·615	43·123	43·670	44·259	29
28	40·518	40·903	41·320	41·774	42·263	42·790	43·359	43·964	28
27	40·107	40·509	40·946	41·418	41·928	42·473	43·058	43·682	27
26	39·718	40·137	40·595	41·084	41·611	42·172	42·776	43·414	26
25	39·353	39·793	40·268	40·774	41·317	41·896	42·514	43·169	25
24	39·022	39·479	39·968	40·492	41·052	41·645	42·278	42·948	24
23	38·728	39·199	39·703	40·242	40·816	41·424	42·071	42·752	23
22	38·462	38·948	39·467	40·020	40·609	41·229	41·887	42·580	22
21	38·211	38·712	39·246	39·813	40·414	41·046	41·715	42·450	21
20	37·961	38·476	39·024	39·601	40·213	40·859	41·538	42·255	20
19	37·699	38·226	38·786	39·376	40·000	40·655	41·346	42·071	19
18	37·402	37·941	38·512	39·111	39·747	40·414	41·116	41·852	18
17	37·070	37·621	38·203	38·818	39·461	40·140	40·851	41·599	17
16	36·721	37·285	37·880	38·503	39·158	39·848	40·571	41·328	16
15	36·373	36·948	37·553	38·188	38·855	39·554	40·285	41·055	15

Annual and Single Premiums, &c.—(continued).

	45	46	47	48	49	50	51	52	
15	1·557	1·525	1·496	1·469	1·444	1·420	1·399	1·379	15
16	1·583	1·552	1·523	1·497	1·471	1·449	1·429	1·409	16
17	1·609	1·579	1·551	1·525	1·501	1·479	1·459	1·440	17
18	1·636	1·606	1·578	1·553	1·530	1·508	1·489	1·471	18
19	1·660	1·631	1·604	1·579	1·557	1·536	1·517	1·501	19
20	1·682	1·654	1·627	1·603	1·582	1·562	1·544	1·527	20
21	1·708	1·676	1·651	1·627	1·606	1·587	1·570	1·555	21
22	1·725	1·698	1·674	1·652	1·631	1·612	1·596	1·582	22
23	1·747	1·721	1·698	1·677	1·658	1·640	1·625	1·611	23
24	1·773	1·748	1·726	1·705	1·687	1·672	1·657	52	
25	1·802	1·779	1·757	1·738	1·720	1·705	51		
26	1·835	1·811	1·791	1·772	1·756	50		27	
27	1·869	1·847	1·828	1·811	49		28		
28	1·906	1·886	1·868	48		29		57·333	48
29	1·946	1·927	47		30		56·229	56·718	47
		46				55·143	55·612	56·136	46
30	1·987			31	54·068	54·517	55·020	55·580	45
	45		32						
		33		53·007	53·439	53·922	54·461	55·055	44
	34		51·971	52·385	52·848	53·366	53·936	54·563	43
		50·958	51·356	51·802	52·297	52·845	53·447	54·106	42
41	49·973	50·354	50·783	51·260	51·787	52·364	52·996	53·686	41
40	49·385	49·796	50·256	50·760	51·315	51·925	52·586	53·302	40
39	48·837	49·276	49·763	50·299	50·881	51·517	52·207	52·949	39
38	48·324	48·790	49·305	49·868	50·476	51·138	51·854	52·624	38
37	47·841	48·336	48·874	49·463	50·098	50·786	51·528	52·320	37
36	47·383	47·905	48·467	49·081	49·742	50·453	51·220	52·038	36
35	46·952	47·495	48·085	48·721	49·408	50·142	50·931	51·770	35
34	46·544	47·113	47·728	48·385	49·093	49·854	50·660	51·522	34
33	46·166	46·757	47·392	48·074	48·805	49·583	50·412	51·292	33
32	45·813	46·425	47·084	47·786	48·534	49·333	50·179	51·077	32
31	45·485	46·117	46·796	47·518	48·286	49·101	49·967	50·878	31
30	45·178	45·831	46·527	47·267	48·053	48·887	49·766	50·695	30
29	44·887	45·557	46·271	47·029	47·832	48·680	49·574	50·518	29
28	44·611	45·298	46·026	46·801	47·619	48·482	49·391	50·346	28
27	44·346	45·048	45·793	46·583	47·415	48·289	49·213	50·182	27
26	44·092	44·812	45·572	46·376	47·220	48·109	49·043	50·026	26
25	43·862	44·596	45·368	46·184	47·043	47·942	48·890	49·880	25
24	43·653	44·401	45·184	46·012	46·883	47·794	48·750	49·751	24
23	43·472	44·229	45·027	45·866	46·745	47·666	48·633	49·644	23
22	43·311	44·081	44·890	45·738	46·626	47·558	48·534	49·553	22
21	43·163	43·941	44·760	45·615	46·515	47·456	48·438	49·466	21
20	43·006	43·796	44·622	45·490	46·396	47·346	48·336	49·370	20
19	42·834	43·632	44·468	45·344	46·259	47·217	48·217	49·259	19
18	42·624	43·431	44·276	45·161	46·087	47·052	48·059	49·110	18
17	42·379	43·198	44·055	44·948	45·880	46·857	47·873	48·930	17
16	42·117	42·948	43·813	44·715	45·660	46·641	47·666	48·735	16
15	41·855	42·692	43·568	44·479	45·432	46·425	47·456	48·534	15
	34	33	32	31	30	29	28	27	

Annual and Single Premiums, &c.—(continued).

	53	54	55	56	57	58	59	60	
15	1·361	1·345	1·329	1·316	1·303	1·293	1·282	1·273	15
16	1·392	1·375	1·362	1·348	1·336	1·327	1·317	60	
17	1·424	1·408	1·394	1·382	1·371	1·361	59		
18	1·455	1·440	1·427	1·415	1·405	58		19	
19	1·485	1·470	1·458	1·448	57		20		
				56				66·913	56
20	1·512	1·500	1·488			21	65·640	66·344	55
21	1·541	1·528	55		22				
22	1·569	54		23		64·385	65·058	65·812	54
	53		24		63·152	63·793	64·512	65·314	53
		25		61·941	62·553	63·237	64·000	64·845	52
			60·752	61·335	61·990	62·718	63·525	64·411	51
	26	59·590	60·149	60·773	61·469	62·240	63·085	64·009	50
49	58·451	58·984	59·585	60·249	60·985	61·793	62·677	63·635	49
48	57·842	58·414	59·052	59·757	60·528	61·376	62·294	63·289	48
47	57·266	57·877	58·550	59·291	60·100	60·980	61·935	62·966	47
46	56·722	57·365	58·075	58·851	59·695	60·609	61·597	62·660	46
45	56·200	56·881	57·624	58·434	59·311	60·257	61·279	62·372	45
44	55·708	56·421	57·199	58·041	58·949	59·928	60·980	62·104	44
43	55·247	55·993	56·803	57·674	58·615	59·625	60·703	61·857	43
42	54·822	55·597	56·436	57·339	58·308	59·346	60·453	61·629	42
41	54·432	55·236	56·104	57·036	58·032	59·095	60·226	61·425	41
40	54·074	54·910	55·804	56·760	57·781	58·868	60·022	61·245	40
39	53·751	54·609	55·530	56·512	57·557	58·667	59·841	61·081	39
38	53·450	54·336	55·279	56·285	57·350	58·478	59·672	60·930	38
37	53·174	54·080	55·046	56·072	57·155	58·303	59·515	60·793	37
36	52·912	53·841	54·825	55·871	56·975	58·140	59·369	60·660	36
35	52·664	53·614	54·618	55·682	56·803	57·985	59·229	60·534	35
34	52·437	53·404	54·427	55·507	56·643	57·839	59·098	60·417	34
33	52·221	53·208	54·246	55·341	56·495	57·706	58·975	60·307	33
32	52·026	53·029	54·082	55·192	56·357	57·580	58·862	60·208	32
31	51·842	52·859	53·928	55·049	56·229	57·464	58·760	60·114	31
30	51·674	52·702	53·786	54·919	56·110	57·356	58·661	60·027	30
29	51·508	52·550	53·645	54·793	55·993	57·252	58·568	59·942	29
28	51·350	52·405	53·512	54·671	55·883	57·149	58·475	59·858	28
27	51·199	52·265	53·381	54·551	55·771	57·051	58·385	59·777	27
26	51·053	52·129	53·256	54·435	55·667	56·955	58·297	59·698	26
25	50·919	52·006	53·142	54·330	55·574	56·867	58·218	59·625	25
24	50·800	51·898	53·043	54·240	55·489	56·791	58·149	59·561	24
23	50·701	51·808	52·961	54·164	55·420	56·731	58·094	59·512	23
22	50·619	51·732	52·891	54·103	55·364	56·677	58·046	59·470	22
21	50·538	51·660	52·828	54·045	55·311	56·631	58·003	59·433	21
20	50·450	51·577	52·751	53·975	55·247	56·573	57·951	59·386	20
19	50·346	51·479	52·658	53·887	55·166	56·498	57·883	59·325	19
18	50·206	51·344	52·533	53·768	55·055	56·392	57·784	59·232	18
17	50·034	51·181	52·376	53·619	54·916	56·262	57·663	59·116	17
16	49·845	51·001	52·204	53·456	54·761	56·116	57·522	58·987	16
15	49·653	50·818	52·029	53·291	54·604	55·967	57·382	58·854	15
	26	25	24	23	22	21	20	19	

Annual and Single Premiums, &c.—(continued).

	18	17	16	15	14	13	12	11	
				72·220	73·146	74·182	75·339	76·618	60
			70·859	71·733	72·712	73·804	75·010	76·335	59
		69·519	70·347	71·273	72·307	73·449	74·701	76·070	58
57	68·206	68·987	69·866	70·845	71·928	73·114	74·412	75·825	57
56	67·655	68·489	69·417	70·443	71·570	72·802	74·145	75·595	56
55	67·137	68·019	68·995	70·067	71·237	72·511	73·894	75·385	55
54	66·650	67·580	68·599	69·715	70·926	72·240	73·661	75·191	54
53	66·197	67·169	68·232	69·385	70·638	71·990	73·449	75·010	53
52	65·771	66·784	67·886	69·080	70·370	71·760	73·250	74·847	52
51	65·378	66·430	67·568	68·800	70·129	71·553	73·075	74·701	51
50	65·013	66·103	67·279	68·547	69·907	71·363	72·916	74·570	50
49	64·679	65·803	67·014	68·317	69·709	71·194	72·773	74·450	49
48	64·367	65·527	66·770	68·101	69·522	71·034	72·638	74·340	48
47	64·075	65·267	66·543	67·903	69·350	70·888	72·517	74·235	47
46	63·802	65·026	66·330	67·716	69·191	70·748	72·397	74·139	46
45	63·546	64·796	66·126	67·538	69·033	70·615	72·284	74·043	45
44	63·304	64·580	65·936	67·370	68·887	70·489	72·178	73·952	44
43	63·082	64·385	65·762	67·218	68·757	70·376	72·080	73·871	43
42	62·878	64·204	65·602	67·078	68·634	70·271	71·993	73·799	42
41	62·698	64·041	65·459	66·956	68·529	70·181	71·913	73·734	41
40	62·535	63·898	65·337	66·845	68·436	70·102	71·849	73·679	40
39	62·392	63·773	65·223	66·750	68·352	70·032	71·791	73·632	39
38	62·259	63·654	65·121	66·663	68·276	69·968	71·736	73·589	38
37	62·136	63·546	65·026	66·577	68·206	69·907	71·686	73·547	37
36	62·016	63·441	64·935	66·499	68·136	69·848	71·640	73·509	36
35	61·906	63·342	64·848	66·424	68·072	69·793	71·593	73·469	35
34	61·801	63·249	64·766	66·353	68·010	69·744	71·550	73·434	34
33	61·702	63·164	64·690	66·286	67·955	69·694	71·509	73·399	33
32	61·615	63·085	64·624	66·228	67·903	69·650	71·470	73·367	32
31	61·530	63·013	64·560	66·172	67·854	69·606	71·433	73·335	31
30	61·454	62·942	64·498	66·120	67·810	69·569	71·398	73·305	30
29	61·379	62·876	64·437	66·065	67·760	69·525	71·363	73·276	29
28	61·303	62·809	64·379	66·013	67·713	69·485	71·328	73·245	28
27	61·227	62·742	64·317	65·958	67·664	69·444	71·294	73·216	27
26	61·155	62·674	64·256	65·905	67·618	69·402	71·256	73·187	26
25	61·091	62·616	64·204	65·858	67·577	69·365	71·226	73·160	25
24	61·032	62·564	64·160	65·818	67·544	69·339	71·203	73·140	24
23	60·988	62·527	64·126	65·789	67·518	69·318	71·185	73·129	23
22	60·954	62·498	64·100	65·768	67·504	69·304	71·177	73·120	22
21	60·922	62·468	64·075	65·748	67·486	69·292	71·165	73·114	21
20	60·877	62·430	64·043	65·722	67·463	69·272	71·148	73·100	20
19	60·822	62·381	63·997	65·677	67·425	69·237	71·119	73·072	19
18	60·738	62·302	63·925	65·611	67·364	69·181	71·069	73·029	18
17	60·627	62·200	63·831	65·527	67·282	69·109	71·004	72·971	17
16	60·508	62·087	63·726	65·427	67·195	69·027	70·929	72·901	16
15	60·382	61·970	63·618	65·328	67·102	68·945	70·854	72·834	15
	18	17	16	15	14	13	12	11	

Annual and Single Premiums, &c.—(continued).

	10	9	8	7	6	5	
60	78·022	79·553	81·223	83·032	84·985	87·088	60
59	77·783	79·359	81·068	82·912	84·898	87·030	59
58	77·561	79·178	80·922	82·801	84·819	86·978	58
57	77·354	79·009	80·791	82·702	84·747	86·928	57
56	77·165	78·854	80·669	82·612	84·683	86·887	56
55	76·990	78·715	80·561	82·527	84·624	86·847	55
54	76·830	78·586	80·459	82·454	84·569	86·812	54
53	76·684	78·470	80·369	82·385	84·519	86·780	53
52	76·550	78·362	80·285	82·324	84·476	86·748	52
51	76·431	78·266	80·211	82·268	84·435	86·722	51
50	76·324	78·181	80·145	82·218	84·402	86·701	50
49	76·227	78·103	80·087	82·175	84·371	86·680	49
48	76·137	78·033	80·032	82·134	84·342	86·660	48
47	76·052	77·966	79·978	82·094	84·313	86·640	47
46	75·971	77·899	79·926	82·055	84·284	86·619	46
45	75·893	77·838	79·877	82·014	84·257	86·602	45
44	75·819	77·777	79·830	81·980	84·228	86·582	44
43	75·750	77·722	79·786	81·948	84·205	86·567	43
42	75·692	77·672	79·748	81·919	84·188	86·556	42
41	75·638	77·631	79·716	81·896	84·169	86·547	41
40	75·595	77·596	79·690	81·877	84·158	86·538	40
39	75·557	77·570	79·670	81·863	84·149	86·532	39
38	75·522	77·544	79·653	81·848	84·140	86·527	38
37	75·490	77·518	79·632	81·834	84·129	86·518	37
36	75·459	77·495	79·612	81·819	84·117	86·512	36
35	75·427	77·468	79·592	81·802	84·106	86·503	35
34	75·397	77·441	79·571	81·787	84·094	86·495	34
33	75·368	77·418	79·550	81·773	84·082	86·485	33
32	75·342	77·395	79·533	81·758	84·074	86·479	32
31	75·315	77·375	79·518	81·747	84·065	86·473	31
30	75·289	77·354	79·504	81·738	84·059	86·470	30
29	75·266	77·337	79·486	81·723	84·051	86·465	29
28	75·240	77·317	79·472	81·712	84·039	86·456	28
27	75·217	77·294	79·454	81·700	84·030	86·450	27
26	75·191	77·273	79·440	81·686	84·022	86·444	26
25	75·169	77·259	79·425	81·677	84·013	86·439	25
24	75·152	77·243	79·417	81·671	84·010	86·439	24
23	75·143	77·237	79·411	81·668	84·010	86·439	23
22	75·138	77·235	79·411	81·668	84·010	86·439	22
21	75·135	77·232	79·411	81·668	84·010	86·439	21
20	75·123	77·223	79·402	81·663	84·007	86·436	20
19	75·100	77·203	79·385	81·647	83·993	86·424	19
18	75·062	77·171	79·356	81·621	83·972	86·410	18
17	75·007	77·121	79·314	81·586	83·942	86·386	17
16	74·948	77·069	79·268	81·548	83·913	86·366	16
15	74·887	77·016	79·224	81·514	83·890	86·352	15
	10	9	8	7	6	5	

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

The Position of Life and other Insurance Associations in relation to their Local Medical Examiners—Is it satisfactory?

By Cornelius Walford, F.I.A., F.S.S., F.R.H.S, *Barrister-at-Law.*

[Read before the Institute of Actuaries, 23 February 1880.]

In a former paper,* I had occasion to make some reference to the question of Local Medical Examiners, in tracing the changes in the practice of life insurance during the present century; but I then purposely avoided entering upon a detailed consideration of the points involved, believing these to be worthy of special attention whenever a suitable occasion should arise.

The important part which local medical examiners are called upon to take in the every-day practice of life, and some other important branches of, insurance business, is too well understood in this Institute to require any specific mention. It is enough for my momentary purposes, to say that a most sacred trust is reposed in them; and it will fall within the limit of these purposes to consider in some detail, on this occasion, the manner in which this trust is on the whole performed.

---

* "A Suggestion towards adding a New Feature of Usefulness to the Institute of Actuaries, &c.", *Assurance Magazine* xxii, p. 1.

The first difficulty which arises in the treatment of a question of this character, is the fear of doing injustice, by implication or otherwise, to a body of gentlemen, a considerable proportion of whom, I trust, do conscientiously perform their important functions. But on this point it has to be understood, that on those who do discharge their duties honorably and efficiently there *can* be no adverse criticism. Our remarks, then, in the nature of the case, can only apply to those who are regarded as not acting up to this high but essential standard; and as to these latter, I venture to think that—in very many cases certainly—the true position of matters has but to be placed in a clear light, in order to lead to very speedy and decided improvement.

Having made this just but necessary explanation, I shall proceed straight to my task, only further premising that the peculiar circumstances of my professional position give me opportunities of information which—while it may be taken for granted I shall not on this, or any other, occasion unfairly betray—I still shall not hesitate to make available for the general good; and without which I should hardly feel myself justified in attempting to elucidate the present question with any hope of practical results.

That the time has arrived when the points involved in the consideration before us are regarded as ripe for discussion, may be inferred from what has within the last few months transpired in the insurance press on each side of the Atlantic.* There have appeared some exhaustive articles on the question, alike characterized by fullness of knowledge and plainness of speech. These, I know, have led a number of persons to reflect upon the subject, and so have prepared the way for calm discussion, and, I trust, for efficient action.

Now, confining our attention for the moment to life insurance business only, let us imagine ourselves back to a state of things as they existed early in the present century—that is, when all proposers for insurance were required to attend "personally" before the board of directors, at the head office, on a board day, in order to be passed.† We know that, if this state of things could happen, the business would revert to very much the proportions which it formerly assumed. Life insurance would thus cease to be available to the great mass of the people. This, however, we know to be impossible. The advantages of the great provident institution of life insurance is as necessary to the requirements of a highly civilized community as is their daily bread. Insurance meets a recognized want in a manner peculiarly its own; and it is in the common interest that its administration be rendered efficient, as also that it be guarded on all hands from the damaging contact of fraud!

---

* Vide *Scottish Banking and Insurance Magazine*, Oct. 1879; the *Chronicle*, N.Y., Sept. 1879; the *Philadelphia Underwriter*, same month; and the *Insurance Times*, and the *Spectator*, N.Y.

† The early "Life Underwriters" at Lloyds' usually required a "personal appearance" (see the case of *Cleeve* v. *Gascoigne*, in the law books, middle of last century).

It follows, then, that life insurance must be taken, as it were, to the homes of the people—it must be placed within the reasonable reach of all; but this must not be at the expense of safety in any of its essential conditions. And what are its essential conditions? Equality among the parties concerned is certainly one of the most prominent, as it is one of the most obvious of these. So long as life offices were simply proprietary establishments, trading for the profit of the proprietors, the considerations were limited. It was the business of the managers of the establishments, as it was their sole pecuniary interest, to see that no lives below the ordinary standard of health should gain admission upon the books. If this event should happen, then the measure of expected profit would certainly be reduced; and, of course, if this were carried to excess —not a very likely event—the permanent stability of the company might become endangered, and thus the common interests of the policyholders become involved. This, however, is a very improbable case; there has been only one near approach to it in this country during the present century. This was in the case of a proprietary office, founded in 1824.

But from the time that policyholders were admitted into participation of profits,—and there is no life office in these realms which does not, and has not for many years now, granted such participation to those who desire it,—I say, from the time participation in profits by policyholders became general, it became the interest of every policyholder to see that this principle of equality was carried out in its entirety. If any member below the ordinary standard of health be admitted into an office without paying an adequate extra rate of premium to compensate for such impairment, an injury, direct in effect (even although it may be remote in point of the time of maturing), is inflicted upon all the participating policyholders in such office—present and future.

If this reasoning be correct, then the position of matters concerning the management of a life office, mutual and mixed,—for in this sense they are as one—is reduced into the shape of a two-fold trust. 1. The policyholders, in the interest of themselves, as of all the surviving members who have preceded them, and of all the participating members who are to follow them, do, in truth, stand in the position of trustees, morally accountable for any transgression of that great principle of equality which constitutes the essential ground-work of the scheme of life insurance. Difference in age is compensated for in the scale of premiums. Difference in the standard of health has to be estimated on the peculiar and distinctive features of each case; and in order to form an equitable estimate of these, there must be full disclosure on the part of the proposers, followed by an efficient medical estimate on the part of the medical examiner. In view, then, to the proper carrying into effect any contract of life insurance, there must exist the elements of good faith, of sufficient skill, and of common honesty.

On the question of sufficient skill, I may pause to make one remark. Half a century ago it may have happened that there were scattered through the provinces of this kingdom medical practitioners who had not enjoyed the advantages of that high standard of training in their calling which has since happily been secured, and is now a distinguishing feature of the entire profession. That might have been a real reason, in many cases, for requiring the examination to be made at the head office. I do not think, as a matter of fact, it ever was asserted as a reason. I do not think unsuitable lives have ever, at least to any appreciable extent, been insured from lack of ability on the part of the examining officer; whenever such an event happens, it is from lack of appreciation of the nature of the trust involved—and this may result from either of two causes—unpardonable indifference, or downright dishonesty.

I now proceed to the consideration of the second branch of the trust, which I hold is implied in the conduct of life insurance business. 2. While it has been shown that the policyholders themselves owe at least the allegiance of good faith towards each other, to the extent of preventing unsuitable lives—unsuitable alike from health or habits—from becoming insured, there is another class of persons upon whom a decided moral liability (capable of becoming, under certain extreme suppositions, extended with a *legal* responsibility) rests,—these are the *directors*. The directors hold office under the trusts imposed by the constitution of the company. They represent (primarily, it is usually assumed,) the stockholders, where there are any; but in every case they represent the interests of the policyholders of all classes—and most especially of the participating policyholders. If we are to take the ratio of participation as a measure of the degree of trust imposed, in the usual case of four-fifths of the profits to the insured, then the degree of responsibility to the policyholders, as against the stockholders, is as four to one. In any case, then, it can but be admitted that the *first duty* of directors is to look after the interests of the policyholders. This is to be regarded as a fundamental proposition. Now let us proceed to consider what flows from this.

It is clear that in the administration of the affairs of an office due diligence must be exercised, namely, that degree of diligence which competent business men are supposed to bring to bear in the conduct of their business affairs. The chief officers they appoint for the conduct of the details of the business must be such as by their training are to be regarded as competent to the efficient performance of the duties required, and among these head office appointments is that of a chief medical adviser to the board. But the duties of directors by no means end here. In the proper conduct of the business of a life office the appointment of provincial agents becomes necessary, and, as a consequence, of provincial medical examiners. Now all these are required to be persons who understand the proper performance of the duties which are to be entrusted to them—which, in fact, fall within the usual scope of such appointments. The directors do not usually personally select these provincial agents and medical examiners. They appoint a proper officer for this purpose, giving him suitable instructions; and they are supposed to carefully watch the results. I have before my mental vision many boards of directors who properly regard every provincial agent appointed, as being a person who should uphold the honour of the office, and to this end such agents must have a just appreciation of the responsibilities which attach to their position. But I also have another vision of boards who seem to care very little whom they appoint as representatives, so long as business speedily follows the appointment!

It hardly requires a momentary pause to determine which of the two boards more faithfully carries out its trusts to policyholders, and all others concerned. It certainly requires no argument to show which office it must, in the nature of things, be more advantageous to insure in. For myself I advisedly say that I would never remain insured in any office whose directors did not first do all that sound management indicates as necessary to secure equitable dealing as among the persons admitted to its ranks; and next, in the event of fraudulent dealings being discovered—and no insurance association, however honourably or ably conducted, can prevent the perpetration of fraud occasionally—take very immediate measures by way of remedy.

It being then admitted that the affairs of all life associations demand as a first principle, efficient administration, it follows that any system which admits of systematic inefficiency must *primâ facia* be wrong. As a rule, the conduct of affairs at the head office is efficient, or is intended by the board to be so. The weak place in life-office management is in truth nearly always with the provincial agencies. There is one notable form of exception (I trust now beginning to be understood), which it will not be out of place here to refer to, and this is where large proposals are put forward on the plea of urgency a few hours before the board, and where the head office medical examinations are consequently hurried; where also the life has been "nursed" and "coached" by those who are no novices in the skilful planting of doubtful lives; and where proposals upon the same life are before half-a-dozen boards on the same day, in the hope, which is too often realized, that if some offices decline, others will accept, and so a sufficient amount of "cover" to compensate for all trouble involved may be secured!

I will not go over the ground of the bad consequences resulting from the appointment of inefficient or dishonest agents. That portion of the subject was dealt with in my previous paper. It is indeed necessary to say here, that the greater part of the injury which results to insurance offices from having inefficient or dishonest agents, is due to the laxity, or worse, which such agents too frequently induce among the local medical examiners. But this, it will be seen, is nearly equivalent to saying that if such medical examiners performed their duties with due regard to the responsibilities of their position, the frauds perpetrated with the connivance, not to say at the instance, of agents, would be well nigh impossible. And can this be denied?

It is seen then that the position at which we have now arrived is, that frauds upon life offices could not, in the great majority of instances, be effectuated without the concurrence of the medical examiner before whom the life appears—which concurrence, however, I believe, is more from laxity and indifference than from inability to discover and prevent, if the proper degree of interest could be aroused; while purely fraudulent design, or even culpable connivance, is, I believe, very rare indeed. But that such frauds should be frequent, seems to me, under any of the assumptions now made, to say, in language not to be mistaken, that the relationship of local medical examiners to life offices is by no means satisfactory.

Now let us, in pursuit of this vital point, next consider what course it is expected that the medical examiner shall take when a life appears before him for examination. He will have appointed a time at which he shall have leisure mentally and physically for the duty involved. He will have received the proposal under cover from the head office or from the local agent; and this he will have read carefully in view of seeing whether the first glance of the proposer realizes the ideal of such a life as is there described. He will have marked any exceptional points, in view of special attention being directed towards them in the examination. He will now take the form supplied for the details of his examination, and he will deliberately ask, in their order, all the questions there enumerated. He will note the place of residence and the occupation, knowing that these two incidents combined often elicit important facts from the teachings of vital statistics; he will note the age, and see how far that may harmonize with appearances. He may also fairly ask how the proposer has ascertained his age. He will deliberately ask the question as to sickness and disease, noting any attempt at wavering or evasion, and quickly follow this up: not passing on until he is fully satisfied, or, if not so satisfied, until he has noted below, as part of his report, any point which he thinks may require further elucidation than he can now obtain. He will have regard to the occupation as affecting any ailments, and rendering them likely to be attended with more than ordinary danger; he will also consider the light which occupation may throw upon probable habits, and the effect of these latter upon any diseases experienced or indicated, even in a slight degree. Next he will reach the important stage of "family history". He will know that upon this much will and does always turn. Personal peculiarities which can in no other way be reconciled may here be elucidated. The proportion of the family dying young, or the reverse, may indicate the degree of health of the parents; and generally a flood of light may be obtained here. The proportions of the physical frame, as to height and weight; the action of the lungs, the heart, the pulse, are all vital points, and any of these may yield an indication of some latent or lurking defect when all else has failed. Finally, the amount and object of the insurance may form the subject of a little friendly discourse, at a time when the proposer, having got through the more serious part of the business, may be more communicative. The real object of the insurance, and the relation of its amount to the personal and social position of the party, is always a matter to be regarded. The proposer gone, the examiner will take a rapid survey of all the circumstances in his mind. The mental purview of an experienced man will sometimes indicate a suspicion when no isolated circumstance has previously done so. This may be accomplished in walking, riding, or driving, and is frequently best accomplished in the still hours of the night, by a man who "thinks" upon his work; and all conscientious workers, I suppose, do think, and reflect upon work in hand. When this reflective process has been performed then the final report will be written, and the forms will be returned to the office—not to the agent. The trust reposed in the medical examiner is by the directors, as part of the efficient management of the office. It is a necessary and important part of the business. The question of the effect of rejection upon the agent, as affecting his commission or emoluments, is not to be regarded; neither has the fear of offending or even injuring the proposer. If any question of *injury* to anyone be mooted, the medical examiner has to remember that in this particular function of examination he is the representative of the office, not the advocate of the proposer, nor even the friend of the agent. The duties delegated to him by the directors, are part of the trust *they* inherit by virtue of their office. If he cannot conscientiously discharge it, he can write to the office and say so; another medical examiner will then be appointed for this particular case; while he (the regular examiner) will be regarded as a man of honour and courage combined; and this is a grand combination in public, as it is in private life.

It will be said, I know, "Yes, but look at the labour, time, and responsibility involved." The answer may be direct and courteous. There is labour in all conscientious work; the responsibility is a necessary element of the confidential relationship. If the examiner shrink from the responsibility of his office, he is not fit for the post. As to the time,—well, that is always a consideration with a busy man. To conduct an examination in the careful manner I have indicated would take, say, twenty minutes, and add for the required reflection upon the case, ten—say, half-an-hour in all. The fee, either half-a-guinea or a guinea —say, on an average, 16*s*. 4*d*. Do we, as professional men, earn our money much more readily? Besides, there are alternatives.

If a case has special features, and the report bears marks of due and skilful regard to these, would a special request for a guinea, instead of half-a-guinea, be disregarded, or even complained of, by the office? It would usually be sent by return of post, with best thanks. The other alternative is more simple. If a man is too busy to perform the work properly, he should not accept the appointment.

I think it very likely that a medical examiner who performs his duties in the conscientious manner I have described, and I am glad to state that I am acquainted with some who have always done so,—that they have gained and retain the esteem of the offices for whom they act, hardly requires to be said,—I say, I think it very likely, that the performance of duties in this unflinching manner would lead to speedy complaints from agents of a certain class. But what must be the logical deduction from such a complaint? The local examiner is retained to make a full and faithful report for the guidance of the board. The medical adviser at the head office will soon discover if the standard of selection set up be unnecessarily high, and he will be the person to indicate the fact to the board, and the chief officer of the company will in turn indicate to the local examiner. The complaint of the agent to the company would be simply an indication to the board that the medical examiner was a more efficient protector of the interests of the office than the agent himself. The agent, in such a case, would probably receive a reply to the effect that he must select a higher standard of lives, or cease to do business for the company. If he should chance to get a reply that the office was about to relax its standard, it might be desirable for him to turn this over in his mind speedily. Herein he might, perchance, read the beginning of the end. Under any circumstances, he would know that a relaxing of the standard of selection might—almost inevitably must—mean a certain diminution of the bonus rate, and hence an injury to all present policyholders, as also to all future ones, except those who chanced to be of the lower class of lives now accepted. In these circumstances, a sagacious and honourable agent would be looking round, and thinking about selecting another office.

The foregoing unquestionably supposes that agents, as a rule, —and certainly medical examiners,—understand that insured life, even with the most careful selection, rarely reaches the average duration of life in the kingdom at large. This fact finds a ready solution in the axiom of a continuous "selection against the office"—which, put into plain English, means that the lives seeking insurance are so far below the average at the time of being accepted, or within a few years become so, as, notwithstanding the effects of selection, still not to come up to the national standard. This last fact does not, indeed, necessarily presuppose any great amount of fraudulent design. It stands to reason that in many cases a life supposed, or known, to be delicate, and upon which pecuniary interests depend, may suggest a *bonâ fide* need of the advantages of life insurance. It further follows that, in such cases, a ready response will be made to the solicitation of the agent. This readiness in itself will awake inquiry in the mind of a really honest agent. In many such cases, the truth will be readily told by the proposer, who would gladly pay the extra premium required to cover the extra risk, for the sake of being insured, unless warped from his purpose by an indiscreet agent. It is precisely at this point that dishonest practice too often commences. The agent seeks to flatter the family, or the proposer, by getting the life accepted as "first-class"—the more so if it chance to have been refused by any other office. Then begins that insidious manipulation of the facts of past ailments and family history, whereby most of the frauds upon life offices are accomplished. Then also begins the attempt to warp the local medical examiner from the strict path of his duty. And if any such attempt once prove successful, the downward course of that agency has commenced.

The *rationale* of medical examination is, then, to keep out the under-average lives, or only to admit them upon the payment of adequate rates of premium. But the determination of this selection falls within the proper functions of the board, and the chief officers of the association. It cannot be left to the agents, who have a direct interest to the extent of their present and prospective commission on the transaction; and who too often constitute themselves the advocate of the proposer, instead of the trusty agent of the office. It is not intended to be left to the local medical officer, whose sole business is to make an examination so complete and exhaustive as to enable the board to determine upon the wisdom or otherwise of accepting the life proposed, without the necessity of that person appearing at the head office.

Just so long as the agents of life offices remain unconvinced, wilfully or otherwise, of the necessity of medical examinations being conducted on the principles I have laid down in this paper, just so long will the vast majority of local medical examiners fail to perform their duties efficiently. We have seen that the medical examiner could defeat, by an efficient discharge of his duty, the schemes of the most wily of agents. But if they do not perform this good work of their own free will, and as the result of a just sense of personal honor, is it likely they will ever be induced to do so by any means at the ordinary command of an office? I think not, and hence I have long had under consideration the question of remedies.

It must, I think, be seen from what has preceded, that nearly all the existing evils have resulted from the interference of the local agent—first with the nomination of the local medical examiner, and afterwards, almost as of course, in the faithful discharge of the duties of his office. The medical examiner, being the nominee of the agent, rarely becomes entirely independent of him. In truth, the agent will naturally select for examiner some one whom he believes he can warp to the extent at least of securing his local influence. This, from an agency point of view, may seem reasonable enough. The medical man will lay himself out to assist the agent, say, in procuring proposals; but what follows from this? Why, he who is to act as the independent referee, from whom the management is to obtain the facts on which to base a judgment on the life proposed, has already become, in fact, a partizan. It is contrary to human nature to suppose that the medical examiner will be willing to report against a life he has been the means of introducing. I have, indeed, learned of one recent case to the contrary. Here the medical examiner introduced several lives to the agent, and reported against them all. It turned out, upon careful inquiry, that this medical examiner was aware that these lives had been proposed elsewhere and rejected; further, that in one or two of the cases he had been the examiner in the previous rejection. He employed this knowledge in the scandalous form of obtaining fees for examinations of lives which he knew beforehand must be rejected!*

I have been careful, until now, not to incorporate in my observations individual instances; but find the force of my arguments much weakened thereby. I must here claim the privilege (based upon the necessities of the case) of recording two other instances, of recent discovery:—1. An agent, operating in a large district, nominated first a practitioner of ability and independence —in every way suited for the work. The percentage of lives rejected was considerable—the district being a bad one, as to habits. On the ground of distance for his proposers to journey, soon afterwards he obtained permission to nominate another medical examiner. The first nomination having turned out well, the second was left to him. He selected a man of fair repute locally, but of low moral status; and the way the agency was henceforth worked was this—The first medical examiner was asked to take a preliminary view in the case of doubtful (*i.e.*, known to the agent to be doubtful) lives. If he formed an unfavourable impression, the lives were forthwith taken to the second medical man, and were passed without scruple—the proposal forms having been manipulated so as to evade in some degree the difficulties encountered by the first examiner. Lives that were personally known to the first referee were often sent to the second. In this way, some scandalous frauds were perpetrated upon the office in the course of a few years. The agent had made an excuse not to pay fees for these preliminary examinations; but the referee kept an account of the cases, and finally sent in a claim to the office, when the manner in which the frauds had been concocted became revealed. The frauds, or some of them, had proclaimed themselves in another form previously. Both the examiners had simply been made tools of by the agent.

---

* The increase of medical examination fees, consequent upon lives being over and over again submitted to various life offices, with varying degrees of concealment in the proposal, or on the chance of one of several offices accepting it, must augment the general expenditure of life offices very considerably.

The last individual case I shall name is this:—In an agency in a large town, for a first-class company, there had been, over a limited period of time, some ten or twelve lives rejected as not being first-class. All these lives had in succession been accepted by another office, of equal repute and standing, as first-class lives. This was managed between the agent and the medical examiner of the last-named office, but, of course, not without the fraudulent connivance of the proposers. I could multiply instances of fraud and dexterous manipulation without end—instances which have come under my own knowledge, either in the pursuit of my profession, or by other reliable means. But this is not necessary. There can be few in this room, engaged in the practical business of life insurance, who cannot endorse, from their own experience, all or much of what I have here stated. I shall assume, then, that I have conclusively shown that the relationship at present existing between life offices and their local medical examiners is not satisfactory.

Accident Insurance Offices.—In the title of this paper I have spoken of "Life and other Insurance Offices." The other offices most prominent in my mind are *Accident* insurance companies.

Health insurance, as carried on by the thousands of friendly and provident societies in this kingdom, and Industrial insurance, as conducted by several large and important companies, have each too often their special grievances, as against the medical examiners in whom they confide. They are the victims of continuous frauds, most of which might be averted by vigorous action on the part of able and independent local medical practitioners. In the general remedies which I shall soon have to consider, the necessities of these important associations will not be disregarded.

The accident insurance companies of the United Kingdom form a very important group of offices, concerning which far too little is known by the actuarial profession generally. They have acquired important data for the conduct of their business; but their chief stock-in-trade, it must be admitted, is extended worldly knowledge. This class of knowledge is, indeed, a very important factor in the successful management of all insurance institutions; but it is pre-eminently so in view of the profitable conduct of accident insurance. Speaking here as the practical founder of "general accident" insurance (as distinguished from "railway accident" insurance only) now some thirty years since, and having been more or less intimately associated with the conduct of this business, alike in this country and other parts of the world ever since,—and here it may be noted that all the companies of this class, at home and abroad, have based their operations upon the rates and conditions (with certain variations, of course) of the *Accidental Death Insurance Company,* founded 24 Jan. 1850,—I say, standing in this position as to this great branch of insurance business, I may claim to speak with authority. However unsatisfactory the relations between life offices and their local medical examiners may be, the accident companies are still in a worse plight in this regard, as will presently appear.

The number of persons insured against accidents in the United Kingdom may be estimated at 120,000. This is perhaps slightly over-stated, but it is approximately correct. One out of every twelve persons insured meets with an accidental injury every year. This gives an average of 10,000 claims per annum. Now in the case of every one of these claims a medical certificate is required, as practically the only test which can be readily supplied that an accidental injury has been sustained, of such a character as wholly or partially to disable the insured to the extent of justifying him in claiming compensation from the funds of the company. The disability must be the direct result of the accident; secondary causes bring into play considerations of physical constitution, bodily health, and personal habits. There is no medical examination when the policy is effected; the annual premium is too small for this—and, besides, the theory is that all persons are *primâ facia* liable to accidents—but there is usually a very strict warranty, as against intemperate habits, and latent infirmities, as varicose veins, &c., &c. When the injury has occurred, the insured is required to send up details of circumstances, together with certificate of medical man, specifying the nature of injuries sustained.

Now the only person who can furnish the medical details here required is the medical attendant of the insured, and his position is undoubtedly very frequently one of great embarrassment. The insured is his patient; and although he may see that a very slight injury, magnified by six pounds per week during its continuance, will in the end assume proportions altogether beyond those usually manifested when there is no insurance, what is he to do? The accident company does not pay him any fee, because it is a condition of the contract that the insured shall furnish *proof of his injury*, as the foundation upon which to base his pecuniary claim. The certificate of the medical man takes the shape of a declaration, or professional assurance, that the insured has been from such a date wholly or partially disabled (as the case may be) as the direct result of certain specified injuries.

To say that the medical man in such cases is very often, too often, the unconscious tool of the claimant is to proclaim a simple truism. But then, unfortunately, the insurance company is the ultimate victim of both. The claimant cannot prove his case without the certificate of the medical man; with it he can go a long way towards enforcing a verdict against the deluded company. But does not all this imply a great responsibility on the part of the certifying doctor? And is it not too often glaringly apparent that this feeling of responsibility; the consciousness of being entrusted where important financial interests are at stake, to perform an almost judicial function, has not been regarded even in the slightest degree? What can be done in such cases; where lies the remedy?

This laxity on the part of certain medical practitioners has indeed worked a large measure of retribution upon the entire profession. It was part of the original scheme of accident insurance to provide a sum of money in order to secure proper medical attendance in every case of accident. Thus, a person insured to receive compensation at the rate of £5 per week, would be entitled to a further claim upon the funds for a sum, not exceeding in the whole £10, for the payment of costs of medical attendance. Here was a wise and prudent forethought, calculated to be in the interest of all concerned. Well, how did it come out in practice? Why, simply that medical men would too often keep their patients in bed from slight injuries, until the entire allowance for medical expenses was exhausted. Here the company was again the victim in a two-fold degree—it had to pay large compensation for a slight injury, in order that the medical practitioner might exhaust the allowance for medical attendance. The system had to be abandoned, and it is probably not an over estimate to say that some £50,000 per annum has been lost to the medical profession, as a whole, through this unprincipled grasping of a not inconsiderable minority of its members!

I shall not stay to deal with cases of more direct fraud, in which it usually happens that the medical attendant has been made the too easy dupe—these constitute an accumulating record. The experience of even one year (instead of thirty) behind the scenes of an accident company with a large business, will not tend to improve one's opinion of human nature; although it will have a decided tendency to enlarge the scope of one's ingenuity. Yet, on the whole, insurance frauds of this class present a wonderfully small amount of real originality!

Proposed Remedies.—It is now time that I should address myself to the question of remedies. This is a point which has long occupied my thoughts, but it is one surrounded with many difficulties. *On the whole, I believe that the really effective plan would be the special training of medical practitioners to the peculiar duties required of them by life and accident insurance companies;* I mean the making of the peculiar duties of medical examiners and certifiers a feature in the educational course of medical students. To this end special lecturers might be appointed at each of the great medical schools; and a special branch of examining for proficiency be instituted. This would constitute a high and attractive branch of study; and the repute of moral rectitude, ability, and independence of character, would become essential factors in gaining the required qualification.

Assuming practitioners thus trained to be available—and they will speedily be forthcoming if a substantial indication of demand go forth—how can their services be made generally available? The answer is not difficult as to the thickly-populated districts of the country. By concert between the life and the accident associations, such men might be gradually located in great centres, as Manchester, Leeds, Liverpool, Bradford, Wakefield, Birmingham, The Potteries, Bristol, Exeter, and Plymouth; also in Edinburgh, Glasgow, and Dublin; and finally in all the principal cities and market towns of the United Kingdom. The life offices deeming an immediate change desirable, would refer their agents in future to this specialist, as the only person authorized to make their examinations in the district. The accident companies would, indeed, be glad of the services of such an independent referee for their special cases. The industrial assurance associations and friendly societies would be very prompt, I suspect, to call in such impartial aid; and so it would seem to be certain that a professional income, from a high class of practice, would be made secure at a very early period. There can, I think, be no doubt that the supply would fully equal the demand in a few years; until finally, every part of the kingdom would be represented by men who would be (1) qualified by special training for the duties they undertook; (2) stand quite independent of local agents and local influences; (3) become familiar with the occupations and prevailing diseases of the locality. It will be apparent that the insurance offices, by comparison of results, would be able to judge and appreciate the efficient performance of their duties; and hence would follow increase of emolument, direct, or by promotion to the more lucrative centres.

Of course, such special insurance practitioners would not be permitted to engage in private medical practice, directly or indirectly; they would require to have fixed hours of the day for being at home, regulated by the requirements of the district, or fixed days in the week, as their duties for the accident insurance companies would involve a certain radius of travelling. Other important functions might speedily fall to their lot, as entirely congenial to their position. In the event of the new Law respecting Coroners requiring a medical assessor to certify the true cause of death—and present coroners' courts seem too often to have but one object in view, this being *to prevent the true cause of death being discovered!* I say, in the event of an independent medical report on the cause of death being required by law—and this is the only way to remedy the abuses of the system as now existing, and utilize the functions of a public prosecutor—here will be men suited by their training and circumstances for the duty.*

---

* I know it will be said that these gentlemen will be clearly in the interest of the insurance offices. Well, granted, but what *more* can they do than get at the real facts? What *less* ought to satisfy any one?

It will be asked how I propose to obtain the united action of the insurance offices—life, accident, &c.—for carrying out the preceding proposal, even if it be viewed favourably. My reply is, that I assume the time cannot be far distant when the life offices of the United Kingdom, like those of Germany, and those of the United States, will be compelled to form an organization for the protection of their common interests. And certainly the accident companies cannot be far from such a defensive combination. This question of the appointment of efficient local medical examiners—only, of course, for the advantage of such offices as may require their aid; their remuneration being by an agreed scale of fees for work done—may be well left to such a body. In its efficient working I see the means of a more immediate and effective check upon the mistaken zeal, or worse motives, of local insurance agents; although I still think the machinery indicated in my former paper, or parts of it, will be also required.

I have referred to the fact of life offices having found it necessary to combine for their mutual protection in other countries; but this is not all. In the United States, from the peculiar difficulties of enforcing any law requiring an acknowledged degree of professional efficiency on the part of medical practitioners, and hence a corresponding social restraint upon their proceedings, the criminal law has almost of necessity been called into operation. The revised Statutes of the State of *Ohio* contain the following:—

> " *Sec.* 7,078.—A medical examiner for any life insurance company, or for any person seeking insurance therein, who knowingly makes any false statement or report to such company, or any officer thereof, concerning the health or bodily condition of any applicant for insurance, or concerning any other matter or thing which might affect the granting of such insurance, shall be fined not more than 500 dollars, or imprisoned not more than three months [69, V. 159, s. 31]."

There are other laws which reach agents and others who attempt insurance frauds. I commend this, and these, to the careful consideration of the framers of our new Criminal Code.

Finally, I have to say that in all the foregoing reasoning, and suggestions founded thereon, I have endeavoured not to go beyond the necessities of the case. I could have said very much more; and might have added time, place, and circumstance, if these had been needed; but that there is occasion for combined action became an admitted fact on the occasion of the discussion on my former paper; and with this I am content. It is for those concerned to bring into operation the remedies proposed, or others. The too exclusive mode of conducting the affairs of life and accident offices has heretofore afforded scope for, and almost encouragement to, fraudulent proceedings: for, when driven by detection from one office, the same operators have found a too ready refuge in another, and perhaps a neighbouring one. Surely the interests of policy-holders require a watchful sagacity to prevent frauds, as also a bold determination to expose and punish wrongdoers, quite as much as they require skilful internal administration? It is in behalf of such vigorous outside protective action that I ventured on a former occasion, and now again venture, to speak in words I trust of sufficiently moderated plainness, but of unmistakeable earnestness.

DISCUSSION.

The PRESIDENT (Mr. A. H. Bailey) remarked that the discussion of such a paper as this requires somewhat more than ordinary caution. In treating on such practical subjects, it is impossible to avoid discussing the practice of companies, but the discussion of the practice of any individual company is strictly prohibited at the meetings of the Institute. Mr. Walford has brought something like a charge against a very important and very honourable profession. But he (Mr. Bailey) had found that, so far from medical examiners being in league with agents, the younger practitioners sometimes, instead of answering the question, "Is there sufficient ground for rejecting this case?" exercise all their skill—their knowledge recently brought from the college and hospital—to write elaborate reports, showing why you should reject this case on rather trivial grounds. As they get more experience, they find that that is not what is desired of them. He did not think that there had ever been a single instance of an insurance company in this country having been seriously injured by an undue mortality. He thought he knew the company mentioned in the early part of the paper, and, if so, he had personally investigated its mortality, and perhaps some day its mortality may be published. It had not got into difficulties on account of undue mortality, but difficulties arose from causes which are at work in the present day, and are much more serious matters than ordinary mortality. He did not know how some of Mr. Walford's suggestions are to be carried into practice. He says, "For myself, I advisedly say that I would never remain insured in any office whose directors did not first do all that sound management indicates as necessary to secure equitable dealing as among the persons admitted to its ranks." Now, if a man in a country town were to have a conversation with an agent who was asking him to insure his life, and were to say, "Do your directors do all that sound management indicates as necessary to secure equitable dealing as among the persons admitted to the ranks?" the agent would of course answer in the affirmative. The agent knows nothing whatever about it; and how the proposer is to find out, it is difficult or impossible to say. He (the President) was an entire sceptic about the question of fraud amongst life insurance companies. As he had said before, it is far too slow a process. Of course frauds will exist as long as human nature exists; but as to frauds on life assurance companies bearing anything like the proportion of frauds on banks and other institutions where money can be obtained at once, he was very sceptical.

Mr. NEWBATT—I cannot see in the special recommendation of Mr. Walford's paper any adequate remedy for the indisputable evils which he points out. The cry for specially-trained examiners is very much like that which has been lately heard for specially-trained preachers—it seeks an unattainable, and even if attainable, a doubtful remedy. Have we not already experience of the very class of persons whom Mr. Walford seeks to create? Have we not all at head-quarters trained medical advisers, skilful, acute, conscientious men, of the very class that he has indicated, and yet they remain advisers—and advisers only—of their companies? I think that no board of directors would ever subordinate their own judgments to the judgment of their medical advisers, and erect them into the position of arbiters. Certainly no actuary who understands his business would do so. The bringing together of the facts, and the interpretation of them, is by no means necessarily a simple function to be performed by one brain. We want the facts, and I am content if a medical examiner will state the facts. Their interpretation, and the weight to be given to physical diagnosis and to the many other points which make up the problem which we have to solve, is a question which, in the ultimate result, must be left to be determined by the actuary and his board of directors, in what proportion it is not necessary now to enquire. I came here to-night, however, to express my sympathy with Mr. Walford in his effort to bring about some concord, some combination, amongst the offices, which will have the effect of remedying the many and undoubted evils which now and on a former occasion he has brought under our notice, and which I think with him are more rampant now than at any former period. I think that we have arrived at a crisis in the history of insurance, when every man who has formulated for himself clear views, should express those views frankly and openly. That crisis is being brought about, as all human affairs are brought about, by more than one cause, external and internal, and with that action and re-action which is always going on by a combination of both. Whether we like it or not, the fact is manifest to the observant that our business is no longer being carried on on the ancient lines. We no longer draw a large portion of our business, as we used to do, from local agents in country places, who added the representation of a life office as a make-weight to their proper business, but it is drawn every day more and more largely from men who make it their sole occupation, and whose very bread indeed depends on their success in it. This arises partly from natural causes, one of which, I think, is very obvious—the greater and greater concentration of our population in towns. I leave it to every gentleman to determine for himself what is the connection between the fact I have mentioned and those forms of wrong-doing which, on a former occasion, as now, Mr. Walford has very clearly placed before us. It is a fact that not only misdoings, which may rightly be termed frauds, but a general looseness and insincerity of statement characterise the dealings of the public with ourselves to an extent without previous parallel. Concealing previous proposals, pitting one office against another by making simultaneous proposals, falsifying every kind of fact—these, I am sorry to say, are becoming matters of every-day experience. If these things be so—and I venture to think that the fact is undeniable—what is the cause? The only answer which I can give to it, is that it lies mainly with ourselves. Competition is the root of this evil. It is not open, manly, honest competition, directed to worthy ends, which, though some may suffer temporarily by it, is fruitful and good; but the present competition is too often tainted by selfishness, and is a competition of a personal nature, seeking personal ends. I doubt whether we shall ever get an adequate remedy until directors of offices awake to the fact that insurance business, like every other form of business, must be weighed as well as counted, and that it is to the profitable issue of the business, and not alone to its magnitude, that they must look. In answer to a like assertion on a previous occasion, I have seen it asked, "What is there in this competition to which you object? Specify the nature of your complaint." To that question I answer, that every lowering of the standard for the purpose of attracting recruits from the ranks of another office, is a competition to be condemned; and if again I am asked, "How am I to judge of motive?" I say that internal evidence in every case enables us to determine it. If, for example, I see one office habitually taking at a lower rate cases surcharged by one of its rivals; if I see another stooping from its high position (of which it is sufficiently boastful) that it takes only "select lives", and accepting at inadequate rates cases which, by its own theory, ought to be declined; if I see systematic interference with the agents of other offices, by the offer, it may be, of higher remuneration, or by the easy acceptance of a proposal on the life of the agent to be enticed away; if I see one rate of commission proclaimed to be that which is paid, and I know that another rate is that which really is paid in secret; in every one of these cases I have a right to determine the motive from the fact, and to say that the motive in all alike is equally obvious—an unworthy desire to live at the expense of others. This leads me naturally to ask what is the remedy for this state of things? and I see none more hopeful than an association of managers, having, as the representatives and mouthpieces of their boards—as delegates indeed—the two-fold function of protecting the body corporate from outward attack, and in some measure controlling its internal operations. Now I should be very much misunderstood if I were thought to be suggesting that the body of insurance is in a corrupt state. Assuming that all those things of which I am complaining are done among us, are they not done thoughtlessly,—it may be, done under pressure, done by subordinates, done even at the worst in order that, as it was thought, a great good might be done at the expense of a little wrong? But these are precisely the classes of wrong-doing which would be remedied and removed if brought to the light, as an association of the nature which I am now advocating, would have the power to bring them to the light. We are all the better if working in the light. In these matters the servant cannot be better than his lord. Take an illustration from Mr. Walford's paper. If he be right in thinking that there is a good deal of laxity among the members of the medical profession—and I fear that there is more laxity than the president is inclined to admit—what is simpler than a process of reasoning like this: if the examiner finds that his recommendations to decline a life is disregarded, is he not likely to cease to make them? If he finds that the directors care little for family history, is it likely that he will make much of that point? All these things spread downward and outward, and therefore I ask you to begin at the beginning. Let us have an association that may in some measure control ourselves, that the weak and weary among us may be kept straight, and even the strong be strengthened. That I may not be accused of indulging merely in sentiment, let me suggest a practical use for such an association. Besides checking unfair competition, let it seek to remove the grounds of it. If all offices were alike good and alike uniform, there would be little room for fraud or unfair dealing. If we go a step further, and suppose, if the idea be not too startling, that all offices were open to accept, subject to some very rude test, every life which presented itself, it is very clear that the cause of fraud and the fraud itself, of which we have heard so much, but not too much, would alike vanish. That, I know, is at present a visionary idea—the world is not yet ripe for it; but is it too much to hope that we might to a certain extent approximate to it? Let the external conditions of our business be made to conform, leaving only the internal contrasts to be in future the bases of competition. Take one illustration. Has not the time come when we might combine to get rid of that potential cause of heartburning, the addition of extra premiums for foreign residence?

The PRESIDENT suggested to Mr. Newbatt that time was passing, and the question before them was that of local medical examiners, and that the question of foreign residence was rather wide of the subject of the evening.

Mr. M. N. ADLER—Our president has made a rather startling assertion, that he cares comparatively little for the effects of medical selection. I think there is a possibility of testing this point, and in this way. Ten years ago—I think it was in 1868—a State insurance scheme was, through the exertions of the Emperor of the French, introduced into France, with State guarantee, both for ordinary and accidental life assurance, which was open to the population in general. I think the amounts assured were not to exceed a thousand francs each person. It was found impossible to institute medical examinations and to have the system of friendly referees, and it was therefore waived altogether, and in lieu it was enacted that no claims should be paid if death happened within the first two years of assurance, but that, in that case, the premiums paid, with 4 per-cent interest, should be returned. I do not know how the scheme has succeeded, but I would ask Mr. Walford, who has paid some attention to provident institutions in France, whether any statistics as to this mode of insurance are available, because we might, in this way, find out whether medical selection is indeed such a great boon. To a certain extent, Mr. Hardy might throw some light on the subject, he having given very much attention to the sickness and mortality experience of friendly societies. It is a pretty general rule among friendly societies that no death payments or benefits are granted within the first year after the persons have entered, and it might be well worth finding out whether there is an excessive mortality attributable to this absence of selection. Mr. Walford suggests that the medical officers should be specialists, acting exclusively for insurance companies, accident companies, and friendly societies, and that they would gradually find the work would pay well. I very much doubt whether, if these officers had to attend to give weekly reports as to whether a member of a friendly society were sick or not, they would find the work profitable.

Mr. A. BADEN—The main defect which my experience shows as existing amongst medical examiners is one which I should hesitate to call incompetency, but which is certainly a defect which most of them acknowledge, and which it would be very greatly for the benefit of the world at large, and not only insurance societies, if they could manage to overcome. Medical examiners, for the most part, skilled or unskilled, honest or dishonest, utterly fail to detect signs of that complaint of which we now hear so much under the name of alcoholism, until it has attained stages when it becomes obvious to any individual passing the subject of it in the street. In many cases which have come within my own personal knowledge, I and other lay persons have had good reasons, from the appearance of the proposer, to imagine that he has taken a great deal more of potent liquid than was good for his constitution, but the doctor has unhesitatingly passed the life. He says, "The man's liver is in order, and of the normal size; his tongue is clear, his eye is clear, and I can find no fault with him", and he passes him; and unless it fortunately happens that, before the acceptance by the board, other enquiries are made, the office is saddled with the risk of many cases which certainly ought not to have been put upon the books.

Dr. OGLE would say a few words in reference to the somewhat heavy and sweeping charge Mr. Walford had made against the medical craft. Of course, in all professions, there are black sheep, and no doubt there were in former times more opportunities for collusion between the country agent and the local medical examiner than there are now. In London the medical referees of insurance societies have to scrutinize the results of examinations by local medical men, and it is really at the present time very seldom that in good offices we find anything to indicate collusion of a disreputable character such as that alluded to by Mr. Walford. What has been stated with regard to the training of medical men for the purpose of making insurance examinations, is of considerable importance. The practice connected with the assuring of lives is of quite a different nature from the ordinary treatment of disease, and it has a special character and interest of its own; and, as things are at present, it is only after a man has been many years connected with an insurance office as a medical officer, that he makes himself acquainted with the "ins and outs" of what is required at his hands. There are many biological facts, and many considerations connected with insurance practice.

which are quite independent of ordinary therapeutics, and quite apart from the every-day medical duties of private practice; and for this reason a special training and preparation of some kind would be advantageous. Regarding the connivance and collusion supposed to exist between agents and medical men, head offices ought to have, and retain in their own hands, the appointment of the local medical officers. The directors or the actuary should request their physician to advise as to making such appointments, and not leave it to the agents to make the selection. The office of insurance medical referee in a town is not unfrequently both a difficult and invidious one. Perhaps he has to report on the banker, or lawyer, or parson, or squire in the place. Each or all may be a friend. Or the examinee may be a private patient of his own. He may thus, without any blame, or without wilfully violating his trust to the office, become insensibly, but very naturally, biassed in the interest of the proposer, and may, in certain cases, lead at some future time to suspicion of collusion with the agent. [The PRESIDENT—He holds, in fact, a brief for him.] Even in London one may have to pass judgment, for insurance purposes, on persons known either as friends or as patients. In former times the applicants for insurance appeared before the board. Very often that would be a very useful and desirable thing, even now. A drinker may scrape his tongue, and may abstain for two or three days, and, so to say, steady himself, and he may go to the medical officer in a state of such preparation as entirely to deceive him; whereas the secretary or a member of the board may meet him at another time, and he may present a totally different condition. Even the hour of the day may make a difference, and sometimes persons select the hour of the day at which they come to a medical referee, in order to appear in a proper state for examination. It is very difficult, in many cases, to be so certain as to be able to state on paper the colour of the tongue, or of the cheeks, or the appearance of the eyes, as bearing on the question of whether the person is really under the influence of alcohol in some shape. The detection of signs of over-indulgence in alcoholic drinks, is one of the chief points that all medical examiners have to be on their guard about. That point, and the "family history", especially as to phthisis, are certainly two of the chief things to be looked to in the reports of local medical men. In connection with the question of "habits", there are other kinds of evil habits besides that of drunkenness, which demand the closest vigilance,—habits of a more intimate and private character, which cause the system to be gravely impaired, and the various tissues of the body seriously to degenerate. These, from their very nature, are apt to elude even the most searching scrutiny, and they constitute a class of evils which, in insurance practice, are very often not sufficiently considered. "Family history" is a most important subject of enquiry, and one which local examiners frequently seem entirely to undervalue, or even to ignore. This may arise from their attention having never been sufficiently called to it. Many offices have on their documents very inadequate allusion to family history; and in country places a medical man who has not been accustomed to conducting insurance examinations is guided solely and absolutely by the printed form. The family history should be brought out by inquiry, not only with regard to grandparents and to the father and mother, brothers and sisters, but also, when feasible, as respects uncles, aunts and cousins. Mr. Newbatt has said that medical officers should exist simply for the purpose of eliciting and communicating purely medical facts to the board, but it not seldom happens that the board and the actuary have to do much more than receive from the physician the dry record on paper of bare medical facts. Many cases occur in which great benefit arises from interviews between the medical and actuarial departments, and the consideration of the mutual influence of many subsidiary and collateral factors besides the purely scientific ones. Such colloquies, as every actuary knows, will frequently include points of great nicety and delicate speculation, and enable the physician to arrive at conclusions as a basis for recommendations to the board which may save the office a large amount of risk.

Mr. R. P. HARDY—I have found medical reports to be singularly candid. I could show a large number, written under very great difficulties, which place a full statement of the details of cases before the office in the most unreserved manner. I do not allow the agents to nominate the medical man, but invariably do that myself. I always choose a gentleman of education and position, neither too old nor too young, but one understanding the responsibilities of life; and I have found the plan successful. I dare say some examinations are made in the hurry of business, but I have not noticed that patent fraud which has been mentioned. It has always occurred to me that the offices have to some extent the remedy in their own hands, if they will only exercise a little wholesome disputing of fraudulent policies, and put down the evil. Of course, if you have not the courage to say you have been taken in, but submit to it, I would almost say you deserve what has happened. I know one institution that had a contempt for the opinions of the mob, and exposed a case in which a gang of persons had been plundering offices very generally. I think if that were done, we should not hear so much of these attempted frauds, which I certainly think do exist with regard to the tampering with proposals, but I do not believe they exist to any extent on the part of the medical profession.

Mr. CAPERN—With regard to the training of special medical examiners for insurance work I do not think that that is at all feasible or possible, as stated. The only way in which the idea could be dealt with would be to get the great medical schools to take up the questions of family history, &c., in the medical course, and so in the general training communicate to the future medical men a knowledge of this very important subject (an increasingly important one to insurance offices); and it would be well if some concerted action, such as Mr. Newbatt has recommended with regard to other points, were taken with regard to this. I think it would be a very good thing if offices were to make provision in their forms for the insertion of antecedent family history on both sides. It is now treated in a somewhat empirical fashion; on no real scientific basis, but as a matter of experience; and that experience is not possessed by everybody. Only some of the more highly educated men in London, and in some of the larger provincial towns, thoroughly understand the bearing of it. I have heard the remark made by more than one medical man, " Well, we will state the facts, and put down at what ages and of what diseases relatives have died; but the bearing of these facts on the longevity of the proposer we must leave to the principal medical officer." With regard to the selection of lives, I think the President must have been exceptionally fortunate in his experience. I believe that medical selection was never more essential than it is now. The temptation to introduce inferior lives into offices is increasing every day, and not the least item in that temptation is the exorbitant commission offered by some offices, which is a premium to agents to propose "shady" lives. I do not agree with Mr. Newbatt with regard to doctors. I do not think either actuaries or directors generally know sufficient of the correlation of diseases to enable them to dispense with the absolute guidance of a skilful physician. One point especially occurs to me, and that is the relation of rheumatism and heart disease, or heart disease as a consequence of rheumatism. It appears to me a very important point to deal with. I believe it would repay the labour if some of the gentlemen here would take out their experience of this matter, and see how many cases of heart disease have been the consequence of rheumatic fever. I believe it is a well-established fact that the earlier a person is attacked by acute rheumatism the more likely is the effect to be lasting. I think that is a point sometimes overlooked by medical officers. With regard to the character of the agents, there ought to be combined action. An unprincipled man, whose only object is to get all the commission he can, will go from one office to another which offers higher terms, until he gets what he thinks is right. If the offices would combine in this matter, and say, " We will not give any more than so much", and if an association, such as Mr. Newbatt suggested, existed, we should gather information from one another, and be able to detect some of these men in their nefarious transactions. [Mr. Baden—The sinners will not combine.] . Just one remark on alcoholism. I do not think examiners generally take sufficient notice of this point. The man to be avoided is the "soaker". I do not mean the man who gets drunk occasionally, and then has a long turn at abstinence, but the fellow who is always sipping. His liver becomes indurated, and he gets cirrhosis, whilst the other man may escape. Special attention ought to be directed towards this subject. It is not sufficient for you to say, " Are you a temperate man ? " There are different opinions respecting temperance. You may hear the remark, " The man is a temperate man", and if you ask " What do you call temperate"? the reply perhaps is, " The man does not get drunk". Put the question, " Suppose that early in the morning a man were to commence taking a 'nip', and to repeat the 'nip' at intervals during the day, and finish up by having a good 'nightcap' after supper, would you call that a temperate man"? " Oh, certainly, he would not be drunk". This is not a hypothetical case, and will serve to show what ideas some people have upon the question of temperance. I think medical officers do not, as a rule, ask proposers sufficiently minute questions on that point. Enquiry should be made as to how much a man drinks, and what he drinks ? A man who drinks spirits is a different man from the one who drinks only ale, and practical suggestions on the subject might be made to the examiners in our forms. Of course, if a man be sent to some first-class physician, you need not tell him what he is to do; but when you send a man, as you may be obliged sometimes, to some one who has just managed to scramble through college, it is very necessary that you should indicate what questions he is to ask, and what diseases to try to detect.

Mr. G. HUMPHREYS—Mr. Walford mentioned, near the end of the paper, in speaking of accidental insurance, that one in every twelve persons insured in the accidental companies meets with an accident. The proportion is so much that it would rather frighten one from being insured in an accidental company. [The PRESIDENT —I think that it is one in twelve who gets compensation for accidents. That makes a little distinction.] With regard to the means of providing medical officers with information, there is a book published by Dr. Guy and Dr. Ferrier, called "Forensic Medicine". It points out many questions of importance which medical men should ask, so as to bring out a proper diagnosis of the case.

Mr. CAPERN—Dr. Sieveking has written a very useful little book, entitled "The Medical Adviser and Life Assurance". It would be very handy to carry about, and very useful to medical officers in the country.

Dr. OGLE said he quite endorsed that opinion.

Mr WALFORD, in reply, said that although the President might fortunately not have an opportunity of knowing a great deal with respect to country agents and the local medical examiners connected with them, many there present could support the statement he had made as to facts within his own knowledge and experience day by day and week by week. He repeated that there never have been, at any time in the history of the country, so many fraudulent attempts, showing such marked ingenuity in many ways, as there have been lately with regard to life assurance companies. The accident companies have had bad turns lately, as they always will when trade is bad; but they have not fared very much worse than the life assurance companies; and this experience is not confined to one class of companies, but extends to companies of the highest and the humblest class. The chief medical officers of boards cannot do their duty really efficiently unless they get the data whereon to do it. If they got true returns of the actual state of the case, they would know how to advise their boards, and they would frustrate more frequently than they now do, the attempts to perpetrate frauds. The local medical examiners being careless, and being manipulated, as they constantly are, by the agents, manage to elude the vigilance of the chief medical officer. The interests of insurance are imperilled by the evils he had described, but by the concerted action of the offices fraud may be frustrated. If not stopped, it is going to have an influence upon the mortality of insured life; to increase the mortality, or decrease the expectancy of life in a very great degree; and to unfit for general use the mortality tables now relied upon. The experience tables have hitherto been based upon data selected from the best offices; but little reliance can be placed upon valuations made from such tables, when applied to offices which systematically lower the standard of selection. He proceeded: Mr. Adler spoke of State insurance in France. It happened that in the preparation of the scheme of the late Emperor Napoleon I had the honour of being consulted, and I took a deal of pains to dissuade those whom he entrusted with the organization of the scheme, from an attempt which they were making as to certain portions of it. They replied that the Emperor was determined. I pointed out that State insurance almost encouraged fraud, and that no one thought it a crime to injure the State, but many indeed almost thought it a virtue, and it would require that they should have very special machinery, and very special men to work such an enterprise. What was the fact? The life portion of the business was almost a failure. The accident insurance scheme showed, as I had predicted, the disastrous results of attempting to enforce working men to insure in a collective form. Anything worse than the result of the experiment, for the first two or three years, could not be conceived. Ultimately the whole thing had to be remodelled; and after ten or twelve years, the general accident assurance association of France has begun to pay. But there were tens of thousands of pounds lost at first by this unguarded experiment, and so it must be wherever the State sets up its machinery. It cannot compete in any way whatever with the machinery which depends upon the self-interest of companies, whether mutual or stock companies, where every officer holds his position subject to the office continuing to exist. In a State company unfortunately that does not prevail. There is one exception to the disastrous consequences to which I have spoken, namely the New Zealand scheme. That is getting on; but I am very anxious to see the result of a mortality investigation, because if the lives are not carefully selected, the more lives accepted the worse the results will be; and great numbers are going into it. Mr. Baden spoke about the question of intemperance. No doubt that is an increasing difficulty, and is a point to which medical men should have their attention directed. I am very much disposed to think, if it could happen that all who are going to insure in any office could be made to appear before a board of directors composed of men of experience, they would form a very good estimate of whether that man was a right life to take. It is very certain that, in the old times, when proposers appeared before the board, the effect, as a matter of selection, was very good. Mr. Hardy said if the offices would dispute the frauds, and not hush them up and make things pleasant in the way of compromise, they would do good. I know a case of a series of frauds committed not very long ago, and the board resolved to resist the payment of such claims as they matured, which some did very rapidly. Well, all the fraudulent policies cleared out of that company at a tremendous rate; and although it did the office some harm temporarily, I am sure it did no real harm. Concerted action is required. Nobody can have any doubt about that. If those who want to conduct their business properly would unite, the others would be obliged to follow; at all events, the offices who did join would know where the mischief is, and in the end the offices which submitted to the frauds would have to knock under. If the matter were taken hold of by thoughtful men, who have a determination to uphold insurance in its integrity, the present evils would speedily stop. The alcoholic question is one of the difficulties, but it is only one of them. The real frauds are concocted mostly in connection with the concealment of family history; and there are agents who go from one office to another, always concocting the same sort of frauds. They become skilled by constant practice in concealing the facts of family history, and especially the causes of death of parents, brothers, and sisters, putting in accidental causes, where consumption was the true cause. The local medical men need not be deceived; but when you find that they are so much in the interests of the agents as to attend his family professionally, free, there is evidence of combination going on, which must end in disaster. It is a conspiracy in a very simple form, but it is a conspiracy, the extent of which is never fully known. In such a case, the medical man will do almost anything the agent wishes him to do, and will go almost anywhere the agent wishes him to go. When I find agents taking proposers fifteen or twenty miles away from their residence into another town, where the local examiner will pass anyone that is brought to him, I know there is the intention of fraud. Whether the managers take up this question now, and try and stop it, or whether they go on disbelieving a good deal of it, the fact remains at the present moment—that frauds are going on all round; agents are being corrupted, and local medical examiners debased.

The reading of Mr. Walford's paper was followed by a correspondence in the *Insurance Record,* from which we have made the following extracts.

Dr. J. F. Arlidge, of Stoke-upon-Trent, writes :—

Local agents are too directly interested in the pecuniary results of their work, and may be supposed to press business with more immediate reference to their own gain than is compatible with the advantage of the office employing them. The more the agent is dependent upon the profits of his agency, the greater will be his temptation to push business—to promote assurances of doubtful character. This he may do, indeed, without intent to betray his trust.

Again, too much patronage and influence are lodged in agents' hands by the selection of medical examiners being delegated to them. If an agent wishes to push business, he must secure the co-operation of a medical practitioner easy in his examinations, and who amiably lends himself to the wishes of his patrons. On the other hand, if the agent finds himself associated with a medical colleague who will act independently, and carry out his examinations strictly, at the sacrifice of many proposals, he is apt to feel himself aggrieved, and to seek to replace the obnoxious examiner.

This is no fancy sketch, but a true picture drawn from experience, and one that can be confirmed by medical men at large who have had much to do with assurance agents and offices.

Further, in the matter of local agents, I doubt if the touting by professed agents produces the profitable results to offices that seem to be supposed. The inroads of agents upon their privacy and business hours are resented by many; and I am inclined to believe that the action of those already assured, who are gratified with the results of the business and the management of the offices they have entered, will go further in bringing new assurers than the operation of touting agents.

There is at the present time an interchange, to some extent, between offices, of information concerning rejected lives. This interchange should be more general and extended, and the scandal of lives rejected at one office being admitted as first-class, possibly, at another, be removed. That such an event can happen, proves that the machinery for regulating admission has been faulty, or else improperly worked in some part.

I would suggest inter-communication, not only in the case of lives rejected, but also in that of proposals withdrawn. Apart from improper motives for seeking an assurance, the withdrawal of a proposal will often imply some known weakness in the life proposed, which it is wished to conceal—a desire that cannot be realized in the first office applied to, but may be so in another.

There is, it seems to me, a want of care and discrimination in selecting agents, and deficient supervision of their transactions on the part of the central offices. From what I observe, there is great difficulty in getting efficient local agents; a competition for them between offices; and a great want of information, prior to appointment, regarding their moral and social character, and their fitness by education and manners for their duties.

Enterprising offices, whose aim is business first and sound business afterwards, are most free and liberal in the appointment of agents. They multiply them in the same locality without reference to the possible profits to be gathered by them; the only qualification or condition to earn the coveted post being an assurance effected by the so-called agent on his own life. Hence a multitude of incompetent, useless, dummy agents, may be found in every town and district. The consequence of this multiplication of so-called agents is the introduction into the assurance society practising it, of a body of assurers who pay an annual premium 5 per-cent less than they ought to pay and the less-favoured assurers do pay, in accordance with the published tables. In addition to this, the society is mulcted by the rewards allotted to their enterprising superintending agents who have achieved such success.

Medical men are often required to examine and to report on the lives of their own patients, and there are offices that have rather made a merit of this proceeding. In my humble opinion, it is a faulty one. The practitioner has too immediate an interest in the life of his patient to guarantee himself against undue bias in his favour, and to act in a strictly independent manner. For illustration sake, the medical attendant will probably be made aware how important it is to his patient and his family to effect an insurance, and compassion alone will be apt to colour his report, alike of family and of individual failings, and unfairly influence his estimate of the life's worth. Here, then, sympathy will operate even when fears of offending and losing a patient do not.

In fine, my experience convinces me that the full, faithful, and truthful discharge of the duties of a medical examiner for assurance of life, is fraught with great risk of losing patients, whenever that officer occupies also the position of their ordinary medical attendant.

On the other hand, it must be admitted that the medical attendant of an individual must or should know most of him, of his family and personal history, his habits and constitution, the nature, severity and effects of past illness, and like matters relating to his bodily integrity. Being thus the possessor of knowledge so important, his information is, or ought to be, of the greatest value to life assurance societies. Indeed, no information attainable by an office is, in all points, equal to that an observant, well-skilled, and conscientious medical attendant can supply. Unfortunately, as just noticed, it is marred by the relations subsisting between patient and medical attendant, and it would seem best to receive it as supplementary only to the report of a skilled medical examiner acting for the office and removed from all extraneous influence.

The well-nigh-universal plan now followed, is to put in the hands of the examining officer a string of questions requiring categorical answers, on points of family and personal history, on the habits and features of the proposer, and on the state of the several organs and functions of his body. Some offices, presumably distrustful of the ability of their referees to conduct an examination, have multiplied questions beyond measure, many of which are little calculated to bring out information possessing real value in reference to a decision respecting the eligibility of a life for assurance; whilst, on the other hand, they are vexatious to the examiner, whose only object is to draw up a faithful and sufficient report.

Moreover, it is common to find not a few questions imposed upon the medical referee which have been already replied to by the answer in his proposal and statement, and which, too, he only can answer. They are questions respecting which the word of the assurer must be accepted; the medical man can do nothing more than re-state the replies already given, at the cost of waste of time and useless trouble alike to assurers and examiner. Nevertheless the medical referee must have before him the complete statement, as made by the proposer, as well as the returns made by his friends acquainted with his history.

Instead of the paper of cut-and-dried questions answered categorically, I contend that, if an office have a medical referee in whom it has full confidence, in respect of his honour and medical skill, it will serve its own interest best by leaving him free to draw up a report in such words and at such length as shall fully satisfy his own mind that it conveys every particular necessary to the determination of the value of the life in question.

Medical men accustomed to clinical observations, and to make notes of cases for themselves in their hospital or private practice, want no such crutches as the offices supply, to enable them to make a report on the life proposed for assurance.

But I go further, and assert that the long-drawn-out questions, however cunningly devised and numerous, will fail to extract essential facts, and will, particularly by reason of the categorical replies to them —in practice usually consisting of "yes" and "no"—serve to conceal matters of importance. In short, all the answers taken together may fail to exhibit a life-like representation of the individual examined, of his weaknesses and diseases, and his prospects of survival. The series of "ayes" and "noes" that abound in the inquisitorial paper, can only give a confused idea of the life to the medical referees who, at the central office, have to pronounce upon it as eligible or not for assurance.

Yet the very general use of such papers of questions infers the conviction, on the part of the life offices, that something in the way of a guide to examination is necessary; particularly so as their numerous medical examiners differ widely in scientific ability and education, in tact and experience. Granting some such need, would it not be sufficiently met by the issue of a code of instructions, calling attention to those particulars in the vital history of individuals which are of most moment in relation to life assurance?

I have proposed that the report of the official medical examiner shall be supplemented by information derived from the medical attendant of the proposer, where such information is to be had. Here, again, instead of submitting to the practitioner a series of questions, I would advocate the sending to him a special paper of instructions or suggestions, leaving him to put his opinion and his knowledge of facts in such shape as may seem best to him.

The first of Mr. Walford's series of remedies is the institution of a special class of medical men wholly devoted to the duties of examiners for life assurance. I cannot concur in this project. Even if practicable, it would not work well. I hold that it would be a mistake to call into existence a class of specialists divorced from the teachings and experience of practice, and prone to view cases from a hard and technical stand-point, without the correcting influence of every-day contact with sickness and the sick. The wider his clinical experience has been, the better qualified is the practitioner for the duties of life assurance inquiries.

Again, I do not appreciate the value of the special course of lectures on life assurance Mr. Walford would have attended by those who would be examiners. Something indeed should be taught in the classes, of forensic medicine and public health, of the principles of life assurance and of the value of life at different ages, of the effects of occupations on the health, and other cognate matters; but an additional special course is assuredly not called for. Besides, medical students of the present day are sadly over-lectured. And, after all, it is not by the lectures endured that a competent life assurance examiner will be made, but rather by diligent clinical work, by sound training in each department of medicine, and by well-cultivated powers of observation, on the part of a man endowed by good sense, and possessing some knowledge of the world—as well in its bad as its good aspects. What is wanted are not youthful specialists, brimful of theoretical knowledge, but experienced physicians, who have learned what life is, and what health is, at the bedside, over a series of years.

If insurance companies can raise the position of their medical examiners, can create offices of sufficient emolument to induce medical men to seek them as important adjuncts to ordinary practice, or as compensatory for laying aside such practice altogether, they will compass the end Mr. Walford has in view, and attract the services of those possessed of special qualifications.

But I question whether medical men would surrender practice entirely to take up the duties of medical examiners; and I further question whether the life assurance offices of the country would make the post of sufficient value to induce practitioners to forego the prospects of practice. Here and there a retired medical man, or one ready to retire if a suitable opportunity occurred, might be found to occupy the special office in question; but assuredly the insurance societies would not like their choice of candidates limited to that class of practitioners.

To secure a class of independent referees, Mr. Walford proposes to limit their number. By so doing he also calculates upon making their appointments so valuable that they shall be interdicted from private practice in any form. He would make the largest towns of the country the centres of their work. What he would do in the case of small towns, and for large country districts, he does not say; but we must assume that he would constitute areas of considerable size, to be served by their special examiners. To carry out such a scheme, an element of expense is introduced in the shape of travelling costs, which would naturally fall on the assurance societies employing the examiner.

Unless Mr. Walford's scheme of large districts is supplemented by arrangements to take the medical examiners within easy reach of intending assurers, it would certainly fail to give that convenience to the public they are used to have and would, without doubt, still insist upon having.

Finally, I cannot concur in Mr. Walford's proposal—to forbid the holders of the appointments to engage in private practice, directly or indirectly. Certainly the regular and efficient performance of the duties of the posts he would institute, would be incompatible with extensive general practice. Yet I do not perceive that it would be so in respect of consulting practice of moderate dimensions. Besides, as remarked previously, it would be a prejudicial arrangement to assign the work of assurance examination to gentlemen having no clinical work, no continuous medical experience. What is wanted is, that assurance societies should take care not to appoint those not having time at their command for the regular and periodical discharge of the duties entrusted to them.

Mr. Sprague wrote:—

I know from my own experience that frauds are occasionally attempted upon life insurance companies; I believe, upon the testimony of others, that such attempts are not unfrequent; and I do not doubt that the state of things which Mr. Walford describes as existing between life office agents and medical examiners, has really come under his notice; but I altogether disbelieve that such a state of things is anything but exceptional, and I cannot help thinking that Mr. Walford's experience in the matter must have been remarkably unfortunate. Furthermore, I altogether disbelieve that the attempted frauds have anything whatever to do with the competition between life offices, except in so far as an undue eagerness to transact a large business, may sometimes lead the head-office officials of a life insurance company to pass the proposals submitted to them without a sufficiently careful scrutiny. Some people express themselves as if they thought it a direct advantage to an agent to pass off a bad life upon his company. They seem to forget that an agent receives only the same commission whether the life is good or bad, and that he receives it for a longer time in the case of a good life than in the case of a bad life. If a company is induced by misrepresentations and fraud to accept a bad life on inadequate terms, it is no advantage to the agent, but rather the contrary; for if the company finds that he has knowingly represented a bad life as eligible for insurance, he will certainly lose his agency and his future commissions. Notwithstanding all that has been said, therefore, I believe that the cases are very rare in which the agents of life offices knowingly attempt to deceive their offices as to the lives proposed for insurance. It may be that, in many cases, the agents do not sufficiently realize the importance of the trust that is reposed in them by their companies. They may reason to themselves that it is their business only to obtain the proposals for insurance, and that it is the business exclusively of the medical officer and the head-office authorities, to judge whether the lives proposed are eligible for insurance. This is certainly taking an inadequate view of the responsibility of an agent; but it may perhaps be considered as only carrying one step further the division of labour which is found so useful in all departments of life; and, at all events, it is a very different thing from attempting deliberate fraud upon a company.

Mr. Newbatt says that he does not object to open, manly, honest competition, directed to worthy ends, but the competition he objects to is "tainted with selfishness", is "of a personal nature, seeking personal ends". These, of course, are mere generalities; and, as such, difficult to deal with. We shall all agree that competition should be open, manly, and honest; but, for myself, I may say that I do not in the least understand the distinction Mr. Newbatt wishes to draw between competition directed to worthy ends and competition directed to personal ends; in fact, I have yet to learn that personal ends are necessarily unworthy. When Mr. Newbatt descends into particulars, he says that every lowering of the standard for the purpose of attracting recruits from the ranks of another office, is a competition to be condemned. There are two distinct questions involved in this sentence. The directors of an office may, for reasons satisfactory to themselves, resolve to reduce the rate of ordinary premiums, or the rate of extra premiums, for residence in certain foreign countries; but who can have a right to say that such lowering of the standard is made for the purpose of attracting recruits from the ranks of another office? Mr. Newbatt feels that there is a difficulty in judging as to the motives of another office, but talks vaguely of internal evidence enabling him to determine what is the motive. He says, that if he sees one office habitually taking at a lower rate cases surcharged by one of its rivals, he has a right to determine the motive from the fact, and to say that the motive is an unworthy desire to live at the expense of others. It seems to me that his inference is altogether unwarranted by the facts. If office A habitually takes at a lower rate cases surcharged by office B, the legitimate conclusion, until the contrary is proved, is that the authorities of A consider the terms demanded by B are habitually too high, although, of course, the authorities of B will be of a different opinion. If an office which professes to take only select lives, accepts at inadequate rates cases which by its own theory ought to be declined, it seems a very improbable conclusion that they have done this for the purpose of interfering with the business of a rival; and a more probable explanation appears to be, that the directors have committed an error of judgment. Here, again, however, it may quite easily happen that it is the critic who is wrong in supposing that the lives ought to have been declined. The possibility of the action of a company being mistaken in this way, suggests a serious doubt whether it is wise for a company to profess to take none but select lives. In the same way, the fact that it is possible for a charge to be brought against a company that one rate of commission is proclaimed to be that which is paid, while another rate is really paid in secret, suggests that it is unwise to make any proclamation whatever on the subject of commission, except to those who are personally interested in the matter, namely, the company's own agents and persons seeking its agency. I have long held the opinion very decidedly, that no outsider has the least right to know what rate of commission an office pays to its supporters; and if, as a matter of courtesy, a person is informed of the rates usually paid by an office, it is in the highest degree improper for him to use the information so given him, as the ground of a charge of bad faith or unfair competition against the company. There remains "systematic interference with the agents of other offices." As to this, I have nothing to say, except that I fully agree with Mr. Newbatt that such conduct is discreditable to any respectable office, and I regret that there are offices standing well with the public that are currently reported to be guilty of this conduct.

Mr. Newbatt's remedy for what he believes to be the evils of competition, is the formation of a combination among the life insurance offices—in other words, of a strict trade-union; and his arguments are exactly those which are used by all conscientious believers in the efficacy of trade-unions. For myself, I do not believe that any good result will follow from the establishment of his suggested combination; and, if it were to regulate the terms upon which competition is to be conducted between different offices, I am satisfied that it would be greatly to the injury of the insuring public. He proposes that the external conditions of life insurance business should be made to conform, leaving only the internal contrasts to be in future the bases of competition. Which are external and which internal conditions, is not explained; whether, for instance, all the offices belonging to the combination shall be taken bound to charge the public the same rate of premium and to allow the same rate of commission to their agents. He suggests, however, the abolition of extra premiums for foreign residence, as the subject which might first be taken up by his proposed combination. I have not thought sufficiently as to this particular proposal to be in a position to express a positive opinion on it; but, so far as I can see at present, I should decidedly object to it in the case of my own company, on the ground that we should lose a certain amount of income without gaining any corresponding advantage. It seems to me, however, an altogether unsound principle, that no new feature should be introduced into life assurance practice until the whole of the offices are agreed upon its propriety. We have during recent years had a great number of experiments made by different offices with varying success. We have had one new proprietary company started on the principle of paying no commission whatever, and we have found that, after a few years' trial, this principle has been abandoned. We have seen a mutual life office adopt the principle of no commission, and abandon it again after a few years' trial. We have seen a company, with a paid-up capital exceeding £250,000, try the experiment of insuring lives resident in India and other tropical climates at home rates. This experiment, like the previous ones, was found unprofitable, and abandoned, although I believe it has had the effect of reducing the rates of extra premium charged by other companies for residence in India. As an example of a recent successful experiment, I may mention the rule which was adopted in the first instance by a single company, then, after an interval of several years, by a few, but which has now become very general—namely, that a policy of five years' standing, when the life has attained the age of 30 and has not incurred extra risk, shall be altogether freed from liability to extra premium. One office, at least, has gone further in the same direction, and its regulations provide that there shall be no liability for extra premium in any case, even when extra risk has been incurred and the profession of the life assured renders it certain (or very probable) that extra risk will again be incurred. It remains to be seen whether this experiment will be successful. But my position is that it is for the benefit of the public generally, and therefore, in the long run, for the benefit of the offices, that experiments of this kind should be freely tried by individual offices. If the experiment is found unsuccessful, it is better that one or two offices should have to confess the mistake and retrace their course, rather than the whole body of offices; and if the experiment succeeds, it is right that the office which has the courage and sagacity to make it in the first instance, should get the advantage of it. Consistently with these views I should say, if Mr. Newbatt has satisfied himself that the time has come to abandon extra premiums, except in the case of those persons whose circumstances at the date of the insurance render it more than usually probable that they will incur risk, then his proper course is to urge upon the directors of his own company to act upon his views.

Not only do I think it undesirable in the interest of the public that any attempt should be made to restrict competition between life offices, but I am also strongly of opinion that any such attempt, if made, is certain to prove unsuccessful. Ordinary trade-unions are not able to enforce their decrees upon unwilling members of the trade, without having recourse to crimes of violence and other illegal measures; and there would be no effective method by which a life insurance trade-union could coerce unwilling members of their trade. This opinion is not based upon merely theoretical grounds, but is supported by all that I have been able to learn as to the operations of the Tariff Fire Offices of the United Kingdom, and of the Association of Managers of the Scottish Life Offices. I quite admit that it is very advantageous for the representatives of offices to meet habitually for consultation upon subjects of common interest, also to have an organization that may be used without delay when occasion arises. Such combinations are no doubt very useful for certain purposes, but the regulation of competition between the different offices is an object which, in my opinion, they cannot aim at, with any prospect of benefiting either the public or themselves.

Mr. G. S. Crisford wrote:—

I think that Mr. Sprague, while clearly pointing out the difficulty of dealing practically with the subject of competition, somewhat passes over that which I believe is clear to a great many minds, namely "That there are existing evils mixed up with the present system of life competition, which it would be very desirable to remedy."

I agree with him that competition is good. A monopoly in any branch of trade or commerce is bad in principle and unsatisfactory in results; and nothing, perhaps, is so likely to encourage enterprise, to stimulate business, and to ensure the best results, as free and open competition.

Competition causes companies to vie with each other in offering the greatest advantages to those who are invited to have dealings with them. It results in policies being granted generally under more liberal conditions than formerly. It tends to remedy existing evils, to encourage liberality, to promote investigation, to stimulate individual efforts, to engender new theories, and generally tends to induce companies to offer the best possible return in exchange for the premium paid, and to make the conditions under which the policies may be issued, as liberal as possible while compatible with safety.

Competition also results in the circulation of a great amount of general information, which is useful to the public and (if rightly used) will throw a great deal of light on the question of life assurance, and also dispel many erroneous impressions existing upon the subject.

Competition also does much for the companies themselves; it tends to show what system of working produces the best results, and what special features prove most attractive to the public; it encourages a deeper study of life assurance by those engaged in the profession; and, as a natural result, new light is constantly being thrown upon it.

But there are evils resulting from the present system of competition which deserve very serious consideration:—the danger of putting forward the features of the companies in such attractive guise as shall be too highly coloured or even misleading to the uninitiated; and the temptation to pick out some one or two seeming or real advantages, while suppressing facts or figures which might tell against the company: the encouragement of a high rate of expenditure, and the payment of very heavy commissions, in order to bid for agents and business, and the granting of agencies to anybody and everybody: the consequent depreciation of the value of life assurance in the eyes of those who do not understand its principles, and the creation of many false impressions upon the minds of a very large section of the community. I think these evils are very real ones, and a serious check to the free development of a science which is even yet not half appreciated as it deserves to be.

When we consider the large number of offices which are sound and good, it is quite unnecessary that misleading statements should be put forward, either in prospectuses or by agents; and whenever competition becomes a mere scrambling for business, it entirely exceeds its legitimate bounds, and, instead of being helpful, is injurious to the cause of life insurance.

The Life Assurance Companies' Act has done an immense amount of good, in checking the tendency to publish misleading statements; and it compels the publication of those figures and particulars which are the real essentials by which the true position of an office may be estimated. At the same time, I am bound to say there are still certain ways of putting forward seeming advantages, and even of quoting Government returns, which are very objectionable, because they tend to mislead and often direct the attention of the enquirer to unimportant facts and apparent benefits, which will not bear the test of investigation by those who are thoroughly acquainted with the subject. The better education of the public on insurance matters, and the free circulation of the full Board of Trade returns will, in my belief, be the best check against these evils just pointed out.

Heavy expenditure is another evil result of competition; and this, I fear, is on the increase. There has been much discussion as to how much may with safety be spent in management; and many opinions are put forward, that offices which expect to transact a large business, and which intend to hold their position in competition with their compeers, must pay for so doing by an increase in their expenditure. To a limited extent only I believe this to be true.

It is very easy to increase the expenditure; but only so far as such increase secures additional efficiency in the management and working, will it be really profitable for obtaining additional business. Connected with this subject is the practice of giving large and promiscuous commission allowances. I believe this is not one of the least evils of the present extensive competition among life offices. Large commissions never yet made good agents, and I believe never will; and there are evil results attending them which I think are worthy of notice. They tend to induce agents to select the office which is the highest bidder, and to consider only how much may be pocketed by a transaction of business.

They encourage an agent to think more of getting cases passed, than of the interest of the office and of the careful selection of lives. They tempt men to apply for agencies, who have no real intention of doing a continuous business, but who do so for the sake of the commission on a single proposal, perhaps on their own assurance. They offer a temptation to an inferior class of men, and encourage a laxity in the morality of some agents, in setting forth the claims of the company they represent, and also in rendering to the office the particulars of the proposals which they secure.

The usual rates of commission granted by the leading companies, in my opinion, afford ample remuneration to any ordinary agent.

Promiscuous commission giving is also an evil which very much interferes with the legitimate competition for business. It is unfair to the industrious agent, and it results in offices having on their books a very large number of unsatisfactory accounts, and on their list of agents a number of utterly useless names, as far as any continuous introduction of business is concerned.

With regard to the depreciation of the value of life assurance in the minds of a certain section of the public, this will continue to be the case whenever competition exceeds its proper limit and becomes a mere scramble after proposals. For, however great the benefits of life assurance may be to individuals, these benefits will be undervalued, and the advantages discredited, if there be too great an anxiety apparent to secure proposals by the different companies. The public should be educated rather to seek the agent, and to enquire after an office, than to run away from insurance agents for fear of being dragged into taking out a life policy.

Competition some thirty years ago encouraged a reckless establishment of companies, there being at one time an idea that the number of companies would not suffice until every town of importance in the Kingdom had its own institution for the assurance of life. In vain were arguments brought forward to show that institutions of the kind could hope for a permanent success only when established on a very large scale. The creation of new offices still continued till the main occupation of many consisted in a futile struggle to make themselves more conspicuous than their neighbours. Between 1851 and 1861 something like 150 companies disappeared; since then, others have followed. Since the passing of the Life Assurance Companies' Act, 1870, however, very few companies have been established; and it is probable that very few fresh ones will be started in the future.

Competition, therefore, at present chiefly tends, not to the creation of fresh companies, but to modify or extend the mode of internal administration of the existing offices; and it is a different class of danger ahead we have to look out for, to that when the establishment of unsound and doubtful companies was an easy matter. Heavy expenditure, misrepresentation and fraud, sum up the danger ahead. While I freely admit companies have a perfect right to pay what they choose for business, and to compete how they please for it; and while I would be one of the last to limit the free and independent action of any office, I do strongly advocate any united effort to suppress that which is unsound, and to battle against that which is damaging to all alike.

Notes on the Practical Application of Mr. Makeham's Formula to the Graduation of Mortality Tables. By GEORGE KING, *of the Alliance Assurance Company,* F.I.A., *and* GEORGE F. HARDY, *of the British Empire Mutual Assurance Company,* A.I.A.

[Read before the Institute, 26 January 1880.]

THIS paper had its origin a year or two ago in an investigation entered upon with an object which has since fallen into abeyance. At that time, it being desired to obtain a thoroughly well-graduated table of mortality giving a favourable view of human life, attention was directed to the Carlisle Table, and sundry attempts were made to remove its irregularities.

Many of the experiments then tried would not now be of any public interest, but the effort to use Mr. Makeham's formula brought out a few facts which it is perhaps worth while to record; and, at any rate, whatever may be thought of the present notes, the authors do not consider themselves to be guilty of any impropriety in bringing forward for discussion once more the practical employment of Mr. Makeham's expression for the law of human mortality.

They are the more emboldened to suggest this subject for debate, in that Mr. Walford, at a late meeting, announced that the formula was found in America to give perfectly satisfactory results when applied to a table formed from the experience of assured lives on that continent.

The first step in the original investigation was to graduate the Carlisle Table (as published by Mr. Milne) by means of Mr. Woolhouse's formula; and as the resulting curve was not entirely free from roughness, the process was repeated. The changes introduced by the second graduation were insignificant, but the table produced was one of very great smoothness, although there was a wave-like form observable in the succession of the values of some of the functions. Perhaps $\Delta \log p_x$ is one of the best functions by which to investigate the progression of a table, and it is not a little remarkable that in the Carlisle Table, with its roughnesses removed, this function has well-defined periodically-recurring maxima and minima. We cannot but think that this is due to the graphic method of graduation employed by Mr. Milne, although of course it does not follow that in all cases the graphic method would produce similar disturbances.

It will be remembered that Mr. Milne, instead of graduating the ratio of deaths to population in the several intervals of age, adjusted by the graphic method the deaths and the population independently, and then proceeded—having inserted ordinates to each curve for yearly intervals of age, instead of decennial as in the original facts—to form the ratio of deaths to population for each year of life, which has been symbolized by Dr. Farr by m_x.* Without any further graduation, seemingly, the column l_x was then formed. Now, bearing in mind that after age 20, and below age 100, the population and deaths on which Mr. Milne constructed his table were distributed into decennial periods, it is not a little suggestive that in the final table $\Delta \log p_x$ exhibits a maximum in every decade of life except between ages 50–60, and 80–90.

We would here remark, that Mr. Woolhouse's formula appears to us to afford an excellent instrument for analyzing curves of mortality. It removes the accidental irregularities which conceal the actual character of the table, without cutting deeply enough to impair in any way that character. In fact, annuity values based upon the Carlisle Table, when it has been subjected to a double process of graduation by Mr. Woolhouse's method, do not differ appreciably from those published in Mr. Milne's work.

Mr. Woolhouse's formula, however, does not give satisfactory results in this case at the young ages. Below age 17 the graduated values of l_x are all too small, and the defect rapidly increases, so that below age 15 the method of adjustment must be abandoned.

In Table A we have given in detail the figures connected with Mr. Woolhouse's graduation, although they have no direct bearing on our present subject. The Carlisle Table is of historic interest, and everything relating to it is worthy of the most careful attention.

---

* It will be remembered that Dr. Farr symbolizes by m_x the ratio between the recorded deaths in the year of age x to $(x+1)$, and the enumerated population in the same year of life. On the assumption that in the life table this ratio will be equal to $d_x \div l_{x+\frac{1}{2}}$, he then forms the column l_x. The ratio m_x is approximately equal to $\mu_{x+\frac{1}{2}}$.

TABLE A.—*Carlisle Table graduated by Mr. Woolhouse's Formula.*

Age.	Original	1st Graduation	2nd Graduation		Age.	Original	1st Graduation	2nd Graduation	
	l_x	l_x	l_x	$-\Delta \log p_x$		l_x	l_x	l_x	$-\Delta \log p_x$
10	6460	6448·8	6439·2	+ 1	59	3749	3740·3	3735·9	+131
11	6431	6421·8	6413·7	17	60	3643	3634·3	3630·5	122
12	6400	6394·6	6388·2	22	61	3521	3519·7	3517·5	107
13	6368	6365·0	6360·3	23	62	3395	3398·5	3398·4	89
14	6335	6332·6	6329·4	22	63	3268	3273·3	3275·2	72
15	6300	6297·2	6295·2	20	64	3143	3146·8	3150·1	63
16	6261	6259·1	6258·0	13	65	3018	3020·8	3024·7	66
17	6219	6218·7	6218·3	9	66	2894	2896·3	2900·1	85
18	6176	6176·7	6176·9	5	67	2771	2773·2	2776·4	118
19	6133	6133·9	6134·5	2	68	2648	2651·0	2652·8	164
20	6090	6090·9	6091·7	− 1	69	2525	2528·0	2527·8	220
21	6047	6048·0	6049·0	+ 1	70	2401	2402·2	2399·6	277
22	6005	6005·6	6006·6	1	71	2277	2271·1	2266·4	327
23	5963	5963·7	5964·5	7	72	2143	2132·9	2127·0	362
24	5921	5921·9	5922·4	11	73	1997	1986·8	1981·2	376
25	5879	5879·8	5879·7	15	74	1841	1833·9	1830·1	376
26	5836	5836·7	5835·9	19	75	1675	1676·9	1675·9	360
27	5793	5791·8	5790·4	22	76	1515	1519·7	1521·5	336
28	5748	5744·3	5742·7	20	77	1359	1366·3	1369·9	314
29	5698	5694·1	5692·6	17	78	1213	1219·8	1223·9	304
30	5642	5641·2	5640·3	12	79	1081	1082·2	1085·6	322
31	5585	5586·3	5586·3	10	80	953	954·0	956·2	359
32	5528	5530·3	5531·2	4	81	837	835·3	836·0	414
33	5472	5474·1	5475·5	6	82	725	725·1	724·9	493
34	5417	5418·2	5419·7	7	83	623	623·2	622·6	571
35	5362	5362·7	5363·9	11	84	529	529·3	528·7	655
36	5307	5307·3	5307·7	15	85	445	443·5	443·1	737
37	5251	5251·3	5250·9	22	86	367	365·6	365·8	815
38	5194	5193·9	5192·8	27	87	296	296·3	296·9	858
39	5136	5134·3	5132·7	30	88	232	235·6	236·5	928
40	5075	5072·0	5070·2	27	89	181	183·6	184·7	953
41	5009	5006·9	5005·1	24	90	142	139·9	141·2	819
42	4940	4939·1	4937·7	16	91	105	104·8	105·6	675
43	4869	4869·3	4868·4	10	92	75	77·0	77·5	394
44	4798	4798·5	4798·4	0	93	54	55·6	56·0	− 130
45	4727	4727·7	4728·4	− 4	94	40	40·1	40·1	− 658
46	4657	4657·9	4659·3	− 8	95	30	29·3	28·8	−1085
47	4588	4590·0	4591·7	− 8	96	23	21·8	21·0	−1321
48	4521	4524·3	4525·9	− 3	97	18	16·7	15·7	− 804
49	4458	4460·6	4461·8	+ 2	98	14	13·2	12·1	− 241
50	4397	4398·3	4399·0	10	99	11	10·6	9·5	+ 897
51	4338	4336·5	4336·8	21	100	9	8·4	7·5	835
52	4276	4274·2	4274·5	34	101	7	6·4	5·8	4635
53	4211	4210·5	4211·1	50	102	5	4·6	4·4	8204
54	4143	4144·7	4145·4	64	103	3	3·0	3·0	8038
55	4073	4075·8	4076·1	88	104	1	1·7	1·9	11919
56	4000	4002·6	4001·9	105	105	...	·8	1·0	20412
57	3924	3923·5	3921·2	122	106	...	·2	·4	...
58	3842	3836·6	3832·8	131	107	...	...	·1	...

To the table as adjusted by Mr. Woolhouse's formula we proceeded to apply Mr. Makeham's. Four sets of constants were tried deduced from the values of l_x at ages 15, 35, 55, 75; 20, 40, 60, 80; 25, 45, 65, 85; and 30, 50, 70, 90. The values of these constants were as follows:—

	First Curve.	Second Curve.	Third Curve.	Fourth Curve.
$\log q$	·0364852	·0411317	·0404469	·0397194
$\log g$	−·0007393	−·0003083	−·0003272	−·0003654
$\log s$	−·0029073	−·0034071	−·0038187	−·0039126
$\log k$	4·8452254	4·8549296	4·8681803	4·8743597

The deviations from the Carlisle Table, adjusted by Woolhouse's formula (for the present purposes the ungraduated table), are given in the following Table B:—

Table B.—*Carlisle Mortality—Deviations in the Values of l_x.*

Age.	Adjusted by Woolhouse's Formula l_x	First Curve. Ages used, 15, 35, 55, 75.		Second Curve. Ages used, 20, 40, 60, 80.		Third Curve. Ages used, 25, 45, 65, 85.		Fourth Curve. Ages used, 30, 50, 70, 90.	
		Excess.	Defect.	Excess.	Defect.	Excess.	Defect.	Excess.	Defect.
15	6295·2	...	...	51·5	...	155·1	...	224·5	...
20	6091·7	...	22·8	...	...	70·0	...	128·9	...
25	5879·7	..	38·5	..	39·1	...	...	48·9	...
30	5640·3	...	32·1	...	50·1	...	39·2	...	...
35	5363·9	...	...	...	28·1	...	42·7	...	12·7
40	5070·2	30·6	...	...	...	...	36·8	...	15·7
45	4728·4	79·9	...	55·0	...	...	...	12·9	...
50	4399·0	73·3	...	62·2	...	...	5·6	...	...
55	4076·1	...	...	8·5	...	...	64·9	...	64·8
60	3630·5	...	28·7	...	...	...	69·0	...	71·6
65	3024·7	11·5	...	52·0	...	...	...	...	0·9
70	2399·6	...	17·3	15·8	...	...	6·3	...	...
75	1675·9	...	...	1·2	...	15·8	...	33·3	...
80	956·2	42·4	...	...	...	42·9	...	70·4	...
85	443·1	19·0	...	...	46·0	...	...	27·9	...
90	141·2	4·3	...	...	42·0	...	16·6	...	...
95	28·8	...	3·3	...	17·8	...	11·9	...	7·2
100	7·5	...	5·7	...	7·2	...	6·8	...	6·4

The next effort was so to combine these four curves as to produce satisfactory final results, and we were led to investigate the different methods which have been employed by others with a like object. Mr. Sutton, in his paper, *J.I.A.*, vol. xx, p. 170, takes, without further adjustment, as the final value of $\log l_x$, the arithmetical mean of the values derived from the several sets of constants; and if the production of a well-graduated mortality table be the only object in view, no objection can be urged to this method.

When, however, it is remembered that the great advantage of Mr. Makeham's formula is the facility it affords for calculating the values of Joint Life Annuities, we think it is to be regretted if that advantage be thrown away; and we therefore prefer to seek another method of preparing the finally adjusted table, wherein that advantage can be retained.

Mr. Woolhouse, in his most useful paper, vol. xv, p. 389, had previously suggested a plan which, in the cases of all the tables with which he dealt, gave most excellent results. By the inspection of the columns l_x, formed from two different sets of constants, he drew a curve lying between them; or, practically, he calculated l_x by a process analogous to taking the arithmetical mean of the several values derived from the distinct sets of constants. But Mr. Woolhouse went further, and having, as above described, formed an idea of what his final curve should be, he combined the logarithms of his constants in the same proportions as he had combined the values of l_x, and from these adjusted constants he prepared his final mortality table. No doubt, seeing that, in the cases he had in hand, the method he adopted left nothing to be desired, he did not think it worth while to push his analysis further.* It is not, however, difficult to show, that to take the arithmetical mean, or any other proportion of the logarithms of distinct sets of Mr. Makeham's constants, is theoretically incorrect. The proof is as follows:—

If q_1, q_2, g_1, g_2, k_1, k_2, s_1, s_2, be the constants derived from two independent applications of Mr. Makeham's formula to the original table, and if q, g, k, s, be the final constants, and if l_x be the finally adjusted value, and be the geometric mean of the values derived from the preliminary sets of constants, we have

---

* When the paper was read at the Institute, we were under the impression conveyed in the text; but in the discussion, Mr. Sutton pointed out that we had done Mr. Woolhouse an injustice. The final table was not formed by assuming a mean of constants, but by assuming a mean for $\log q$, $\log l_{10}$, $\Delta \log l_{10}$, and $\Delta^2 \log l_{10}$; and by calculating in the usual way the final constants from these values. Mr. Woolhouse, unfortunately, has not described his process very lucidly, and others beside ourselves have misunderstood him. It is almost a natural inference from his explanation, that he took a mean of constants; and if a reader come to that conclusion, and test it, he will find his opinion confirmed, because two of the four final constants of Mr. Woolhouse are exactly means of the preliminary constants, and the other two differ from the mean by only 3 in the seventh decimal place.

$$\log l_x = \tfrac{1}{2}(\log k_1 + \log k_2) + \frac{x}{2}(\log s_1 + \log s_2) + \tfrac{1}{2}(q_1^x \log g_1 + q_2^x \log g_2)$$

$$= \log k + x \log s + \tfrac{1}{2}(q_1^x \log g_1 + q_2^x \log g_2)$$

$$\Delta \log l_x = \log s + \tfrac{1}{2}\{(q_1^x(q_1-1)\log g_1 + q_2^x(q_2-1)\log g_2\}$$

$$\left.\begin{array}{l} \Delta^2 \log l_x = \tfrac{1}{2}\{q_1^x(q_1-1)^2 \log g_1 + q_2^x(q_2-1)^2 \log g_2\} \\ \text{also} \\ \Delta^2 \log l_x = q^x(q-1)^2 \log g \end{array}\right\} \quad . \; . \; . \; . \; . \quad (1)$$

From which it follows that unless $q_1 = q_2 = q$, $\log g$ cannot be equal to $\frac{1}{2}(\log g_1 + \log g_2)$.

It therefore appears, that to take the arithmetical mean of the logarithms for each of the four constants is not justifiable unless $q_1 = q_2$. In the Institute Experience Tables, the different values of q brought out are so nearly alike as to be practically identical; but in other cases a very considerable error may arise. We are justified in writing $\log k = \frac{1}{2}(\log k_1 + \log k_2)$ and $\log s = \frac{1}{2}(\log s_1 + \log s_2)$, but to combine q_1 with q_2, and g_1 with g_2, we must proceed differently. Suppose that, instead of having only two curves, we have four that we wish to combine. Then

$$\log k = \tfrac{1}{4}(\log k_1 + \log k_2 + \log k_3 + \log k_4)$$

$$\log s = \tfrac{1}{4}(\log s_1 + \log s_2 + \log s_3 + \log s_4).$$

If l_x denote the fourth root of the product of the four values of l derived from the four curves, we have

$$\log l_x = \log k + x \log s + \tfrac{1}{4}(q_1^x \log g_1 + q_2^x \log g_2 + q_3^x \log g_3 + q_4^x \log g_4).$$

Also, if we take this value as the final value of the numbers living, we must have

$$\log l_x = \log k + x \log s + q^x \log g\,;$$

whence

$$q^x \log g = \tfrac{1}{4}(q_1^x \log g_1 + q_2^x \log g_2 + q_3^x \log g_3 + q_4^x \log g_4) \quad . \; . \quad (2)$$

In the same way, we can form an equation for any other age $x+t$, and from the two equations calculate the values of the two unknown quantities q and g. It is desirable so to choose the ages x and $x+t$ as to have the value of l_x in each case as near as possible to the original ungraduated value. This mode of combining the constants may be briefly called the "combined method." By it a fifth table for the Carlisle Table was formed, using ages 25 and 85 to find g and q, with very satisfactory results.

The deviations of l_x are given in Table C, and are compared with the deviations of the curve styled the "Faulty Curve", formed by merely taking an arithmetical mean for the logarithm of each constant. This latter curve is seen to be very defective indeed. The process which could be successfully applied to the Institute Tables does not suit the Carlisle.

TABLE C.—*Carlisle Mortality—Deviations in the Values of l_x.*

Age.	Adjusted by Woolhouse's Formula l_x	COMBINED CURVE.		FAULTY CURVE.	
		Excess.	Defect.	Excess.	Defect.
15	6295·2	107·3	...	106·6	...
20	6091·7	43·8	...	42·7	...
25	5879·7	...	7·2	...	9·3
30	5640·3	...	30·6	...	33·8
35	5363·9	...	21·6	...	26·6
40	5070·2	...	6·7	...	14·7
45	4728·4	34·7	...	22·1	...
50	4399·0	29·0	...	9·4	...
55	4076·1	...	35·3	...	65·0
60	3630·5	...	49·0	...	92·6
65	3024·7	7·7	...	...	53·0
70	2399·6	...	10·2	...	88·5
75	1675·9	5·9	...	...	83·9
80	956·2	35·2	...	...	50·3
85	443·1	...	0·7	...	61·5
90	141·2	...	13·7	...	41·1
95	28·8	...	10·3	...	16·3
100	7·5	...	6·6	...	7·0

The constants from which the above curves were formed are as follows:—

	"Combined" Curve.	"Faulty" Curve.
$\log q =$	·0392061	·0394458
$\log g =$	−·0004262	−·0004351
$\log s =$	−·0035114	−·0035114
$\log k =$	4·8606738	4·8606738

An excellent example of the point now before us, and one showing with greater emphasis the error which, in some cases, will result from adopting, as the final values of the logarithms of the constants, the arithmetical mean of the values found independently, is afforded in Mr. Sorley's short paper, *J.I.A.*, vol. xx, p. 340. Mr. Sorley adjusted a mortality table by three independent sets of constants, obtained from the numbers living at ages 20, 40, 60, 80; 30, 50, 70, 90; 40, 60, 80, 100; and in the following Table D are the resulting deviations.

TABLE D.—*Mr. Sorley's Table—Deviations in the Values of* l_x.

Age.	Unadjusted l_x	First Curve. Ages, 20, 40, 60, 80.		Second Curve. Ages, 30, 50, 70, 90.		Third Curve. Ages, 40, 60, 80, 100.	
		Excess.	Defect.	Excess.	Defect.	Excess.	Defect.
30	100,000	168	...	...	...	309	...
40	91,315	...	...	1,190	...	...	...
50	83,030	...	1,352	...	...	...	1,408
60	68,981	...	...	617	...	...	...
70	49,953	...	967	...	...	...	866
80	20,613	...	...	4,601	...	...	...
90	5,726	...	3,940	...	...	...	4,009
100	0	...	...	213	...	...	...

Mr. Sorley then went on to say, speaking of curves (2) and (3) of Table D:—"The consideration of these differences led to the conclusion that a nearer approximation to the truth would be found by taking something like a mean of the two tables. In order to attain this object, therefore, without sacrificing the property of uniform seniority, a mean of constants was taken, and 3"—namely, the second curve of Table E of the present paper—"constructed therefrom. In justification of this course may be quoted the authority of Mr. Woolhouse, who, in graduating the H^{MF} table by Gompertz's formula, adopted a similar process with entire success. Having first constructed two adjusted life-lines from independently ascertained constants, he next made a note of their differences from the ungraduated numbers. Then, noticing from these differences that a new table, giving the living at each age $x = \frac{2}{3} l_x$ of the first curve $+ \frac{1}{3} l_x$ of the second curve, would be a better representation of the facts, he simply took a set of constants in the same proportions respectively of the old sets, and from it produced a table standing to the two others in almost exactly the relation desired."

Acting on this theory, Mr. Sorley made two attempts to form an average table, taking the mean of the constants of curves (1) and (2) and curves (2) and (3) of Table D respectively.

The resulting deviations, given in Table E, could not easily be worse, and regarding them Mr. Sorley remarked that "although such a close agreement was hardly expected in the present case, still it occasioned considerable surprise to find curve (3) falling so wide of the mark, as from the table it will be seen to do. The remaining curves proving equally unsuccessful, made it apparent that the formula would not apply to the case in hand."

TABLE E.—*Mr. Sorley's Table formed from* "mean" *of Constants—Deviations in the Values of* l_x.

Age.	Mean of Constants of 1st and 2nd Curves.		Mean of Constants of 2nd and 3rd Curves.	
	Excess.	Defect.	Excess.	Defect.
30	...	336	...	304
40	...	624	...	742
50	...	3,865	...	4,222
60	...	7,005	...	7,790
70	...	13,461	...	14,769
80	...	10,418	...	11,454
90	...	5,321	...	5,431
100	...	0	...	0

These wild figures are not difficult to account for when we consider the values of q on which they depend.

Curve No. 2 of Table D is combined successively with curves Nos. 1 and 3. Now the value of log q in curve No. 2 is ·0355756, while for curves Nos. 1 and 3 the values are respectively ·0479901 and ·0486767; and as it has been shown above that only when the several values of q are practically identical can the mean of constants be taken, it is not surprising, with the above divergent values, that curves were obtained so wide of the mark.

If, instead of taking the *mean* of constants, we combine them in the way explained on page 196, using ages 30 and 60 to find g and q, the results, given in Table F, are much more satisfactory. It will be seen that only at ages 80 and 90 is there any very serious error, and that error is explained by a reference to Mr. Sorley's ungraduated table. The probability of living ten years contrasts as follows with the same function of the H^{MF} Table.

Age.	Mr. Sorley.	H^{MF}.
30	·9133	·9152
40	·9093	·8841
50	·8307	·8112
60	·7241	·6528
70	·4127	·3787
80	·2778	·0962

Throughout, Mr. Sorley's table shows favourable mortality, but between ages 80 and 90 the vitality is quite abnormal. Not having seen the original data, we cannot give a decided opinion on this point, but we surmise that the cause is to be found in a very limited number of observations; and if that be so, possibly a good table could be formed by taking in all the experience up to 80, and neglecting all above that age.

TABLE F.—*Mr. Sorley's Table formed from "Combined" Constants of three Curves—Deviations in the Values of* l_x.

Age.	Excess.	Defect.
30	159	...
40	350	...
50	...	1,028
60	203	...
70	254	...
80	4,122	...
90	...	1,225
100	63	...

Hitherto we have considered the graduation of tables by means of constants formed from isolated values of l_x, and we think it appears that, by combining in a scientific manner different sets of constants, practically useful results may in most cases be arrived at. The process, however, is not a little laborious, at any rate when many sets of constants are used, and skill is required in selecting them, and in deciding on the proportions to be taken of the several sets. It is necessary, moreover, that the table should be subjected to a preliminary graduation by some such method as Mr. Woolhouse's, before Mr. Makeham's formula can be applied with the best effects. Fortunately, by (so far as we are aware) a new method of finding the constants, all necessity for calculating and combining different sets is obviated, preliminary graduation also being dispensed with, and a final set, giving the average of the whole table, is at once obtained.

If by $\Sigma_m^n \log l_x$ we denote the sum of the values of $\log l_x$ from age m to age n inclusive, we have the following equations:—

$$\Sigma_x^{x+t-1}\log l_x = t\log k + \frac{t}{2}(2x+t-1)\log s + q^x\frac{(q^t-1)}{q-1}\log g \quad . \quad (3)$$

$$\Delta\Sigma_x^{x+t-1}\log l_x = t^2\log s + q^x\frac{(q^t-1)^2}{q-1}\log g \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

and

$$\Delta^2\Sigma_x^{x+t-1}\log l_x = q^x\frac{(q^t-1)^3}{q-1}\log g \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

$$\Delta^2\Sigma_{x+t}^{x+2t-1}\log l_x = q^{x+t}\frac{(q^t-1)^3}{q-1}\log g \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

By taking the logarithms of the members of equation (5) from those of the members of equation (6) we have $t \log q$ (and thence $\log q$), and from the value thus found, working backwards by means of equations (5), (4), and (3), we find the other constants.

In applying this method, which for brevity's sake may be called the "Aggregate Method", the only care necessary is to avoid the two extremes of the table. Seeing that usually there is not much confidence to be placed in a table above age 90, it will, as a rule, be desirable to omit the data above that age; and as the law of mortality seldom appears to remain uniform for many years below 20, we must be cautious in including the younger ages. For the Carlisle Table, using the ungraduated values of $\log l_x$, we have tried, as an experiment, various methods of grouping, which are compared in Table G. It will be seen that they all give

TABLE G.—*Carlisle Mortality—Deviations in the Values of l_x.*

Age.	Milne's Table Unadjusted l_x	FIRST CURVE. 20 years' Intervals from Age 15 to Age 94.		SECOND CURVE. 19 years' Intervals from Age 15 to Age 90.		THIRD CURVE. 18 years' Intervals from Age 18 to Age 89.		FOURTH CURVE. 18 years' Intervals from Age 15 to Age 86.		COMBINED CURVE.	
		Excess.	Defect.	Excess.	Defect.	Excess.	Defect.	Excess.	Defect.	Excess.	Defect.
15	6,300	96·8	...	96·5	...	136·8	...	91·7	...	102·5	...
20	6,090	36·0	...	34·5	...	63·7	...	31·1	...	45·5	...
25	5,879	...	18·8	...	21·3	...	2·0	...	23·3	...	6·5
30	5,642	...	46·4	...	49·2	...	38·3	...	49·7	...	32·3
35	5,362	...	34·3	...	36·9	...	32·8	...	35·8	...	19·7
40	5,075	...	25·5	...	27·0	...	28·1	...	24·4	...	11·5
45	4,727	25·0	...	24·8	...	20·5	...	28·9	...	36·1	...
50	4,397	24·5	...	26·4	...	20·9	...	31·6	...	31·0	...
55	4,073	...	31·8	...	27·9	...	32·6	...	22·4	...	32·2
60	3,643	...	52·1	...	47·2	...	49·3	...	42·9	...	61·5
65	3,018	33·8	...	37·7	...	38·6	...	38·0	...	14·4	...
70	2,401	16·7	...	15·9	...	18·5	...	9·2	...	...	11·6
75	1,675	39·1	...	30·2	...	31·6	...	14·8	...	6·8	...
80	953	67·0	...	50·0	...	47·2	...	28·2	...	38·4	...
85	445	15·3	...	...	3·4	...	9·6	...	23·7	...	2·6
90	142	...	7·8	...	19·4	...	24·4	...	30·4	...	14·5
95	30	...	10·4	...	13·7	...	15·2	...	16·4	...	11·5
100	9	...	8·0	...	8·3	...	8·5	...	8·5	...	8·1
Total Error.	...	589·3		570·3		618·6		551·0		486·7	

sufficiently satisfactory results, and that there is not much to choose between them. The following are the constants for the different "aggregate" curves. Those for the "combined" curve have already been given.

	First Curve.	Second Curve.	Third Curve.	Fourth Curve.
$\log q$	·0396671	·0403962	·0411339	·0408787
$\log g$	−·0003798	−·0003350	−·0002894	−·0003118
$\log s$	−·0035813	−·0036137	−·0037622	−·0036038
$\log k$	4·8611702	4·8615013	4·8663017	4·8609564

The curve formed by the combined method comes out slightly better than any of the others, and we have therefore used it in the annuity values given at the end of this paper.

The Carlisle Table is not the best that can be found for adjustment by Mr. Makeham's formula. The H^M Table is very tractable, and produces surprisingly accurate results. In this case also we have tried various arrangements of grouping, and Table H enables a comparison to be instituted between them. Columns have been added for the constants given by Mr. Woolhouse, *J.I.A.* vol. xv, p. 408. Curve No. 4 is that which we have adopted for our final figures, and it will be seen that it runs a trifle nearer to the original than does Mr. Woolhouse's. The constants of our curves are—

TABLE H.—H^M *Mortality—Deviations in the Values of* l_x.

Age.	Adjusted by Woolhouse's Formula l_x	FIRST CURVE. 19 years' Intervals from Age 15 to Age 90.		SECOND CURVE. 18 years' Intervals from Age 19 to Age 90.		THIRD CURVE. 17 years' Intervals from Age 20 to Age 87.		FOURTH CURVE. 18 years' Intervals from Age 17 to Age 88.		MR. WOOLHOUSE'S CURVE.	
		Excess.	Defect.	Excess.	Defect.	Excess.	Defect.	Excess.	Defect.	Excess.	Defect.
20	96,223	...	65	240	...	278	...	120	...	107	...
25	93,061	...	130	74	...	98	...	...	5	...	6
30	89,865	...	254	...	138	...	124	...	180	...	173
35	86,281	...	175	...	132	...	126	...	142	...	131
40	82,284	...	4	...	16	...	14	. .	3	11	...
45	77,919	23	...	...	25	...	22	5	...	15	...
50	72,726	...	77	42	...	44	...	75	...	78	...
55	66,513	81	...	36	...	37	...	60	...	54	...
60	58,866	...	2	...	14	...	20	...	14	...	27
65	49,297	32	...	63	...	36	...	24	...	11	...
70	38,124	...	113	...	48	...	112	...	132	...	131
75	25,691	...	22	46	...	...	64	...	72	...	43
80	13,930	157	...	191	...	55	...	74	...	128	...
85	5,422	156	...	152	...	40	...	73	...	128	...
90	1,460	...	138	...	152	...	204	...	182	...	153
95	135	5	...	...	...	...	10	...	5	1	...
100	0	4	...	4	...	3	...	3	...	4	...
Total Error.	...	1,438		1,373		1,287		1,169		1,201	

	First Curve.	Second Curve.	Third Curve.	Fourth Curve.
$\log q$	·0393257	·0396739	·0398878	·0396390
$\log g$	−·0004856	−·0004515	−·0004361	−·0004581
$\log s$	−·0026247	−·0027249	−·0027420	−·0026864
$\log k$	5·0384488	5·0416663	5·0421083	5·0403916

Just as this paper is on the point of being handed to the printers, Prof. C. F. McCay's article has appeared, vol. xxii, p. 24. We have not had time to give his work much study, but the section on the "Determination of the Constants" has caught our attention, and we confess we are not able altogether to follow his reasoning. He says:—

"The methods used by Mr. Woolhouse and Prof. Bartlett, to find the four constants in the formula, require the use of three or four rates, separated by equal intervals of time. But it is easy to find them from the total observations, so that the errors in the rates at any one age will be more or less neutralized by the excess of some balancing the deficiency of others."

He then goes on to show that, since $\log p_x = (q-1)q^x \log y + \log s$, therefore

$$\Sigma_{20}^{43} \log p_x = 24 \log s + \overline{q-1} \log g (q^{20} + q^{21} + \ldots + q^{43}),$$

and similarly for $\Sigma_{44}^{67} \log p_x$ and $\Sigma_{68}^{91} \log p_x$, and that by differencing these sums and taking the ratio of the first differences we get q^{24}.

Now the only construction we can place upon his sentences, is that he intends the process he sketches out as above to supplant that used by Mr. Woolhouse, because his own process takes into account the total observations, and Mr. Woolhouse's does not. But surely the two processes are identical! If, instead of taking the sums of three groups, each containing twenty-four values of $\log p_x$, and then differencing *once*, Prof. McCay had taken *four* values of $\log l_x$ at intervals of 24 years, and then differenced *twice*, his final results would have been exactly the same as those he has obtained, since

$$\Sigma_x^{x+23} \log p_x = \log l_{x+24} - \log l_x.$$

In fact the *second* differences in Mr. Woolhouse's method are identical with the *first* differences in Prof. McCay's, and therefore the former gentleman takes account of the whole of the observations quite as much as the latter. There is only one slight advantage, as far as we can see, gained by Prof. McCay, and that is, that, by taking the sum of the ungraduated logarithms of p_x, derived from the observations and not from the life table, he practically uses the ungraduated column of l_x true to seven places of figures, instead of the table containing the values true only to the nearest integer.

We have already remarked that the law of mortality seldom appears to remain uniform much below age 20. It is therefore not possible to carry down the table far below that age by means of the constants used for adult life; but Mr. Makeham, in a most useful practical paper (vol. xvi, p. 344), shows how, by a modification or, rather, duplication of his formula, the desired result may be attained.

We follow his example, but, in the case of the Carlisle Table, the principles he enunciates we apply in a somewhat different manner. We carry down the column l_x—at this part of the table called, for distinction, λ_x—to the youngest age which is to be tabulated, by means of the constants applicable to the higher ages, calling the result the "normal" curve; and then prepare a column of the ratios of the ungraduated values (styled the original curve) to the values so found. This ratio, which may be denominated r_x, we then graduate by means of Mr. Makeham's formula, as if it were an independent life table (but in doing so we assume the value of q to be the same as in the normal curve), and to get the final values of l_x we multiply the normal values by the graduated r_x. Perhaps the process will be better understood by means of the actual figures, given in the following scheme:—

Age.	(1) "Original" Curve. $\log l_x$	(2) "Normal" Curve. $\log \lambda_x$	(3) $\log r_x$ Ungraduated. =(1)−(2)	(4) $\log r_x$ Graduated.	(5) Final Curve. $\log l_x$ =(2)+(4)
10	4·81023	4·82451	$\bar{1}$·98572	$\bar{1}$·98572	4·81023
11	·80828	·82089	·98739	·98716	·80805
12	·80618	·81727	·98891	·98855	·80582
13	·80400	·81364	·99036	·98990	·80354
14	·80175	·81000	·99175	·99119	·80119
15	·79934	·80635	·99299	·99243	·79878
16	·79664	·80268	·99396	·99361	·79629
17	·79372	·79900	·99472	·99472	·79372
18	·79071	·79530	·99541	·99576	·79106
19	·78767	·79158	·99609	·99671	·78829
20	·78462	·78784	·99678	·99758	·78542
21	·78154	·78409	·99745	·99836	·78245
22	·77851	·78031	·99820	·99903	·77934
23	·77546	·77651	·99895	·99958	·77609
24	·77240	·77268	·99972	0·00000	·77268

To find the constants for the graduation of r_x only three equidistant values of r_x are required, since one of the constants q is already known. The third value should generally be that most nearly equal to zero in the ungraduated series; the first, that for the youngest tabulated age, and the second, intermediate between the two. We have used the values for ages 10, 17, and 24. The resulting constants, employing Greek letters to distinguish them from those of the normal curve, are as follows:—

$$\log\kappa = -\cdot 02838$$
$$\log\sigma = +\cdot 00189$$
$$\log\gamma = -\cdot 00194$$

and $$\log r_x = \log\kappa + x\log\sigma + q^x\log\gamma.$$

In our final table, given at the end of the paper, we therefore have, for the young ages, the formula

$$l_x = (k\kappa)(s\sigma)^x(g\gamma)^{q^x} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Mr. Makeham, in the paper above mentioned, graduates the difference (instead of the ratio) between l_x of the original curve and l_x of the normal curve, so that, for the young ages, his method gives the equation

$$l_x = ks^xg^{q^x} + \kappa\sigma^x\gamma^{q^x} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The two methods, as regards monetary values, usually give results practically identical, and the only advantage we can claim for ours is that by it, in cases where it can be successfully applied, we retain, in a partial form, even to the youngest age, the law of uniform seniority, and have greater facilities for exhibiting the effect of the adjustment on joint-life annuity values. That effect we shall immediately proceed to investigate.

The younger ages of the H^M Table can on the whole be more satisfactorily treated by Mr. Makeham's process, which we have therefore adopted. Taking the difference between l_x of the normal curve (which we have called λ_x) and l_x of the original curve, and, in Mr. Makeham's nomenclature, styling it the Residual Series, we base the graduation of this series on the ages 10, 15, and 20, the resulting constants being

$$\log\kappa = +3\cdot 75346$$
$$\log\sigma = +\ \cdot 19275$$
$$\log\gamma = -\ \cdot 89103$$

and if λ'_x signify the number in the Residual Series opposite age x, we have

$$\log\lambda'_x = \log\kappa + x\log\sigma + q^x\log\gamma.$$

The following scheme gives the actual figures:—

Age.	(1) Original Curve l_x	(2) Normal Curve λ_x	(3) Residual Series, Ungraduated λ'_x =(2)−(1)	(4) Residual Series, Graduated λ'_x	(5) Final Curve l_x =(2)−(4)
10	100000	102893	2893	2893	100000
11	99510	102233	2723	2767	99466
12	99113	101575	2462	2525	99050
13	98784	100918	2134	2191	98727
14	98496	100263	1767	1796	98467
15	98224	99609	1385	1385	98224
16	97942	98956	1014	999	97957
17	97624	98301	677	689	97612
18	97245	97649	404	413	97236
19	96779	96997	218	234	96763
20	96223	96343	120	120	96223
21	95614	95691	77	55	95636
22	94971	95034	63	23	95011
23	94321	94378	57	8	94370
24	93683	93720	37	3	93717
25	93061	93057	− 4	1	93056

Table K gives the deviations in each case, of l_x from the original curves.

Table K.—*Deviations in the Values of l_x.*

	Carlisle Table.			H^M Table.		
Age.	Original Curve l_x	Graduated Curve.		Original Curve l_x	Graduated Curve.	
		Excess.	Defect.		Excess.	Defect.
10	6460·0	...	...	100000	...	...
11	6431·0	...	3·4	99510	...	44
12	6400·0	...	5·3	99113	...	63
13	6368·0	...	6·8	98784	...	57
14	6335·0	...	8·1	98496	...	29
15	6300·0	...	8·1	98224	...	...
16	6261·0	...	5·1	97942	15	...
17	6219·0	...	...	97624	...	12
18	6176·0	5·0	...	97245	...	9
19	6133·0	8·7	...	96779	...	16
20	6090·0	11·3	...	96223	...	...
21	6047·0	12·7	...	95614	22	...
22	6005·0	11·4	...	94971	40	...
23	5963·0	8·6	...	94321	49	...
24	5921·0	3·9	...	93683	34	...
25	5879·0	...	6·5	93061	...	5

In order to test more satisfactorily the effects on the rates of mortality of the graduation of the H^M Table, the "Exposed to

Risk" at each age, as given in the volume of "Mortality Experience", page 244, were multiplied by the corresponding q_x of the finally adjusted table, and the results compared with the actual deaths. The deviations throughout were very small, and at all parts of the table were tolerably well distributed between the positive and negative signs, so that at no point did the accumulated error reach considerable proportions. The total expected deaths, 20,487, differ by only 27 from the total actual deaths, 20,514.

A summary of these calculations is given in Table L, where the expected deaths, the actual deaths, and the error, are shown for each quinquennial period of life. We submit that, for practical purposes, our graduation possesses every necessary accuracy.

TABLE L.—*Comparison of "Expected" with "Actual" Deaths, H^M Mortality.*

Age.	Expected Deaths.	Actual Deaths.	Error.	
			Excess.	Defect.
10 to 14	10	10	...	...
15 „ 19	35	36	...	1
20 „ 24	229	232	...	3
25 „ 29	705	659	46	...
30 „ 34	1,253	1,273	...	20
35 „ 39	1,697	1,772	...	75
40 „ 44	2,005	1,980	25	...
45 „ 49	2,178	2,209	...	31
50 „ 54	2,302	2,271	31	...
55 „ 59	2,349	2,334	15	...
60 „ 64	2,279	2,312	...	33
65 „ 69	2,031	2,040	...	9
70 „ 74	1,627	1,609	18	...
75 „ 79	1,057	1,054	3	...
80 „ 84	520	527	...	7
85 „ 89	172	160	12	...
90 „ 94	38	36	2	...
Total...	20,487	20,514	152	179

Unfortunately, the shape in which the data of the Carlisle Table have come down to us does not admit of a similar test, and we shall content ourselves with merely comparing annuity values on single and joint lives. Such a comparison is given in Table M (pp. 208-9) for both mortality tables. As was to be expected, the Carlisle Table does not come out quite so favourably as the other, but still the deviations are not of much practical importance.

It will now be convenient to investigate the effect on annuity values of the rectification of the earlier portions of the tables.

Let $_tp_x$=probability of living t years by the final table,

l_x=living by the final table,

λ_x= do. by the normal table,

$r_x=l_x : \lambda_x$;

and therefore

$$\log l_x=\log\lambda_x+\log r_x.$$

Then

$$\begin{aligned}\log {}_tp_{xy}&=\log l_{x+t}+\log l_{y+t}-\log l_x-\log l_y\\&=\log\lambda_{x+t}+\log\lambda_{y+t}-\log\lambda_x-\log\lambda_y\\&\quad+\log r_{x+t}+\log r_{y+t}-\log r_x-\log r_y.\end{aligned}$$

But in the normal table the law of seniority holds; and therefore, if s be so taken that $q^x+q^y=2q^s$, then $\log {}_tp_{xy}=2\log {}_tp_s$, and, making corresponding substitutions in the above equation, and collecting into one expression the terms involving r, we have

$$\log {}_tp_{xy}=2\log\lambda_{s+t}-2\log\lambda_s+\log\frac{r_{x+t}r_{y+t}}{r_xr_y}$$

TABLE M.

Specimen Values of Annuities—Single Lives, 3 *per cent.*

H^M				CARLISLE.			
Ages.	Original.	Graduated.	Deviation.	Ages.	Original.	Graduated.	Deviation.
10	24·148	24·138	−·010	10	23·512	23·487	−·025
15	23·158	23·148	−·010	15	22·582	22·588	+·006
20	22·043	22·033	−·010	20	21·694	21·623	−·071
25	21·038	21·025	−·013	25	20·665	20·646	−·019
30	19·867	19·897	+·030	30	19·556	19·644	+·088
35	18·587	18·616	+·029	35	18·433	18·493	+·060
40	17·176	17·179	+·003	40	17·143	17·189	+·046
45	15·594	15·593	−·001	45	15·863	15·734	−·129
50	13·896	13·881	−·015	50	14·303	14·141	−·162
55	12·094	12·075	−·019	55	12·408	12·439	+·031
60	10·236	10·226	−·010	60	10·491	10·670	+·179
65	8·418	8·398	−·020	65	8·917	8·890	−·027
70	6·657	6·659	+·002	70	7·123	7·167	+·044
75	5·061	5·077	+·016	75	5·512	5·566	+·054
80	3·742	3·705	−·037	80	4·365	4·148	−·217

TABLE M—(*continued*).

Specimen Values of Annuities—Two Joint Lives, 3 per cent.

H^M					Carlisle.				
Ages.		Original.	Graduated.	Deviation.	Ages.		Original.	Graduated.	Deviation.
20	20	18·582	18·564	−·018	20	20	17·993	17·872	−·121
	30	17·315	17·301	−·014		30	16·748	16·680	−·068
	40	15·424	15·393	−·031		40	15·131	15·052	−·079
	50	12·809	12·766	−·043		50	12·995	12·738	−·257
	60	9·650	9·631	−·019		60	9·782	9·867	+·085
	70	6·394	6·394	...		70	6·790	6·781	−·009
	80	3·646	3·611	−·035		80	4·225	4·000	−·225
30	30	16·373	16·423	+·050	30	30	15·784	15·922	+·138
	40	14·816	14·837	+·021		40	14·449	14·558	+·109
	50	12·469	12·471	+·002		50	12·551	12·466	−·085
	60	9·486	9·489	+·003		60	9·529	9·732	+·203
	70	6·328	6·338	+·010		70	6·662	6·729	+·067
	80	3·623	3·595	−·028		80	4·168	3·983	−·185
40	40	13·710	13·720	+·010	40	40	13·481	13·567	+·086
	50	11·818	11·809	−·009		50	11·954	11·859	−·095
	60	9·171	9·170	−·001		60	9·224	9·429	+·205
	70	6·203	6·210	+·007		70	6·515	6·598	+·083
	80	3·582	3·554	−·028		80	4·102	3·941	−·161
50	50	10·543	10·526	−·017	50	50	10·942	10·680	−·262
	60	8·486	8·474	−·012		60	8·720	8·768	+·048
	70	5·915	5·919	+·004		70	6·338	6·309	−·029
	80	3·484	3·456	−·028		80	4·054	3·835	−·219
60	60	7·199	7·190	−·009	60	60	7·295	7·536	+·241
	70	5·301	5·301	...		70	5·565	5·693	+·128
	80	3·258	3·237	−·021		80	3·695	3·606	−·089
70	70	4·223	4·224	+·001	70	70	4·556	4·607	+·051
	80	2·801	2·789	−·012		80	3·228	3·133	−·095
80	80	2·049	2·051	+·002	80	80	2·459	2·351	−·108

Returning now to the final table, this becomes

$$\log {}_tp_{xy} = 2\log l_{s+t} - 2\log l_s - 2\log r_{s+t} + 2\log r_s + \log\frac{r_{x+t}r_{y+t}}{r_x r_y}$$

$$= \log {}_tp_{ss} + \log\frac{r_{x+t}r_{y+t}r^2_s}{r_x r_y r^2_{s+t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If the second term of the right-hand member of equation (9) vanishes—that is, if the fraction of which the logarithm is taken be equal to unity—then ${}_tp_{xy} = {}_tp_{ss}$, and $a_{xy} = a_{ss}$, and, where the graduation has been applied to r_x itself, this actually does happen in the case of a temporary annuity for such a term that at the end of it neither of the lives x and y exceeds the age at which the rectification of the normal table ceases.

The case of a whole-life annuity is different, and we now proceed to find an expression for the error involved in the assumption that at young ages, as well as in adult life, $a_{xy} = a_{ss}$. To do so, we must make some supposition as to the seniority of the lives. Let $x > y$, from which it follows $x > s > y$, and, z being the age at which the rectification ceases, let $z - x = m$, $x - s = n$, and $s - y = w$. The whole-life annuity may be divided into four parts, and we may write

$$a_{xy} = {}_{|m}a_{xy} + {}_{m|n}a_{xy} + {}_{m+n|w}a_{xy} + {}_{m+n+w|}a_{xy} \quad . \quad . \quad . \quad . \quad (10)$$

and we can proceed to find the value in succession of each portion.

We have already seen from equation (9) that when r_x has been graduated, as in the Carlisle Table,

$${}_{|m}a_{xy} = {}_{|m}a_{ss} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11a)$$

but when λ'_x has been graduated, as in the H^M Table, this becomes, summing for t,

$${}_{|m}a_{xy} = \Sigma_1^m . v^t . {}_tp_{ss} . \frac{r_{x+t} . r_{y+t} . r^2_s}{r_x . r_y . r^2_{s+t}} \quad . \quad . \quad . \quad . \quad . \quad (11\beta)$$

From the same equation we observe that

$${}_{m|n}a_{xy} = v^m {}_mp_{ss} \Sigma_1^n . v^t . {}^m_tp_{ss} . \frac{r_{y+m+t} r^2_s}{r^2_{s+m+t} r_x r_y} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

since in the fraction, r_{x+m} has become equal to unity, and may be omitted. The value of r varies at every age, and therefore the value of the annuity in equations (11β) and (12) can be found only by calculating separately the value of each payment, and taking the sum.

In the same way, for the remaining portions of expression (10), we have

$${}_{m+n|w}a_{xy} = v^{m+n} . {}_{m+n}p_{ss} . \Sigma_1^w . v^t . {}^{m+n}{}_tp_{ss} . \frac{r_{y+m+n+t} r^2_s}{r_x r_y} \quad . \quad . \quad . \quad (13)$$

$${}_{m+n+w|}a_{xy} = v^{m+n+w} {}_{m+n+w}p_{ss} \Sigma_1^\infty . v^t . {}^{m+n+w}{}_tp_{ss} . \frac{r^2_s}{r_x r_y} \quad . \quad . \quad . \quad (14)$$

The true value of a_{xy} will then be found by adding together the values of its four parts in equations (11) to (14). These expressions are, no doubt, a little alarming in their appearance, but that is, to a great extent, due to the unavoidably awkward notation. It must, however, be confessed that the calculations would be somewhat laborious, on account of the variations in r_x. The periods m, n, and w must, however, in all cases be short, and for practical purposes it would be sufficient to take an average value of r. Thus, in expression (12), we might write, for all values of t, $r_{y+m+t}=r_{y+m+\frac{n}{2}}$, and, similarly, for r_{s+m+t}, and for the corresponding functions in expressions (11β) and (13). The fractional coefficients in all the expressions would then become constants, which we might write R_1, R_2, and R_3, and we should have finally

$$a_{xy}={}_{|m}a_{ss}+{}_{m|n}a_{ss}R_1+{}_{m+n|w}a_{ss}R_2+{}_{m+n+w|}a_{ss}R_3 \quad . \quad . \quad . \quad . \qquad (15a)$$

or

$$a_{xy}={}_{|m}a_{ss}R_0+{}_{m|n}a_{ss}R_1+{}_{m+n|w}a_{ss}R_2+{}_{m+n+w|}a_{ss}R_3 \quad . \quad . \quad . \qquad (15\beta)$$

In each of the factors R there occurs the common coefficient $\frac{r^2_s}{r_xr_y}$. If we neglect the other component parts of R we can simply write

$$a_{xy}=a_{ss}\times\frac{r^2_s}{r_xr_y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

That this approximation is quite accurate enough for all practical purposes, is shown by examples 1 and 2 at the end of the paper.

Perhaps it is desirable here to directly call attention to the fact, that the foregoing enquiry into the value of a_{xy} relates only to the young ages. For all ages above 23, in both tables, the equation is rigidly accurate, $a_{xy}=a_{ss}$ when $q^x+q^y=2q^s$, or, what is the same thing, when $\mu_x+\mu_y=2\mu_s$.

It is necessary to know the value of the ratio r_x in order to calculate the effect on joint-life annuity values of the special adjustment of the mortality table at the young ages; and therefore, for reference, and for use when necessary with the tables at the end of the paper, we give, in Table N, the values of $\log r_x$ for both the Carlisle and the H$^{\text{M}}$ Tables. It will be remembered that r_x is the ratio by which l_x of the *normal* curve must be multiplied, in order to get l_x of the final curve.

TABLE N.—log r_x.

Age.	CARLISLE.	H^M
10	$\bar{1}$·98572	$\bar{1}$·98761
11	·98716	·98807
12	·98855	·98906
13	·98990	·99047
14	·99119	·99215
15	·99243	·99392
16	·99361	·99560
17	·99472	·99694
18	·99576	·99816
19	·99671	·99895
20	·99758	·99946
21	·99836	·99975
22	·99903	·99989
23	·99958	·99996
24	0·00000	·99999

At the end of the paper, in Tables O to T, are given, for both Carlisle and H^M, tables of $\log l_x$, l_x, and μ_x; also, at 3 per-cent interest, the values of annuities on one, two, and three lives, with the corresponding D and N columns.

The primary function is $\log l_x$, and not l_x. It was constructed directly from the constants, and on it the monetary values depend. Thus it happens that, at the older ages, l_x does not correspond absolutely with $\log l_x$ or with the monetary values, but is only the nearest integral number that the logarithm will give. Also, since by Gompertz's function, l_x can never become zero, we have been obliged, in order to make the annuities on old lives correct to the last decimal place, to carry the calculations further than it has been deemed necessary to print them. Hence it arises that, for a few years at the end of the tables, the equation $a_x = N_x \div D_x$ is only approximately correct. To obviate unnecessary figures, the column D has in all cases been cut down to five places. This has the effect of reducing the radix of the joint-life tables.

The column μ_x was formed by a continuous process from the constants. The computation of this function at the younger ages is perhaps worthy of notice, on account of the special adjustment of the mortality table superadded to the usual graduation.

In the case of the Carlisle Table, there was no difficulty encountered, because the first section of the table, ages 10 to 24, follows Makeham's law equally with the remainder, as shown in formula (7). We have, for ages below 24,

$$\mu_x = -\log_e(s\sigma) - \{\log_e(g\gamma)\log_e q\}q^x \quad . \quad . \quad . \quad . \quad (17)$$

and for higher ages

$$\mu_x = -\log_e s - \{\log_e g \log_e q\}q^x \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The two portions of the column of μ_x were therefore calculated by an identical process.

In the case of the H^{M} Table, the form of l_x below age 24 being different, a different course had to be followed. Retaining the notation already employed, we have, by formula (8),

$$l_x = ks^x g^{q^x} + \kappa\sigma^x\gamma^{q^x}$$
$$= \lambda_x + \lambda'_x;$$

and therefore

$$\mu_x = -\frac{1}{l_x}\left\{\frac{d\lambda_x}{dx} + \frac{d\lambda'_x}{dx}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

But the series λ and λ' both follow Makeham's law, and have forces of decrement analogous to the force of mortality. These forces of decrement were easily calculated, and from them the differential coefficients of formula (19).

We have tabulated the function μ_x because it is useful for many purposes, and principally in the computation of joint-life annuities. We believe that the equivalent ages of joint lives have hitherto been usually determined by means of powers of q, but this operation is troublesome unless a complete table of q^x have been prepared beforehand, and such a table cannot be turned to any other profitable purpose. But since μ_x is of the form $\text{A} + \text{B}q^x$, it follows that if $2q^s = q^x + q^y$, then also $2\mu_s = \mu_x + \mu_y$, and a table of μ_x answers our purpose equally with a table of q^x.

It must, however, be noted that, although throughout the whole extent of each of our mortality tables, q remains constant, and the equivalent age found by means of q must always be correct, yet, because of the change in the other constants, μ fails us below age 24. We are not, for all that, precluded from using the force of mortality. The property of uniform seniority comes to our aid. If x be the younger life, and if $y - x = t$, we have

$$q^s = \tfrac{1}{2}(q^x + q^y)$$
$$= q^x \tfrac{1}{2}(1 + q^t)$$
$$= cq^x \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where c is a constant depending, not on x or y, but on their difference, t. Therefore

$$s=x+\frac{\log c}{\log q} \quad . \quad . \quad . \quad . \quad (21)$$

It thus appears that, to find the age s equivalent to the joint-lives x and y, we must add to the younger age a constant depending on the difference in age of the proposed lives. Should, therefore, x or y, or both, be less than 24, we can substitute for them any other two ages we please having the same difference, and by that means find the addition to be made to x, the younger of the proposed lives.

As an example, let it be required to find the age equivalent to 12 and 20, H^M Table. These lives, being both below 24, we must move up the table. Let us use 32 and 40.

$$\begin{array}{r}
\mu_{32}=\cdot 00798 \\
\mu_{40}=\cdot 00990 \\
\hline
2)\ \cdot 01788 \\
\hline
\mu_s=\cdot 00894
\end{array}$$

By interpolating in the table, this value is found to correspond to age 36·72, and we conclude that the age equivalent to 12 and 20 is 16·72.

Exactly similar reasoning applies to cases involving three or more lives.

It will not be inappropriate to close this paper by one or two numerical examples illustrative of the use of the tables:—

1st. Let it be required to find $a_{12:20}$ by the H^M Table. The common age, as given above, is 16·72. If we apply formulas (11) to (14), writing first $s=16$, and then $s=17$, we find $a_{16.16}=19{\cdot}626$ and $a_{17.17}=19{\cdot}471$, whence by interpolation $a_{12.20}=19{\cdot}514$. This value is absolutely correct as regards our graduated table. If we adopt the approximation of formula (15β), the value comes out 19·530. Both these calculations are, however, laborious, and not likely to be undertaken by anyone, except perhaps out of curiosity, since formula (16) gives sufficient accuracy without undue trouble. We subjoin the work in full, that it may be seen how few figures are required:—

$$\begin{array}{rr} 2\log r_{16} = & 99120 \\ \text{Colog}\, r_{12} = & 01094 \\ ,,\quad r_{20} = & 00054 \\ \log a_{16:16} = & 29132 \\ \hline & 29400 \end{array} \qquad \begin{array}{rr} 2\log r_{17} = & 99388 \\ \text{Colog}\, r_{12} = & 01094 \\ ,,\quad r_{20} = & 00054 \\ \log a_{17:17} = & 28526 \\ \hline & 29062 \end{array}$$

$$\begin{array}{rr} \text{Corrected, } a_{16:16} = & 19{\cdot}679 \\ ,,\qquad a_{17:17} = & 19{\cdot}526 \\ \hline \Delta \quad = & {\cdot}153 \\ & 27 \\ \hline & 107 \\ & 3 \\ \hline \text{Deduction} = & {\cdot}110 \end{array}$$

$$\begin{array}{rr} a_{16:16} = & 19{\cdot}679 \\ \text{Deduct} & {\cdot}110 \\ \hline a_{12:20} = & 19{\cdot}569 \end{array}$$

The value of $a_{12:20}$ as given in the Institute volume, is 19·523.

2nd. To find $a_{10:15}$. Using ages 30 and 35, we find the annuity we seek to be that on two lives both aged 12·77. By formulas (11) to (14), the true value is found to be 20·391. Formula (16) gives us 20·416. In the Institute volume, the value is 20·405.

3rd. To find $a_{60:45}$. We give this and the next example *in extenso* :—

$$\begin{array}{rr} \mu_{60} = & 02920 \\ \mu_{45} = & 01204 \\ \hline 2) & 04124 \\ \hline \mu_{54\cdot 88} = & 02062 \end{array} \qquad \begin{array}{rr} a_{54:54} = & 9{\cdot}183 \\ \text{Deduct} & {\cdot}297 \\ \hline a_{60:45} = & 8{\cdot}886 \end{array} \qquad \begin{array}{rr} \Delta = & {\cdot}337 \\ & 88 \\ \hline & 270 \\ & 27 \\ \hline \text{Deduction} = & {\cdot}297 \end{array}$$

This value is identical with that in the Institute volume.

4th. To find $a_{30:40:50}$:—

$$\begin{array}{rr} \mu_{30} = & 00768 \\ \mu_{40} = & 00990 \\ \mu_{50} = & 01543 \\ \hline 3) & 03301 \\ \hline \mu_{42\cdot 86} = & 01100 \end{array} \qquad \begin{array}{rr} a_{42:42:42} = & 11{\cdot}039 \\ \text{Deduct} & 251 \\ \hline a_{30:40:50} = & 10{\cdot}788 \end{array} \qquad \begin{array}{rr} \Delta = & {\cdot}292 \\ & 68 \\ \hline & 234 \\ & 17 \\ \hline \text{Deduction} = & {\cdot}251 \end{array}$$

These examples might be multiplied endlessly; but those already given are sufficient, along with the specimens of annuities in Table M, to prove the practical utility of our tables.

Table O.

Institute H^M *Table.*

Constants.

log q = ·0396390
,, g = − ·0004581
,, s = − ·0026864
,, k = 5·0403916

Age x	log l_x	l_x	μ_x	Age x	log l_x	l_x	μ_x
10	5·00000	100000	·00596	58	4·79340	62144	·02536
11	4·99767	99466	·00476	59	·78199	60533	·02719
12	·99585	99050	·00369	60	·76976	58852	·02920
13	·99444	98727	·00289	61	·75661	57097	·03140
14	·99329	98467	·00250	62	·74247	55268	·03380
15	·99222	98224	·00254	63	·72722	53361	·03644
16	·99104	97957	·00297	64	·71078	51378	·03933
17	·98950	97612	·00358	65	·69303	49321	·04250
18	·98783	97236	·00449	66	·67383	47188	·04597
19	·98571	96763	·00526	67	·65306	44984	·04977
20	·98328	96223	·00590	68	·63055	42712	·05394
21	·98062	95636	·00637	69	·60616	40379	·05850
22	·97777	95011	·00668	70	·57969	37992	·06350
23	·97483	94370	·00689	71	·55094	35558	·06898
24	·97182	93717	·00701	72	·51971	33091	·07498
25	·96874	93056	·00712	73	·48574	30601	·08155
26	·96563	92391	·00722	74	·44879	28105	·08876
27	·96248	91723	·00732	75	·40856	25619	·09665
28	·95928	91050	·00743	76	·36475	23161	·10529
29	·95603	90371	·00755	77	·31700	20749	·11476
30	·95272	89685	·00768	78	·26495	18406	·12514
31	·94936	88994	·00782	79	·20818	16150	·13650
32	·94593	88294	·00798	80	·14625	14004	·14896
33	·94243	87585	·00815	81	·07865	11985	·16260
34	·93886	86868	·00833	82	·00484	10112	·17755
35	·93520	86139	·00854	83	3·92424	8399	·19403
36	·93144	85396	·00876	84	·83619	6858	·21197
37	·92759	84643	·00901	85	·73999	5495	·23153
38	·92362	83873	·00928	86	·63485	4314	·25306
39	·91953	83086	·00957	87	·51991	3311	·27665
40	·91530	82281	·00990	88	·39425	2479	·30250
41	·91093	81457	·01025	89	·25684	1807	·33081
42	·90640	80612	·01064	90	·10655	1278	·36184
43	·90169	79743	·01106	91	2·94216	875	·39583
44	·89679	78848	·01153	92	·76231	579	·43306
45	·89167	77924	·01204	93	·56554	368	·47386
46	·88632	76970	·01260	94	·35021	224	·51856
47	·88072	75984	·01321	95	·11456	130	·56751
48	·87484	74962	·01388	96	1·85665	72	·62115
49	·86866	73903	·01462	97	·57435	38	·67993
50	·86214	72801	·01543	98	·26532	18	·74432
51	·85525	71656	·01631	99	0·92702	8	·81486
52	·84797	70464	·01727	100	·55664	3	·89214
53	·84024	69221	·01834	101	·15113	1	·97681
54	·83203	67925	·01950	102	$\bar{1}$·70712	0	1·06957
55	·82330	66573	·02077	103	·22093	...	1·17120
56	·81398	65160	·02216	104	$\bar{2}$·68854	...	1·28254
57	·80404	63685	·02369	105	·10552	...	1·40449

Table P.

H^{M} 3 *per-cent—Values of Annuities.*

Age x	One Life a_x	Two Lives a_{xx}	Three Lives a_{xxx}	Age x	One Life a_x	Two Lives a_{xx}	Three Lives a_{xxx}
10	24·138	20·992	18·797	58	10·967	7·845	6·188
11	23·997	20·855	18·675	59	10·597	7·516	5·897
12	23·820	20·661	18·480	60	10·226	7·190	5·609
13	23·615	20·419	18·219	61	9·856	6·869	5·327
14	23·388	20·143	17·916	62	9·489	6·550	5·050
15	23·148	19·851	17·591	63	9·123	6·238	4·779
16	22·908	19·558	17·266	64	8·759	5·931	4·514
17	22·678	19·287	16·975	63	8·398	5·629	4·256
18	22·449	19·020	16·687	66	8·041	5·333	4·005
19	22·235	18·782	16·441	67	7·688	5·045	3·762
20	22·033	18·564	16·221	68	7·340	4·764	3·527
21	21·833	18·356	16·017	69	6·996	4·490	3·299
22	21·636	18·156	15·826	70	6·659	4·224	3·080
23	21·436	17·956	15·635	71	6·328	3·966	2·869
24	21·232	17·753	15·443	72	6·004	3·717	2·667
25	21·025	17·547	15·248	73	5·688	3·477	2·474
26	20·811	17·334	15·046	74	5·378	3·246	2·289
27	20·592	17·115	14·839	75	5·077	3·024	2·113
28	20·366	16·890	14·625	76	4·784	2·811	1·945
29	20·134	16·659	14·406	77	4·501	2·607	1·786
30	19·897	16·423	14·181	78	4·227	2·412	1·635
31	19·654	16·179	13·950	79	3·961	2·227	1·493
32	19·404	15·929	13·713	80	3·705	2·051	1·360
33	19·147	15·674	13·470	81	3·459	1·884	1·234
34	18·884	15·412	13·221	82	3·223	1·726	1·116
35	18·616	15·144	12·966	83	2·997	1·577	1·005
36	18·341	14·871	12·706	84	2·781	1·437	·903
37	18·059	14·591	12·440	85	2·574	1·305	·807
38	17·772	14·305	12·169	86	2·378	1·181	·718
39	17·478	14·015	11·894	87	2·192	1·066	·636
40	17·179	13·720	11·614	88	2·015	·958	·560
41	16·873	13·418	11·329	89	1·847	·857	·491
42	16·561	13·112	11·039	90	1·690	·764	·427
43	16·244	12 801	10 747	91	1·540	·678	·363
44	15·922	12·487	10·450	92	1·401	·598	·317
45	15·593	12·168	10·151	93	1·270	·525	·269
46	15·260	11·845	9·850	94	1·148	·458	·227
47	14·922	11·520	9·545	95	1·032	·397	·189
48	14·579	11·191	9·239	96	·926	·342	·154
49	14·231	10·860	8·931	97	·827	·292	·115
50	13·881	10·526	8·623	98	·736	·247	·076
51	13·526	10·192	8·314	99	·651	·207	·037
52	13·167	9·856	8·006	100	·573	·171	...
53	12·806	9·519	7·698	101	·502	·140	...
54	12·441	9·183	7·391	102	·437	·113	...
55	12·075	8·846	7·086	103	·377	·089	...
56	11·707	8·511	6·784	104	·323	·070	...
57	11·337	8·177	6·485	105	·275	·053	...

TABLE Q.

Institute H^M *Table—One Life—Interest* 3 *per-cent.*

Age x	D_x	N_x	Age x	D_x	N_x
10	74410	1796141	58	11190	122722
11	71856	1724285	59	10583	112139·3
12	69470	1654815	60	9989·0	102150·3
13	67230	1587585	61	9408·9	92741·4
14	65098	1522487	62	8842·2	83899·2
15	63046	1459441	63	8288·4	75610·8
16	61044	1398397	64	7748·0	67862·8
17	59057	1339340	65	7221·2	60641·6
18	57116	1282224	66	6707·6	53934·0
19	55182	1227042	67	6208·1	47725·9
20	53276	1173766	68	5722·8	42003·1
21	51409	1122357	69	5252·8	36750·3
22	49585	1072772	70	4798·2	31952·1
23	47816	1024956	71	4360·0	27592·1
24	46102	978854	72	3939·3	23652·8
25	44444	934410	73	3536·9	20115·9
26	42841	891569	74	3153·8	16962·1
27	41292	850277	75	2791·0	14171·1
28	39795	810482	76	2449·7	11721·4
29	38349	772133	77	2130·7	9590·7
30	36949	735184	78	1835·0	7755·7
31	35596	699588	79	1563·3	6192·4
32	34287	665301	80	1316·0	4876·4
33	33023	632278	81	1093·5	3782·93
34	31798	600480	82	895·74	2887·19
35	30613	569867	83	722·33	2164·86
36	29465	540402	84	572·60	1592·26
37	28354	512048	85	445·47	1146·79
38	27278	484770	86	339·50	807·29
39	26235	458535	87	252·96	554·33
40	25224	433311	88	183·89	370·44
41	24244	409067	89	130·11	240·334
42	23294	385773	90	89·370	150·964
43	22371	363402	91	59·424	91·540
44	21476	341926	92	38·130	53·410
45	20606	321320	93	23·533	29·877
46	19761	301559	94	13·916	15·961
47	18940	282619	95	7·8526	8·1083
48	18141	264478	96	4·2098	3·8985
49	17364	247114	97	2·1337	1·7648
50	16607	230507	98	1·0169	·74791
51	15869	214638	99	·45302	·29489
52	15151	199487	100	·18746	·10743
53	14450	185037	101	·07154	·03589
54	13766	171271	102	·02499	·01090
55	13099	158172	103	·00792	·00298
56	12448	145724	104	·00226	·00072
57	11812	133912	105	·00057	·00015

TABLE Q—*(continued).*

Institute H^M *Table—Two Lives—Interest* 3 *per-cent.*

Age x	D_{xx}	N_{xx}	Age x	D_{xx}	N_{xx}
10	74410	1561963	55	8720·7	77141·8
11	71471	1490492	56	8110·9	69030·9
12	68810	1421682	57	7522·3	61508·6
13	66374	1355308	58	6954·1	54554·5
14	64100	1291208	59	6405·9	48148·6
15	61927	1229281	60	5878·7	42269·9
16	59797	1169484	61	5372·0	36897·9
17	57646	1111838	62	4886·9	32011·0
18	55538	1056300	63	4422·7	27588·3
19	53396	1002904	64	3980·8	23607·5
20	51264	951610	65	3561·6	20045·9
21	49165	902475	66	3165·2	16880·7
22	47111	855364	67	2792·7	14088·0
23	45123	810241	68	2444·4	11643·6
24	43206	767035	69	2121·0	9522·6
25	41357	725678	70	1822·9	7699·7
26	39581	686097	71	1550·4	6149·3
27	37875	648222	72	1303·6	4845·7
28	36233	611989	73	1082·3	3763·43
29	34656	577333	74	886·38	2877·05
30	33138	544195	75	715·04	2162·01
31	31678	512517	76	567·37	1594·64
32	30274	482243	77	442·11	1152·53
33	28922	453321	78	337·75	814·78
34	27622	425699	79	252·48	562·30
35	26369	399330	80	184·30	378·00
36	25162	374168	81	131·06	246·942
37	23999	350169	82	90·577	156·365
38	22879	327290	83	60·672	95·693
39	21798	305492	84	39·269	56·424
40	20754	284738	85	24·480	31·944
41	19748	264990	86	14·645	17·2988
42	18778	246212	87	8·3749	8·9239
43	17839	228373	88	4·5585	4·3654
44	16933	211440	89	2·3505	2·0149
45	16057	195383	90	1·1422	·87274
46	15210	180178	91	·52015	·35259
47	14391	165782	92	·22059	·13200
48	13598	152184	93	·08654	·04546
49	12832	139352	94	·03117	·01429
50	12090	127262	95	·01022	·00407
51	11371	115891	96	·00303	·00104
52	10676	105215	97	·00080	·00024
53	10002	95213·3	98	·00019	·00005
54	9350·8	85862·5	99	·00004	·00001

Table Q—*(continued)*.

Institute H^M *Table—Three Lives—Interest* 3 *per-cent.*

Age x	D_{xxx}	N_{xxx}	Age x	D_{xxx}	N_{xxx}
10	74410	1398705	53	6923·7	53297·9
11	71089	1327616	54	6351·6	46946·3
12	68155	1259461	55	5805·6	41140·7
13	65530	1193931	56	5285·1	35855·7
14	63118	1130813	57	4790·6	31065·1
15	60828	1069985	58	4321·6	26743·5
16	58576	1011409	59	3877·7	22865·8
17	56269	955140	60	3459·7	19406·1
18	54003	901137	61	3067·3	16338·8
19	51668	849469	62	2700·8	13638·0
20	49328	800141	63	2360·0	11278·0
21	47020	753121	64	2045·3	9232·7
22	44760	708361	65	1756·6	7476·1
23	42582	665779	66	1493·6	5982·5
24	40491	625288	67	1256·3	4726·2
25	38485	586803	68	1044·0	3682·2
26	36570	550233	69	856·47	2825·72
27	34740	515493	70	692·56	2133·16
28	32991	482502	71	551·29	1581·87
29	31319	451183	72	431·37	1150·50
30	29719	421464	73	331·21	819·29
31	28192	393272	74	249·12	570·17
32	26730	366542	75	183·19	386·98
33	25331	341211	76	131·41	255·570
34	23994	317217	77	91·734	163·836
35	22714	294503	78	62·164	101·672
36	21487	273016	79	40·776	60·896
37	20314	252702	80	25·809	35·087
38	19189	233513	81	15·708	19·3788
39	18111	215402	82	9·1593	10·2195
40	17077	198325	83	5·0960	5·1235
41	16086	182239	84	2·6930	2·4305
42	15137	167102	85	1·3452	1·08529
43	14226	152876	86	·63174	·45355
44	13352	139524	87	·27726	·17629
45	12512	127012	88	·11300	·06329
46	11707	115305	89	·04246	·02083
47	10935	104370	90	·01460	·00623
48	10194	94176·9	91	·00455	·00168
49	9483·1	84693·8	92	·00128	·00040
50	8801·6	75892·2	93	·00032	·00008
51	8148·0	67744·2	94	·00007	·00001
52	7522·6	60221·6	95	·00001	·00000

TABLE R.

Carlisle Table.

CONSTANTS.
log q= ·0392061
,, g=− ·0004262
,, s=− ·0035114
,, k= 4·8606738

Age x	log l_x	l_x	μ_x	Age x	log l_x	l_x	μ_x
10	4·81023	64600	·00495	58	4·57693	37751	·02475
11	·80805	64276	·00507	59	·56586	36801	·02632
12	·80582	63947	·00519	60	·55406	35815	·02804
13	·80354	63612	·00533	61	·54149	34793	·02993
14	·80119	63269	·00548	62	·52806	33733	·03199
15	·79878	62919	·00565	63	·51369	32635	·03425
16	·79629	62559	·00583	64	·49830	31499	·03672
17	·79372	62190	·00602	65	·48179	30324	·03942
18	·79106	61810	·00624	66	·46404	29110	·04238
19	·78829	61417	·00648	67	·44495	27858	·04562
20	·78542	61013	·00674	68	·42439	26570	·04917
21	·78245	60597	·00702	69	·40222	25248	·05305
22	·77934	60164	·00733	70	·37829	23894	·05730
23	·77609	59716	·00767	71	·35243	22513	·06195
24	·77268	59249	·00804	72	·32445	21108	·06703
25	·76882	58725	·00894	73	·29416	19686	·07260
26	·76492	58200	·00902	74	·26135	18254	·07870
27	·76099	57675	·00910	75	·22577	16818	·08537
28	·75702	57151	·00920	76	·18715	15387	·09267
29	·75300	56624	·00930	77	·14522	13971	·10066
30	·74894	56097	·00942	78	·09966	12579	·10941
31	·74482	55567	·00955	79	·05013	11224	·11898
32	·74065	55036	·00968	80	3·99625	9914	·12945
33	·73641	54502	·00983	81	·93761	8662	·14092
34	·73211	53965	·01000	82	·87376	7478	·15347
35	·72773	53423	·01018	83	·80421	6371	·16720
36	·72327	52877	·01038	84	·72843	5351	·18223
37	·71872	52326	·01060	85	·64581	4424	·19869
38	·71407	51769	·01083	86	·55572	3595	·21669
39	·70932	51206	·01109	87	·45746	2867	·23640
40	·70445	50635	·01137	88	·35024	2240	·25797
41	·69945	50055	·01168	89	·23322	1711	·28158
42	·69430	49465	·01202	90	·10548	1275	·30742
43	·68901	48866	·01239	91	2·96600	925	·33569
44	·68354	48255	·01280	92	·81367	651	·36665
45	·67789	47631	·01324	93	·64729	444	·40053
46	·67204	46994	·01373	94	·46552	292	·43760
47	·66597	46341	·01426	95	·26691	185	·47818
48	·65966	45673	·01484	96	·04986	112	·52258
49	·65308	44986	·01548	97	1·81265	65	·57120
50	·64621	44280	·01618	98	·55335	36	·62440
51	·63902	43558	·01694	99	·26989	19	·68262
52	·63149	42805	·01778	100	0·95998	9	·74635
53	·62358	42032	·01870	101	·62102	4	·81611
54	·61525	41233	·01970	102	·25048	2	·89243
55	·60647	40408	·02079	103	$\bar{1}$·84526	1	·97599
56	·59718	39553	·02199	104	·40209	0	1·06742
57	·58736	38669	·02331	105	$\bar{2}$·91739	...	1·16751

TABLE S.

Carlisle—3 per-cent—Values of Annuities.

Age x	One Life a_x	Two Lives a_{xx}	Three Lives a_{xxx}	Age x	One Life a_x	Two Lives a_{xx}	Three Lives a_{xxx}
10	23·487	19·925	17·482	58	11·382	8·164	6·446
11	23·315	19·729	17·279	59	11·027	7·849	6·167
12	23·136	19·530	17·073	60	10·670	7·536	5·891
13	22·957	19·329	16·865	61	10·313	7·224	5·618
14	22·774	19·125	16·655	62	9·956	6·916	5·349
15	22·588	18·920	16·443	63	9·600	6·610	5·085
16	22·399	18·712	16·231	64	9·244	6·309	4·825
17	22·208	18·502	16·017	65	8·890	6·011	4·570
18	22·015	18·292	15·803	66	8·539	5·719	4·321
19	21·820	18·083	15·591	67	8·191	5·432	4·078
20	21·623	17·872	15·379	68	7·846	5·151	3·842
21	21·425	17·663	15·171	69	7·504	4·876	3·612
22	21·227	17·455	14·966	70	7·167	4·607	3·389
23	21·028	17·250	14·765	71	6·835	4·345	3·173
24	20·830	17·049	14·570	72	6·508	4·091	2·965
25	20·646	16·875	14·412	73	6·187	3·845	2·765
26	20·457	16·697	14·250	74	5·873	3·606	2·572
27	20·262	16·511	14·082	75	5·566	3·375	2·388
28	20·062	16·321	13·907	76	5·267	3·152	2·212
29	19·856	16·124	13·728	77	4·974	2·940	2·043
30	19·644	15·922	13·542	78	4·690	2·734	1·883
31	19·426	15·714	13·351	79	4·414	2·538	1·731
32	19·202	15·499	13·153	80	4·148	2·351	1·586
33	18·972	15·278	12·951	81	3·889	2·172	1·450
34	18·736	15·052	12·741	82	3·641	2·002	1·321
35	18·493	14·819	12·527	83	3·401	1·841	1·200
36	18·245	14·580	12·306	84	3·171	1·688	1·086
37	17·990	14·336	12·080	85	2·950	1·544	·980
38	17·729	14·085	11·848	86	2·739	1·408	·880
39	17·462	13·829	11·611	87	2·538	1·280	·788
40	17·189	13·567	11·369	88	2·346	1·159	·702
41	16·909	13·299	11·121	89	2·164	1·047	·622
42	16·624	13·027	10·870	90	1·991	·942	·549
43	16·332	12·749	10·612	91	1·827	·844	·481
44	16·036	12·466	10·352	92	1·672	·753	·419
45	15·734	12·179	10·087	93	1·526	·669	·363
46	15·425	11·886	9·817	94	1·389	·591	·312
47	15·112	11·590	9·546	95	1·260	·519	·265
48	14·793	11·290	9·270	96	1·140	·454	·224
49	14·469	10·986	8·992	97	1·027	·394	·187
50	14·141	10·680	8·712	98	·922	·340	·154
51	13·808	10·371	8·430	99	·825	·290	·126
52	13·471	10·059	8·146	100	·734	·245	·101
53	13·131	9·745	7·862	101	·651	·206	·080
54	12·786	9·429	7·577	102	·574	·171	·062
55	12·439	9·113	7·293	103	·503	·140	·045
56	12·089	8·797	7·009	104	·438	·114	·031
57	11·736	8·480	6·726	105	·378	·091	·018

TABLE T.

Carlisle Table—One Life—Interest 3 per-cent.

Age x	D_x	N_x	Age x	D_x	N_x
10	48068	1129020	58	6797·7	77372·5
11	46434	1082586	59	6433·5	70939·0
12	44852	1037734	60	6078·7	64860·3
13	43317	994417	61	5733·4	59126·9
14	41829	952588	62	5396·8	53730·1
15	40385	912203	63	5069·1	48661·0
16	38984	873219	64	4750·1	43910·9
17	37626	835593	65	4439·8	39471·1
18	36307	799286	66	4137·8	35333·3
19	35026	764260	67	3844·5	31488·8
20	33782	730478	68	3559·9	27928·9
21	32574	697904	69	3284·3	24644·6
22	31399	666505	70	3017·7	21626·9
23	30257	636248	71	2760·4	18866·5
24	29146	607102	72	2512·8	16353·7
25	28047	579055	73	2275·3	14078·4
26	26987	552068	74	2048·2	12030·2
27	25965	526103	75	1832·1	10198·1
28	24979	501124	76	1627·4	8570·7
29	24028	477096	77	1434·6	7136·1
30	23111	453985	78	1254·1	5882·0
31	22226	431759	79	1086·4	4795·58
32	21372	410387	80	931·65	3863·93
33	20548	389839	81	790·28	3073·65
34	19753	370036	82	662·35	2411·30
35	18985	351101	83	547·90	1863·40
36	18244	332857	84	446·77	1416·63
37	17528	315329	85	358·62	1058·01
38	16836	298493	86	282·94	775·07
39	16168	282325	87	219·08	555·99
40	15522	266803	88	166·17	389·82
41	14898	251905	89	123·22	266·600
42	14293	237612	90	89·148	177·452
43	13709	223903	91	62·775	114·677
44	13143	210760	92	42·916	71·761
45	12595	198165	93	28·406	43·355
46	12065	186100	94	18·147	25·208
47	11551	174549	95	11·152	14·0560
48	11052	163497	96	6·5684	7·4876
49	10569	152928	97	3·6933	3·7943
50	10100	142827·7	98	1·9737	1·82065
51	9645·2	133182·5	99	·99765	·82300
52	9203·2	123979·3	100	·47449	·34851
53	8774·1	115205·2	101	·21108	·13743
54	8356·6	106848·6	102	·08781	·05012
55	7950·8	98897·8	103	·03334	·01678
56	7555·8	91342·0	104	·01167	·00511
57	7171·8	84170·2	105	·00371	·00140

Table T—*(continued)*.

Carlisle Table—Two Lives—Interest 3 *per-cent.*

Age x	D_{xx}	N_{xx}	Age x	D_{xx}	N_{xx}
10	31052	618695	55	3212·8	29278·2
11	29846	588849	56	2988·5	26289·7
12	28682	560167	57	2773·3	23516·4
13	27555	532612	58	2566·2	20950·2
14	26465	506147	59	2367·6	18582·6
15	25410	480737	60	2177·1	16405·5
16	24388	456349	61	1994·8	14410·7
17	23400	432949	62	1820·5	12590·2
18	22441	410508	63	1654·3	10935·9
19	21512	388996	64	1496·2	9439·7
20	20612	368384	65	1346·3	8093·4
21	19739	348645	66	1204·5	6888·9
22	18891	329754	67	1071·0	5817·91
23	18068	311686	68	945·87	4872·04
24	17269	294417	69	829·20	4042·84
25	16471	277946	70	721·04	3321·80
26	15706	262240	71	621·44	2700·36
27	14975	247265	72	530·40	2169·96
28	14275	232990	73	447·91	1722·05
29	13606	219384	74	373·88	1348·17
30	12965	206419	75	308·13	1040·04
31	12350	194069	76	250·41	789·63
32	11763	182306	77	200·43	589·20
33	11199	171107	78	157·76	431·44
34	10660	160447	79	121·93	309·513
35	10143	150303·8	80	92·363	217·150
36	9646·9	140656·9	81	68·453	148·697
37	9171·9	131485·0	82	49·528	99·169
38	8716·1	122768·9	83	34·907	64·262
39	8279·0	114489·9	84	23·906	40·356
40	7859·6	106630·3	85	15·865	24·491
41	7457·1	99173·2	86	10·172	14·3187
42	7070·2	92103·0	87	6·2815	8·0372
43	6699·0	85404·0	88	3·7220	4·3152
44	6342·1	79061·9	89	2·1082	2·2070
45	5999·3	73062·6	90	1·1366	1·07036
46	5669·7	67392·9	91	·58048	·48988
47	5352·8	62040·1	92	·27944	·21044
48	5048·0	56992·1	93	·12610	·08434
49	4754·8	52237·3	94	·05301	·03133
50	4472·5	47764·8	95	·02062	·01071
51	4200·8	43564·0	96	·00737	·00334
52	3939·4	39624·6	97	·00240	·00095
53	3687·9	35936·7	98	·00071	·00024
54	3445·7	32491·0	99	·00019	·00005

TABLE T—*(continued).*

Carlisle Table—Three Lives—Interest 3 *per-cent.*

Age x	D_{xxx}	N_{xxx}	Age x	D_{xxx}	N_{xxx}
10	20060	350678	53	1550·1	12186·7
11	19184	331494	54	1420·8	10765·9
12	18342	313152	55	1298·2	9467·7
13	17528	295624	56	1182·1	8285·6
14	16745	278879	57	1072·4	7213·21
15	15987	262892	58	968·77	6244·44
16	15257	247635	59	871·30	5373·14
17	14552	233083	60	779·70	4593·44
18	13871	219212	61	694·05	3899·39
19	13213	205999	62	614·13	3285·26
20	12577	193422	63	539·90	2745·36
21	11961	181461	64	471·30	2274·06
22	11366	170095	65	408·26	1865·80
23	10790	159305	66	350·63	1515·17
24	10232	149072·5	67	298·36	1216·81
25	9672·3	139400·2	68	251·32	965·49
26	9140·9	130259·3	69	209·35	756·14
27	8636·9	121622·4	70	172·29	583·85
28	8158·5	113463·9	71	139·90	443·95
29	7704·1	105759·8	72	111·96	331·985
30	7272·8	98487·0	73	88·176	243·809
31	6862·8	91624·2	74	68·246	175·563
32	6473·7	85150·5	75	51·820	123·743
33	6103·8	79046·7	76	38·530	85·213
34	5752·5	73294·2	77	28·001	57·212
35	5418·5	67875·7	78	19·845	37·367
36	5101·1	62774·6	79	13·685	23·6824
37	4799·3	57975·3	80	9·1569	14·5255
38	4512·2	53463·1	81	5·9293	8·5962
39	4239·4	49223·7	82	3·7035	4·8927
40	3979·7	45244·0	83	2·2239	2·6688
41	3732·7	41511·3	84	1·2792	1·38955
42	3497·3	38014·0	85	·70186	·68769
43	3273·6	34740·4	86	·36571	·32198
44	3060·3	31680·1	87	·18010	·14188
45	2857·5	28822·6	88	·08337	·05851
46	2664·4	26158·2	89	·03607	·02244
47	2480·6	23677·6	90	·01449	·00795
48	2305·6	21372·0	91	·00537	·00258
49	2139·0	19238·0	92	·00182	·00076
50	1980·4	17252·6	93	·00056	·00020
51	1829·6	15423·0	94	·00015	·00005
52	1686·2	13736·8	95	·00004	·00001

DISCUSSION.

The PRESIDENT (Mr. A. H. Bailey) had hoped that the result of the studies of which Gompertz was the pioneer, and of which Mr. Makeham was a very active and successful disciple, would have resulted in the discovery of some normal law of mortality which could be expressed by a curve, the equation to which, if we assume the abscissa to be the age, and the ordinate the number living, might have been written down in mathematical symbols like the equation to a parabola or any other known curve, and which, when we had got certain raw materials, would have enabled us to deduce tables of mortality with a moderate amount of labour. He had heard it said that if the rate of mortality be assumed to increase annually at the rate of 3 per-cent from 15 to 55, and at the rate of 8 per-cent afterwards, a very fair table of mortality would result, and he believed this to be true. But no such simple process as that appears to be available; and, therefore, we are much indebted to gentlemen like Mr. King, Mr. Hardy, and others, who give their thought and attention to this subject, and enable us thereby to obtain such tables as we require for practical use.

Mr. SUTTON pointed out that the authors of the paper were mistaken as to the process followed by Mr. Woolhouse in his paper in volume xv of the *Journal*. Having worked out the figures, he had found that Mr. Woolhouse took his $\log q$, constructed in the manner indicated by him, and then, in a similar manner, constructed his $\log l_{10}$, $\Delta \log l_{10}$, and $\Delta^2 \log l_{10}$, and by means of these values obtained in the usual way the values of the other constants. The values differ slightly from those brought out by the method described by Mr. King and Mr. Hardy, but he had himself got exactly the results indicated by Mr. Woolhouse. He had been struck with the somewhat new method adopted by Mr. King and Mr. Hardy of comparing their graduated tables with the tables they have graduated. They take in all cases the difference—excess, or deficiency—of l_x, the number living. But he did not quite see, at first sight, to what extent that helps us to satisfy ourselves that the table is near, or near enough, to the original facts from which it proceeds. He had been accustomed, himself, to compare the actual deaths at each age, with the expected according to the new table constructed. He had made such a comparison between the unadjusted Institute H^M experience, and the same experience as adjusted by Mr. Woolhouse, and found that from age 10 to age 96 the actual deaths recorded are 20,517; while Mr. Woolhouse gives 20,520—a difference of 3. That result will be admitted to be near enough for all practical purposes. Mr. Sutton then read the following figures:—

Ages inclusive.	DEATHS.		All ages (inclusive) from 10 up to	DEATHS.	
	Actual.	By Woolhouse's H^M Table.		Actual.	By Woolhouse's H^M Table
10–14	10	9	14	10	9
15–19	36	34	19	46	43
20–24	232	226	24	278	269
25–29	659	671	29	937	940
30–34	1,273	1,261	34	2,210	2,201
35–39	1,772	1,757	39	3,982	3,958
40–44	1,980	2,009	44	5,962	5,967
45–49	2,209	2,213	49	8,171	8,180
50–54	2,271	2,302	54	10,442	10,482
55–59	2,334	2,326	59	12,776	12,808
60–64	2,312	2,286	64	15,088	15,094
65–69	2,040	2,006	69	17,128	17,100
70–74	1,609	1,622	74	18,737	18,722
75–79	1,054	1,072	79	19,791	19,794
80–84	527	526	84	20,318	20,320
85–89	160	160	89	20,478	20,480
90–94	36	36	94	20,514	20,516
95–96	3	4	96	20,517	20,520
10–96	20,517	20,520			

As far as he could gather, nowhere throughout the paper were the unadjusted results used. Mr. King and Mr. Hardy had apparently compared their results with the table as originally graduated by Mr. Woolhouse—the one on which the Institute annuity tables are constructed. They would be doing good service if they were to adopt their "aggregate method" to adjust not Mr. Woolhouse's results, but the unadjusted results published by the Institute.

Mr. **Berridge** thought that the authors' object had not been simply to produce a table more carefully graduated than Mr. Woolhouse's—for that would be almost a useless labour—but to produce a table which would be not only as well graduated as Mr. Woolhouse's, but should have certain other specific qualities. When Mr. Makeham introduced his method of graduation, it was deemed that he had made a very great step in advance in the graduation of tables, in having produced a table by which joint life annuities could be calculated independently from one table, without the enormous labour of a number of tables or different combinations of tables. But as Mr. Makeham's table depends upon the values of l_x at widely-placed intervals, it was felt that hardly a sufficient number of ages were taken to render his table a sufficiently exact representation of the original facts, and consequently Mr. Woolhouse's system was adopted by the Institute—a system which takes into account all the original facts. So, in a certain sense, does Mr. Makeham's, but not in the same way; and Mr. Makeham's principle was sacrificed to the necessity of having a table which would go closer to the facts. Now, Messrs. King and Hardy had made a very great step in developing Mr. Makeham's formula, in a manner which combines at once Mr. Woolhouse's exactitude, or correspondence, and Mr. Makeham's principles.

Mr. G. F. **Hardy** was not in a position to say whether Mr. Woolhouse did actually take the mean of the constants in the way supposed—that is, by adding $\frac{2}{3}$ of the first set to $\frac{1}{3}$ of the second. He certainly did so in the case of log q, and conveyed the impression in his paper that he had done so in the case of the other constants. If his values of log a and log k are combined in the same way, the results agree with his final values to the last figure; log q, however, comes out with an error of 3 in the seventh place. With regard to the accordance between the number of actual and expected deaths being the real test of the goodness of a graduation, that must depend partly on the nature of the graduation. Some methods of graduation, like Mr. Woolhouse's, do not assume any law of mortality whatever; while others, which may be represented by Mr. Makeham's, *do* assume that a certain formula will represent the law of mortality throughout the table. In this latter class, therefore, the errors in the graduated table as compared with the original are of two kinds, first, those due to accidental irregularities in the original table, and, secondly, those due to a difference between the actual law of mortality in the original table and that assumed by the formula. As far as the former are concerned, we might take some such method as that used by Professor McCay, giving to each age a weight proportioned to the number of deaths at that age. But as regards the second class of errors, this does not appear to be always applicable. For instance, if we take Professor McCay's paper, we shall see that, in his Table No. 2, page 40, where he uses the whole of the experience after the first ten years of assurance, giving to each age a weight proportioned to the number of deaths at that age, we get only 42 deaths below 30, while we have 58 above 90. Now, I think it will be admitted that it is at least as important for the purposes of a life company that the graduated table should agree with the original between ages 20 and 30, as after 90. We cannot get any really perfect scientific method for determining these constants, but we are bound to use methods more or less arbitrary, and even to cut off entirely that portion of our experience below, say, age 18 or 20, in order that we may not spoil our graduation of the remaining portion of the table to which Mr. Makeham's formula can be applied. Although the method of Mr. Woolhouse is perfect with regard to the H^{M} table and the H^{MF} table, yet it does not apply with the Carlisle and other tables, such as that graduated by Mr. Sorley. The real test, when we are taking the mean of constants, will be found to be when the differences between our values of log q are such that their 2nd and higher powers may be neglected. In Mr. Woolhouse's graduation of the H^{MF}, the values only differ in the fourth place, and, of course, if we square this difference, we shall see that it only affects the seventh place of decimals; so that it is a quantity that may be altogether neglected. Mr. Sorley's values of log q differ in the first significant figure, that is, in the 2nd place of decimals, and, of course, the square of this quantity cannot be neglected without serious error.

Mr. WALFORD had always felt that Mr. Gompertz's hypothesis stood much in need of improvement, and Mr. Makeham had been successful in his attempt to introduce smooth working into it. His formula worked remarkably well with certain mortality tables, but not equally well with all. The labour bestowed on the endeavour to improve the graduations of mortality tables, is for the common good of the profession. The true aim must be to find a means of scientific graduation, which will improve and not distort the original facts.

Mr. H. W. Manly thought that when a table is to be graduated, it is desirable to adopt such a method that the graduated results shall follow as closely as possible the original facts. But when we weigh the two questions; whether it is better to have a graduation which shall be as exact as Mr. Woolhouse's, for instance, and be compelled to calculate all the various problems by the old lengthy and complicated methods; or, by adopting such a modification of Mr. Makeham's method, as is used by Mr. King and Mr. Hardy, to be able to obtain the values of joint lives and other problems by the tabulation of a simple function; there can be little doubt that we can afford to sacrifice the closeness of the curve for the great benefits to be obtained. There is a paper in the *Journal* which, years ago, struck him very forcibly as likely one day to become very important—that is, one by Mr. Makeham upon the solution of problems in three joint lives. Mr. Milne occupied many pages of his work in treating of these problems, and produced the most formidable formulas. Mr. Makeham solved them very briefly by introducing the value of a certain benefit, but he left us as far off as ever, because that value is practically insoluble. Perhaps when Mr. King and Mr. Hardy have time, they will turn to the paper, and see whether that function can be solved by means of this method of graduation.

Mr. King, in reply, said that he and Mr. Hardy had gone very carefully into the question whether Mr. Woolhouse took the mean of constants, and they came to the conclusion stated in the paper. If they were mistaken, he would heartily apologize to Mr. Woolhouse. But Mr. Sorley had apparently arrived at the same conclusion. With regard to taking the difference between the two values of l_x, as a test of the nearness of the graduated table to the original, they had followed Mr. Woolhouse in his method of displaying the deviations, as given in his paper in the fifteenth volume. It is a very fair method, and gives a good idea how values run; and it is worked out very easily, although, perhaps, it is not a highly scientific method. As to whether the table is a well-adjusted one, there are many ways, of course, of testing the accuracy of graduation; but, from the life office point of view, the closeness of the values of annuities is one of the most satisfactory criterions, although, perhaps, it is not the most scientific. Then, as to adopting the unadjusted values of the logarithm of l_x to work upon, seeing the nearness of Mr. Woolhouse's graduation, and seeing that a group of 17 or 20 values of $\log l_x$ was taken, the two results would be, so far as practical purposes are concerned, identical. He did not think that there would be anything to choose between them. Mr. Makeham's formula applies very well indeed to both the Carlisle Table and the H^M Table, but great caution must be used in applying it throughout the whole of such a table as Mr. Sorley's. The table has a very sudden twist in it, where the rate of mortality relatively suddenly decreases; and of course Mr. Makeham's formula does not take accurate account of such a fact. It would be very interesting to know more of the data upon which that table was constructed.

Mr. Sorley requests us to print the following remarks:—

The particulars of my " unsuccessful attempt " to use Makeham's method were given in the *Journal* to illustrate the impossibility (as I held) of applying that formula to every case. The curves constructed from mean constants were, I need scarcely say, added merely as a matter of curiosity, to contrast with Mr. Woolhouse's (supposed) similarly-constructed curve. That the artifice in question has no pretensions to a mathematical law, needs no demonstration; the fact must be evident to anyone who knows that arithmetical and geometrical progression are two different things.

Mr. King and Mr. Hardy appear to disagree with my conclusion " that the formula would not apply to the case in hand ", and remark of their new curve (see Table F), "that only at ages 80 and 90 is there any very serious error", and go on to say that, if the experience above 80 is of limited extent, "possibly a good table could be formed by taking in all the experience up to 80, and neglecting all above that age". Now opinions will, of course, vary, within certain limits, as to what is, and what is not, a "very serious error"; but in the present instance, I venture to think that these gentlemen will scarcely, on re-consideration, adhere to their statement. They have probably been misled by looking too exclusively at the deviations in the column l_x. This test, although useful as indicating that certain curves can never be satisfactory, has rather a tendency to conceal the faults of others. A better plan is to contrast the adjusted with the unadjusted decrements in moderate intervals. This is done, in the following table, for periods of ten years:—

Ages.	SUM OF DECREMENTS.		Deviation per-cent of Adjusted.
	Unadjusted.	Adjusted. King & Hardy.	
30– 40	8,685	8,494	− 2·2
40– 50	8,285	9,663	+ 16·6
50– 60	14,049	12,818	− 8·8
60– 70	19,028	18,977	− 0·3
70– 80	29,340	25,472	− 13·2
80– 90	14,887	20,234	+ 35·9
90–100	5,726	4,438	− 22·5

The last column of the table gives the percentage of deviation, and, from it, it will be found that, not to speak of the mortality above age 80, there is in the decade 40-50, for example, the very large error of over *sixteen per-cent.* Indeed, it may safely be said, that the curve is likely to represent quite as closely half-a-dozen other mortality tables as the one from which it is constructed. It seems, on the whole, to be even less successful than some of those obtained by myself.

The suggestion that the experience above 80 might be omitted is based on the mistaken supposition that the mortality in the decade following age 80 is "quite abnormal". My statistics related to female annuitants in a certain annuity society, and, therefore, should not be judged of according to the standard of the H^{MF} Table, where only about eleven per-cent of the lives are female. A more appropriate comparison may be made with the Peerage Females, and the Government Female Annuitants. The following table shows the probability of living ten years according to the four tables. It will be seen that the ungraduated ${}_{10}p_{80}$ falls between the Peerage and the Government Annuitants, while the H^{MF} Table differs widely from the other three.

Age.	Ungraduated Table.	Peerage Females.	Government Female Annuitants.	H^{MF}.
30	·9133	·9116	·9066	·9152
40	·9093	·8881	·8894	·8841
50	·8307	·8416	·8484	·8112
60	·7241	·6994	·7295	·6528
70	·4127	·5324	·4747	·3787
80	·2778	·2881	·1655	·0962

I am thus, I think, warranted in holding—(1), that it is impossible in this case to represent the actual mortality, even under age 80, by Mr. Makeham's formula; (2), that it would be improper to set aside the actual mortality above 80, even were the exposures (which they were not) relatively unimportant. It appears to me that, in endeavouring to make use of an empirical law of mortality, it should be accepted as an axiom that, if any real conflict between the formula and the facts be found to exist, it is the formula, and not the facts, that must give way.

CORRESPONDENCE.

FORMULA FOR THE FORCE OF MORTALITY.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—May I ask the privilege of sufficient space in your *Journal* to observe, that the formula for computing force of mortality from numbers living,

$$\mu=\frac{8(l_{-1}-l_{+1})-(l_{-2}-l_{+2})}{12l},$$

given by Mr. W. S. B. Woolhouse at p. 64 of the *Journal* for April last, is identical with one which had been published by me some years before (*Smithsonian Report* of 1873, p. 343). My notation was

$$\mu_3 = \frac{8(y_2-y_4)-(y_1-y_5)}{12y_3},$$

and I remarked upon it as follows:—"Thus, if we have found by observation the numbers of persons annually attaining any five consecutive birthdays, or, what amounts to the same thing, the numbers living at those five ages out of a given number of persons born, the above formula will give the intensity of mortality at the middle birthday."

I am, Sir,
Your obedient Servant,
E. L. DE FOREST.

Watertown, Connecticut, U.S.A.,
1 *Oct.* 1878.

In connection with the above letter, it may be useful to remind our readers of the more exact formula for the force of mortality given by Mr. Sprague in vol. xiii, p. 210, and applied by him in vol. xvii, p. 268, namely

$$-\mu_x l_x = a_0 - \frac{c_0}{6} + \frac{e_0}{30} - \frac{g_0}{140},$$

where a_0, c_0, e_0, g_0, are the central differences of the first, third, fifth, and seventh orders, their values in terms of the l's being

$$a_0 = (l_{+1}-l_{-1}) \div 2$$
$$c_0 = \{l_{+2}-l_{-2}-2(l_{+1}-l_{-1})\} \div 2$$
$$e_0 = \{l_{+3}-l_{-3}-4(l_{+2}-l_{-2})+5(l_{+1}-l_{-1})\} \div 2$$
$$g_0 = \{l_{+4}-l_{-4}-6(l_{+3}-l_{-3})+14(l_{+2}-l_{-2}-l_{+1}+l_{-1})\} \div 2.$$

Taking two terms in the value of μl, we get by substitution,

$$\mu = \{8(l_{-1}-l_{+1})-(l_{-2}-l_{+2})\} \div 12l,$$

which is the formula given by Mr. Woolhouse. Taking three terms, we similarly get

$$\mu = [9\{5(l_{-1}-l_{+1})-(l_{-2}-l_{+2})\}+l_{-3}-l_{+3}] \div 60l.$$

Ed. *J.I.A.*

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

Remarks as to the Influence of Marriage on the Death Rates of Males and Females; abridged from Dr. James Stark's *Letters to the Registrar-General of Births, Deaths, and Marriages in Scotland.*

I.—Females.

From the Seventh Detailed Annual Report (relating to the year 1861).

A MOST important enquiry, and one which from want of facts has not hitherto been investigated, is, "What is the effect of marriage on female life, both during the period of child-bearing, and after that period of life has passed?"

In order to throw light on this interesting subject, Table XXV has been prepared. It gives the numbers, first, of married women (wives) and widows, and, secondly, of unmarried women above 15 years of age, living in Scotland, at every quinquennial period of life, on the 30th of June 1861; also the deaths during the year 1861, first, among the married and widowed, and, secondly, among the unmarried at the same periods; and, lastly, the various percentages of deaths among the two classes.

TABLE XXV.—*Population and Deaths of the Married and Unmarried Females at Different Ages in Scotland in* 1861; *and the Proportion of Deaths to the respective Populations at each Age.*

Ages above 15 Years.	MARRIED AND WIDOWED WOMEN IN 1861.			UNMARRIED WOMEN ABOVE 15 YEARS IN 1861.		
	Population.	Deaths.	Per-cent of Deaths to Population.	Population.	Deaths.	Per-cent of Deaths to Population.
15– 20	3,356	33	0·983	153,877	1,064	0·691
20– 25	39,662	361	0·910	113,443	889	0·783
25– 30	71,171	651	0·928	60,650	548	0·903
30– 35	77,010	714	0·927	34,732	337	0·941
35– 40	71,503	798	1·116	24,213	286	1·181
40– 45	69,230	773	1·116	19,568	217	1·109
45– 50	58,568	744	1·270	14,568	212	1·455
50– 55	54,153	785	1·449	13,890	219	1·576
55– 60	41,214	820	1·989	10,156	211	2·077
60– 65	42,151	1,117	2·649	11,341	808	2·715
65– 70	26,094	1,114	4·269	6,838	307	4·489
70– 75	20,233	1,341	6·627	5,515	368	6·672
75– 80	12,126	1,267	10·448	3,213	312	9·741
80– 85	7,454	1,126	15·105	1,977	321	16·236
85– 90	2,636	611	23·179	579	154	26·597
90– 95	759	244	32·147	181	72	39·778
95–100	202	73	36·138	48	17	35·416
100 and above	34	17	50·000	23	6	26·087
Not stated	...	17	...	...	15	...
Total, 15–105 yrs.	597,556	12,606	2·109	474,812	5,863	1·240
In Two Groups,						
15– 45	331,932	3,332	1·003	406,483	3,341	0·821
45–105	265,624	9,274	3·495	68,329	2,522	3·691

The first striking fact which that table seems to bring to light is, that the general death-rate among the married women is very much higher than that among the unmarried; so that, while only 1,240 unmarried women above 15 years of age died during the year out of every 100,000 unmarried women, 2,109 married women died out of a like number of married women. But this difference is caused by the number of unmarried women being far greatest at that period of life when the mortality is very low, namely, from 15 to 20 years of life, whereas the number of married women is greatest at those periods of life when the mortality is high, seeing that the mortality increases with age.

When the females are arranged in the two groups: 1st, Females at the ages when they are capable of bearing children, namely, from 15 to 45 years of age; and, 2nd, Females past the child-bearing age, namely, from 45 to the close of life, we arrive at the second striking fact which the table reveals, namely, that the increased mortality of the married female is limited to the child-bearing age (15 to 45 years), whereas, above that age, the married female dies at a considerably lower rate than the unmarried female.

These facts, then, prove that it is child-bearing which is the cause of the high mortality of the married female at the child-bearing ages; but that as soon as she gets past that period of life, her chances of life are superior to that of the unmarried female.

Comparing the mortality of the married and unmarried females at every quinquennial period of life, it is seen that the high death-rate of the married female appears to be confined to the ages under 30 years; but that from 30 to 35, and again from 35 to 40 years, the death-rate of the married female falls below that of the unmarried.

But the striking fact is, that the mortality of the married females between the ages of 15 and 20 years is higher than that of the married females even during the next three quinquennial ages, namely, it is higher than the mortality of the married females at the ages 20 to 25, 25 to 30, and 30 to 35 years. This seems to indicate that marrying before the frame has acquired its full development, causes the woman to run greater risk of her life than if marriage had been delayed till the full growth was completed.*

The following tables, published in the Second Detailed Annual Report, will assist us to understand why the mortality of the married female is so much higher than that of the unmarried at the three quinquennial ages, 15–20, 20–25, 25–30.

---

* This inference, reasonable as it seems, and fairly deduced from the figures, is not borne out by the corresponding figures for the year 1862, printed below; and we can only conclude that the number of married women under 20 is too few for us to reason with certainty as to their mortality.—Ed. *J.I.A.*

TABLE VI.—*Number and Ages of Mothers, and Number of Children which they had on the Birth of their last Child—Edinburgh City,* 1855.

Number of Children at Birth of Mother's Last Child.	NUMBER AND AGES OF MOTHERS.						Total Mothers.	Total Children.
	15–20	20–25	25–30	30–35	35–40	40 and above		
1 Child	85	438	290	109	52	15	989	989
2 Children	13	242	230	135	46	13	679	1,358
3 "	2	124	304	157	61	16	664	1,992
4 "	...	36	209	167	81	20	513	2,052
5 "	...	7	103	190	95	18	413	2,065
6 "	...	...	56	142	88	37	323	1,938
7 "	...	1	19	87	81	40	228	1,596
8 "	...	...	8	34	77	37	156	1,248
9 "	...	...	1	22	51	36	110	990
10 "	...	...	...	5	29	20	54	540
11 "	...	...	...	3	20	18	41	450
12 "	...	...	1	...	7	11	19	228
13 "	...	...	...	...	3	10	13	169
14 "	...	...	...	...	...	2	2	28
15 "	...	...	...	...	...	1	1	15
16 "	...	...	...	...	...	...	1	16
17 "	...	...	...	...	...	...	1	17
18 "	...	...	...	...	...	1	1	18
Total Mothers	100	848	1,221	1,051	693	295	4,208	15,709

TABLE VII.—*Number and Ages of Mothers, and Number of Children which they had on the Birth of their last Child—City of Glasgow,* 1855.

Number of Children at Birth of Mother's Last Child.	NUMBER AND AGES OF MOTHERS.						Total Mothers.	Total Children.
	15–20	20–25	25–30	30–35	35–40	40 and above		
1 Child	269	1,483	729	222	72	17	2,792	2,792
2 Children	30	898	844	291	75	14	2,152	4,304
3 "	4	436	919	419	119	34	1,931	5,793
4 "	...	113	702	486	171	36	1,508	7,032
5 "	...	27	384	515	212	42	1,180	5,900
6 "	...	8	191	447	240	51	937	5,622
7 "	...	1	40	254	285	65	645	4,515
8 "	...	...	25	157	236	88	506	4,048
9 "	...	...	2	72	147	110	331	1,979
10 "	...	...	...	15	75	84	174	1,740
11 "	...	...	...	10	50	59	119	1,309
12 "	...	...	...	2	14	31	47	564
13 "	...	...	...	1	2	19	22	286
14 "	...	...	...	...	4	9	13	182
15 "	...	...	...	1	...	4	5	75
16 "	...	...	...	...	...	2	2	32
17 "	...	...	...	...	...	1	1	17
Total Mothers	303	2,966	3,836	2,892	1,702	666	12,365	47,180

From these tables it appears that, in Edinburgh and Glasgow, in 1855, of 16,573 mothers, 9,274 were under 30 years of age, and 7,299 above that age. That is to say, that, speaking in a very general way, nearly as many women above 30 years as under that age, bore children in these towns during 1855. Now it may be assumed that the relative proportions of women under and above 30 years of age, who annually bear children in Scotland, are much the same.

If nearly an equal number of women above and under 30 years of age bear children, and if it be found, as Table XXV proves, that the mortality of the married female is only greater than that of the unmarried female below 30 years, while, above that age, fewer married females die than unmarried; the conclusion seems irresistible, that the greater mortality of the married female under 30 years of age cannot be owing simply to child-bearing, because it does not increase the mortality of the married female above 30 years of age. But that table just as distinctly proves that it is some danger connected with married life, because the unmarried females under 30 years of age are not subject to it; and that it is a danger which is greatest during the earlier years of marriage, and which disappears as life advances. Every medical practitioner seeing these facts would at once suggest, as the explanation, that this additional danger during the earlier years of marriage is the birth of the first child. It is a well-known fact, that the risk to the mother is far greater at the birth of her first child, than at any subsequent delivery; and it is extremely probable that the whole extra mortality of the married female under 30 years of age, over that of the unmarried at the same ages, may be caused by the greater dangers which attend the delivery of the first child. This can almost be demonstrated to be the true explanation.

The above tables may be thrown into another form, as follows:—

Ages of Mothers.	Total Number of Mothers.	Number of Mothers bearing their First Child.	Proportion of Mothers bearing their First Child to every 100 Mothers.
15–20 years	403	354	87·8
20–25 „	3,814	1,921	50·3
25–30 „	5,057	1,019	20·1
30–35 „	3,943	331	8·3
35–40 „	2,395	124	5·1
40 and above	961	32	3·3
All ages .	16,573	3,781	...

From these figures it may be concluded that the excessive mortality in the married female, as compared with that of the unmarried at the same ages, is almost solely due to the superadded dangers which attend the birth of the first child; that, in fact, after the birth of her first child, the married female, even during the rest of her child-bearing life, has an equal chance of life with the unmarried, and has a better life than the unmarried after she has passed her 30th year.

In order fully to have investigated this subject, the enquiry ought to have been carried one step further, so as to ascertain positively whether a higher death-rate does attend the birth of the first child. But the schedules now in use in Scotland for the registration of births and deaths, do not furnish us with all the information required for settling this question; so that, in the meantime, we must rest satisfied with the above approximations.

From the Eighth Detailed Report (1862).

In last Report, attention was directed to the influence of marriage on the death-rate of the female; and a table was published which showed the different death-rates which prevailed at each quinquennial period of life among those who were or had been married, and those who were unmarried. A similar table has been prepared for 1862; but as this series of tables will not be continued in subsequent reports, in consequence of the long time required for their preparation, the percentage of deaths for 1861 is given alongside that of 1862; and a Summary Table is added, showing the mean death-rates of the married and the unmarried at every quinquennial period of life.

TABLE XXIII.—*Number of the Living, and Deaths of the Married and Unmarried Females, at different Ages, in Scotland in* 1862; *and the Proportion of Deaths to the respective Populations at each Age. Also the Proportions for* 1861.

Ages above 15 Years.	MARRIED AND WIDOWED WOMEN IN 1862.			Per-cent of Deaths in 1861.	UNMARRIED WOMEN ABOVE 15 YEARS IN 1862.			Per-cent of Deaths in 1861.
	Number Living.	Deaths.	Per-cent of Deaths.		Number Living.	Deaths.	Per-cent of Deaths.	
15– 20	3,384	25	0·738	0·983	154,829	1,075	0·694	0·691
20– 25	39,924	364	0·912	0·910	114,180	895	0·783	0·783
25– 30	71,649	683	0·953	0·928	61,044	506	0·829	0·903
30– 35	77,503	764	0·985	0·927	34,954	367	1·049	0·941
35– 40	71,962	806	1·120	1·116	24,368	300	1·231	1·181
40– 45	69,660	806	1·171	1·116	19,690	225	1·142	1·109
45– 50	58,934	785	1·332	1·270	14,659	222	1·507	1·455
50– 55	54,479	826	1·516	1·449	13,973	269	1·926	1·576
55– 60	41,462	886	2·136	1·989	10,216	244	2·389	2·077
60– 65	42,390	1,200	2·830	2·649	11,405	370	3·241	2·715
65– 70	26,242	1,212	4·732	4·269	6,875	350	5·090	4·489
70– 75	20,337	1,342	6·519	6·627	5,544	434	7·828	6·672
75– 80	12,190	1,424	11·681	10·448	3,230	359	11·114	9·741
80– 85	7,490	1,278	17·062	15·105	1,986	332	16·717	16·236
85– 90	2,646	708	26·757	23·179	582	161	27·663	25·597
90– 95	763	282	36·972	32·147	181	55	30·386	39·778
95–100	203	94	46·305	36·138	48	20	41·666	35·416
100 & above	34	17	50·000	50·000	23	3	13·042	26·087
Not stated	...	11	...	...	...	3	...	...
Total, 15–105 yrs.	601,252	13,543	2·252	2·109	477,787	6,190	1·295	1·240
In Two Groups,								
15– 45	334,082	3,448	1·032	1·003	409,065	3,368	0·820	0·821
45–105	267,170	10,095	3·778	3·495	68,722	2,822	4·107	3·691

TABLE XXIV.—*Mean Annual Percentage of Deaths among the Married and Unmarried Women of Scotland, at different Ages, during the Two Years* 1861 *and* 1862.

Ages.	PERCENTAGE OF DEATHS TO THE NUMBER LIVING.		Whether Proportion of Unmarried is greater or less than that of the Married.	Ages.	PERCENTAGE OF DEATHS TO THE NUMBER LIVING.		Whether Proportion of Unmarried is greater or less than that of the Married.
	Married and Widowed.	Unmarried.			Married and Widowed.	Unmarried.	
15–20	0·860	0·692	–	75– 80	11·064	10·427	–
20–25	0·911	0·783	–	80– 85	16·083	16·476	+
25–30	0·940	0·866	–	85– 90	24·968	26·630	+
30–35	0·956	0·995	+	90– 95	34·559	35·082	+
35–40	1·118	1·206	+	95–100	41·221	38·541	–
40–45	1·143	1·125	–	100 & above	50·000	19·064	–
45–50	1·301	1·481	+				
50–55	1·482	1·751	+	Total .	2·180	1·267	–
55–60	2·062	2·233	+				
60–65	2·739	2·979	+	In Two Groups,			
65–70	4·500	4·789	+	15– 45	1·017	0·820	–
70–75	6·588	7·250	+	45–105	3·636	3·899	+

By these tables it will be seen, that though the proportion of the deaths among the married and the unmarried women at each age differed slightly during each year, there is no difference whatever in the results brought out by each table; for, excepting at the extreme ages 80 to 85, and again 90 to 95 years, the results of the tables of both years are absolutely identical. [See, however, the note on p. 235.] In the Summary Table, plus (+) and minus (—) marks have been appended to the proportions at each age, to indicate whether the deaths among the unmarried at that age were greater (+) or less (—) than those of the married at the same age.

When the mean annual percentage of deaths in the married and unmarried females at each quinquennial period of life is compared, it is found that the married die at a higher ratio during the three quinquennial periods of life, 15 to 30 years; but that during the latter portion of their child-bearing life, when the half of their children are born, namely, between 30 to 40 years of age, the married die at a lower rate than the unmarried. At the age when the usual "change of life" occurs, namely, from 40 and 45 years, the mortality of the married female again slightly exceeds that of the unmarried,—a result which might have been expected, seeing that the fatigues of child-bearing, and the harder labour connected with the nursing and rearing of her family, somewhat weakens the system, and renders that critical period of life more trying to the married than to the unmarried woman. From that period, however, to old age—that is, to 75 years—the married and the widowed die in smaller proportions than the unmarried. Above that age the numbers are too small to allow any certain conclusions to be drawn; still, from 80 to 95 years the chances of life seem still to be decidedly in favour of the married woman.

From the Eighth Decennial Census of Scotland.

The higher death-rate suffered by the married female from the 15th to the 30th year of life is, in all probability, caused solely by the greater danger to life which, in a civilized state, attends the bearing of the first child. Every medical man knows that the risk to the mother is far greater at the birth of her first child than at any subsequent delivery, and that the danger is greater just in proportion to her delicate uprearing. The following table will render this view almost a certainty. It shows the number of mothers at each quinquennial period of life who gave birth to children in Scotland in 1855, and the number and proportion of those at each age who gave birth to their first child.

TABLE LIX.

Ages of Mothers.	Total Number of Mothers.	Number bearing their First Child.	Percentage bearing their First Child.
15–20	2,589	1,424	55·00
20–25	19,230	7,650	39·78
25–30	25,679	4,418	17·32
30–35	21,317	1,502	7·04
35–40	15,070	544	3·61
40 and above	7,153	54	0·75
All Ages	91,038	15,622	...

From this it appears that it was only at the ages when a very high proportion of the married women were giving birth to their first child that their death-rate rose higher than that of the unmarried. But the moment that age was attained, when the great majority of the married women had got over the birth of their first child, namely, above 30 years, the mortality of the married women fell below that of the unmarried. This fact is the more remarkable, seeing that very nearly half of the women bearing children in Scotland were above 30 years of age; and yet these tables prove that the additional burden of child-bearing thrown on them did not increase their death-rate above that of their unmarried sisters, but appeared to give them a better chance of life.

[It is deserving of consideration how far the higher rate of mortality shown by the above figures among the unmarried women, may be due to extensive understatement of their ages. If the unmarried women appearing in the returns as between the ages of 45 and 65 were, on the average, all 2 or 3 years older than they represented themselves to be, and the ages of the married women were correctly given, this would probably fully account for the greater apparent mortality among the unmarried women at those ages. We cannot help thinking that Dr. Stark's figures would have been more useful if the widows had been included with the unmarried women instead of with the married. The placing of them along with the latter certainly disturbs all the conclusions as to the mortality attendant on child-bearing. The questions raised by Dr. Stark seem to require further discussion;

and we believe that the statistics of the Peerage families, if carefully compiled and examined, would throw much light on these as well as many other interesting points.—Ed. *J.I.A.*]

II.—Males.

From the Ninth Detailed Report (1863).

In the Seventh and Eighth Detailed Reports attention was directed to the influence of marriage on the death-rate of the female.

Desiring to complete the enquiry, the influence of marriage on the death-rate of the male sex has been examined into for 1863, and the results are so startling that they merit the greatest attention.

Table XVIII shows the number of married and unmarried men in Scotland at each quinquennial age, the deaths at the same ages, and the proportion of deaths to the living at each age. By that table it appears that, at every age from 20 to 85 years, the death-rate of the married men is very much smaller than that of the unmarried; in fact, it proves that the influence of marriage on the male sex is much more potent than it is on the female—potent to an extent never previously imagined.

Table XVIII.—*Married and Unmarried Men at each Quinquennial Age, in Scotland in* 1863, *the Deaths at the same Ages, and the Proportion of Deaths to the Living at each Age.*

Ages.	Husbands and Widowers.			Unmarried Men.		
	Number Living.	Deaths.	Per-cent of Deaths.	Number Living.	Deaths.	Per-cent of Deaths.
20– 25	22,946	137	0·597	106,587	1,251	1·174
25– 30	54,221	469	0·865	48,618	666	1·369
30– 35	66,153	600	0·907	25,962	383	1·475
35– 40	63,858	690	1·080	15,857	253	1·595
40– 45	62,645	782	1·248	12,311	208	1·689
45– 50	54,505	869	1·594	8,824	179	2·028
50– 55	49,591	880	1·774	7,636	205	2·684
55– 60	38,006	929	2·444	5,550	142	2·558
60– 65	35,920	1,216	3·385	5,242	227	4·330
65– 70	22,021	1,134	5·149	2,848	156	5·477
70– 75	16,029	1,291	8·055	2,021	205	10·143
75– 80	9,716	1,135	11·681	1,081	157	14·542
80– 85	5,477	953	17·400	513	101	19·688
85– 90	1,708	488	28·571	151	32	21·192
90– 95	449	137	30·512	50	21	42·000
95–100	103	40	38·835	6	3	50·000
100 & above	28	15	53·571	3	...	...
Not stated	...	4	...	...	5	...
All Ages	508,376	11,769	2·338	243,259	4·194	1·723

Here, between the ages of 20 and 25 years, the death-rate of the bachelors was exactly double that of the married men. As the persons became older, this excessive difference in the death-rates of the married and unmarried decreased; but it decreased slowly and regularly, showing the difference in favour of the married men at every period of life. It is thus proved that bachelorhood is more destructive to life than the most unwholesome trades, or than residence in an unhealthy house or district, where there has never been the most distant attempt at sanitary improvements of any kind.

The above facts are brought out by a comparison of the deaths with the living, where the correctness of the conclusion rests on the truthfulness of the numbers of the married and unmarried as ascertained at the census. But the correctness of the conclusion may be tested by estimating the mean age at death of the married and the unmarried; and if the two results correspond, or corroborate each other, the greatest confidence may be placed in the conclusion deduced from the facts.

If we reckon the mean age at death of the whole married and unmarried in the above table, namely, from 20 years to the close of life, we find that the mean age at death of the married men was $59\frac{7}{10}$ years, whereas the mean age at death of the bachelors was only 40 years. Reckoning from the twentieth year of life, therefore, it would appear that married men have a chance of living 19½ years longer than those who remain bachelors.

In the general population, however, nearly the same number of men are married and unmarried at the twenty-fourth year of life. It will afford, therefore, a more unexceptionable average if we strike from the table all persons under 25 years of age, and only calculate the mean age at death from the twenty-fifth year to the close of life. When this is done, it is found that the mean age at death of the married man is $60\frac{2}{10}$ years, while that of the bachelors is only $47\frac{7}{10}$ years, giving $11\frac{5}{10}$ years of life in favour of the married men.

In ignorance of the above remarkable facts, it has been the custom to compare the mortality in our army—which is almost entirely composed of bachelors—with the mortality of the males at the same ages in the civil population, more than the half of whom are married. This made the mortality of our soldiers to appear comparatively much higher than it really is. These tables prove, that in civil life, bachelors die in much larger proportions than the married; and to ascertain whether the mortality among our soldiers was excessive, the comparison ought to have been made with the mortality of bachelors living in towns, and not with that of the whole male population at the particular ages. Even comparing the mortality of the soldiers with the whole bachelors of the country at the particular ages, would yield a false conclusion: because it has been proved that the mortality in towns, where human beings are congregated in masses, is much higher than it is in the rural districts; and soldiers, from being congregated in masses, and usually occupying barracks in towns, must be considered as a town, and not as a rural population.

On looking over the above table (XVIII), it is seen that, at each quinquennial age, the death-rate of the married was greatly below that of bachelors. One would, therefore, naturally imagine that a comparison of the total death-rate at all ages would lead to the same conclusion—that, in fact, the death-rate of each class at all ages would be a trustworthy and correct summary of the death-rate at each age. The very reverse is the case. The summary of the table contradicts, and is opposed to the conclusions which are afforded by each quinquennial comparison, and makes it appear that the bachelors at all ages, from 20 years to the close of life, only died at the rate of 1723 annually in every 100,000 bachelors, while the married men died at the rate of 2338 in a like number of married men.

This anomaly is explained by looking at the numbers dying at each age. Very nearly the half of all the deaths of the bachelors occurs before the thirtieth year of life. As the mortality under 30 years, as compared with the persons living at that age, is very much lower than it is at more advanced periods of life, when the whole deaths at all ages are thrown together and compared with the total bachelors living, the general mortality seems to be little higher than that due to the earlier periods of life. But all this is reversed with regard to the married men. With them, the greatest number of deaths occurs between the sixtieth and seventy-fifth years of life, at which period the mortality is high as compared with the number living. Consequently, when the total deaths of the married at all ages are compared with the total living, a high mortality seems to have prevailed; because the persons were all so much older when they died than were the bachelors. The summary of the columns, therefore, and comparing the total deaths of the married at all ages with the total deaths of the bachelors, necessarily leads to a false conclusion. The comparison, to be correct, must be limited to comparing the mortality of each class at each several age, or to the mean age at death, both of which lead to the same conclusion, and that a true one.

From the Tenth Detailed Report (1864).

In the Ninth Detailed Report, attention was directed to the striking difference which prevailed in Scotland between the death-rates of the married and unmarried men at each quinquennial period of life. The difference in favour of the married men was so unexpected, and so great, that it was deemed necessary to investigate the subject further; and accordingly the Registers for 1864 have been carefully gone over, in order to prepare Table XXX, which shows the deaths of all males 20 years of age and upwards, who died in Scotland during the year 1864. These deaths are arranged in two columns, according as the persons had been married or unmarried, and according to their ages—the ages being arranged in quinquennial periods, in order to correspond with the ages of the living as ascertained at the Census of 1861.

It is satisfactory to find that the facts relative to the very different death-rates of the married and unmarried men in the whole population of Scotland for the year 1863, which were for the first time brought before the notice of the public in last Detailed Report, are in every point corroborated by the researches into the mortality of the married and unmarried for the year 1864. On a point of this kind, however, which had never been previously investigated for a whole population, it was desirable to give the fullest information possible. Accordingly, the whole details for 1864 are given in Table XXX, and an additional table is added (Table XXXI), showing the percentage of deaths in the married and unmarried men at each quinquennial period of life during 1863 and 1864, and also the mean mortality at each age for these two years. The remarks may therefore be confined to the mean of the two years, as being a nearer approximation to the truth than any one year's results can be, though it must be confessed that the two years' tables corroborate each other to a most surprising extent.

Above the age of 85 the numbers for both classes were too small to yield trustworthy or steady results; and so few attain these extreme ages, that no discrepancies in the results at such ages, could affect the conclusions deduced from the mortality at the other ages. Small, however, as are the numbers who attain such extreme ages, the difference, on the whole, preponderates in favour of the married men.

TABLE XXX.—*Married and Unmarried Men at each Quinquennial Age, in Scotland in* 1864, *the Deaths at the same Ages, and the Proportion of Deaths to the Living at each Age.*

Ages above 20 Years.	Husbands and Widowers in 1864.			Unmarried Men in 1864.		
	Number Living.	Deaths.	Per-cent of Deaths.	Number Living.	Deaths.	Per-cent of Deaths.
20– 25	23,656	155	0·655	106,587	1,373	1·288
25– 30	54,790	428	0·781	48,618	788	1·620
30– 35	66,657	683	1·024	25,962	445	1·714
35– 40	64,285	807	1·255	15,857	287	1·809
40– 45	63,042	988	1·567	12,311	244	1·982
45– 50	54,833	995	1·814	8,824	195	2·209
50– 55	49,874	1,065	2·135	7,636	205	2·684
55– 60	38,212	1,064	2·784	5,550	185	3·351
60– 65	36,108	1,351	3·741	5,242	240	4·578
65– 70	22,123	1,203	5·438	2,848	187	6·566
70– 75	16,094	1,329	8·257	2,021	208	10·291
75– 80	9,756	1,161	11·890	1,081	154	14·246
80– 85	5,496	955	17·376	513	100	19·493
85– 90	1,713	508	29·655	151	40	26·490
90– 95	451	161	35·698	50	14	28·000
95–100	103	32	31·068	6	3	50·000
100 & above	28	9	32·166	2	1	50·000
Not stated	...	11	...	...	9	...
All Ages	507,221	12,905	2·544	243,259	4,678	1·923

TABLE XXXI.—*Mortality of Married and Unmarried Men in Scotland in* 1863 *and* 1864, *and Percentage of Deaths to the Numbers Living at each Age.*

Ages.	Husbands and Widowers.			Unmarried Men.		
	1863.	1864.	Mean Annual Per-cent of Deaths.	1863.	1864.	Mean Annual Per-cent of Deaths.
20– 25	0·597	0·655	0·626	1·714	1·288	1·501
25– 30	0·865	0·781	0·823	1·369	1·620	1·494
30– 35	0·907	1·024	0·865	1·475	1·714	1·594
35– 40	1·080	1·255	1·167	1·595	1·809	1·602
40– 45	1·248	1·567	1·407	1·689	1·982	1·835
45– 50	1·594	1·814	1·704	2·028	2·209	2·118
50– 55	1·774	2·135	1·954	2·684	2·684	2·634
55– 60	2·444	2·784	2·614	2·558	3·351	2·854
60– 65	3·385	3·741	3·563	4·330	4·578	4·454
65– 70	5·149	5·438	5·293	5·477	6·566	6·021
70– 75	8·055	8·257	8·156	10·143	10·291	10·217
75– 80	11·681	11·890	11·785	14·542	14·246	14·394
80– 85	17·400	17·376	17·388	19·688	19·493	19·540
85– 90	28·571	29·655	29·113	21·192	26·490	23·841
90– 95	30·512	35·698	33·105	42·000	28·000	35·000
95–100	38·835	31·068	34·952	50·000	50·000	50·000
100 & above	53·571	32·166	42·868	00·000	50·000	25·000
All Ages	2·338	2·544	2·441	1·723	1·923	1·823

From the Eighth Decennial Census of Scotland.

TABLE LX.—*Mortality of Married and Unmarried Men in Scotland during the years* 1863 *and* 1864, *and the mean Annual Percentage of Deaths to the Living at each Age.*

Ages.	MARRIED MEN AND WIDOWERS.				UNMARRIED MEN.			
	Mean Number Living at each Age.	Number who Died in		Mean Annual Percentage of Deaths at each Age.	Mean Number Living at each Age.	Number who Died in		Mean Annual Percentage of Deaths at each Age.
		1863.	1864.			1863.	1864.	
20– 25	21,499	137	155	0·680	109,316	1,251	1,373	1·200
25– 30	54,753	469	428	0·820	50,779	666	788	1·432
30– 35	66,795	600	683	0·960	26,684	383	445	1·551
35– 40	64,347	690	807	1·162	16,027	253	287	1·685
40– 45	63,171	782	988	1·401	12,522	208	244	1·805
45– 50	54,996	869	995	1·695	8,880	179	195	2·106
50– 55	50,076	880	1,065	1·943	7,701	205	205	2·662
55– 60	38,712	929	1,064	2·573	5,590	142	185	2·934
60– 65	36,132	1,216	1,351	8·553	5,251	227	240	4·437
65– 70	22,861	1,134	1,203	5·113	3,022	156	187	5·658
70– 75	17,031	1,291	1,329	7·692	2,219	205	208	9·328
75– 80	9,854	1,135	1,161	11·750	1,138	157	154	13·620
80– 85	5,461	953	955	17·469	548	101	100	18·431
85– 90	1,719	488	508	28·970	164	32	40	21·951
90– 95	448	137	161	33·259	47	21	14	38·298
95–100	100	40	32	36·000	6	3	3	50·000
100 & above	30	15	9	40·000	2	...	1	25·000
Not stated	...	4	11	...	...	5	9	...

These statistics seem to prove that the married state is the condition of life best fitted for mankind; and that the prolongation of life which attends that state, is a special provision of nature to protect the father of a family, in order that he may provide for his offspring, and superintend their rearing. It is quite true, however, that this special prolongation of life is based on fixed laws of nature. The married man is, in general, not only more healthy, vigorous, and free from disease than the unmarried, but he is also more regular in his habits, is better housed, better fed, and better attended to. Indeed, the married men may, in one sense, be accounted as selected lives; for the weak, the delicate, those suffering from disease of any kind, the dissipated, the licentious, do not marry, so that all such are found among the ranks of the unmarried men; and as all such are known to die at a much higher ratio than the robust and temperate, we have what may be called the natural explanation of the difference which exists in the death-rates of the married and unmarried.

Even this, however, would only satisfactorily account for the difference in the death-rates of the married and unmarried during the earlier years of life—say from 20 to 40 years of age—but quite fails to explain the difference in the death-rates at the higher ages. Thus, almost all men who remain unmarried from natural infirmity of constitution, or from being addicted to intemperate or licentious habits, die before they have attained their fortieth year. Scarcely one such survives the critical climacteric age of seven times seven. All men, therefore, who have survived their fortieth year, and certainly all those above 50 years of age, must be considered, for all practical purposes, as selected lives; and if a difference should be found to exist in the death-rates of the married and unmarried above that period of life, it can rationally be only referred to causes connected with their being married or unmarried. But Table LX proves that at every separate quinquennial age, from 40 years up to extreme old age, married men died in a much lower proportion than the unmarried. It is, therefore, only reasonable to conclude that the lower death-rate of the married men can be alone attributed to the influences of marriage, and to the habits which attend marriage in this country.

Form of Life Policy in the year 1721.

THROUGH the kindness of Mr. Bailey, we are able to lay before our readers a copy of one of the first life policies granted by the London Assurance Corporation:—

Government Stamp,
3s. 10d.

No. 12.

BY THE GOVERNOR AND COMPANY OF THE LONDON ASSURANCE OF HOUSES AND GOODS FROM FIRE.

In the Name of God, Amen. MR. THOMAS BALDWIN of St. Margarets Westminster Doth make Assurance, and causeth himself to be assured upon the Natural Life of MR. NICHOLAS BOURNE of St. Margarets Westminster for and during the Term and Space of Twelve Calendar Months, to commence the First day of December next in the Year of Our Lord One Hundred Thousand Seven Hundred and Twenty-one and fully to be compleat and ended. And it is Declared, that this Assurance is made to and for the Use, Benefit, and Security of the said Thomas Baldwin, his Executors, Administrators, and Assigns, in Case of the death of the said Nicholas Bourne within the Time aforesaid, which the above Governor and Company do allow to be a good and sufficient Ground and Inducement for the making this Assurance, and do agree that the life of him the said Nicholas Bourne is and shall be rated and valued at y^{e} Sum Assured without any farther Account to given to them for the same: The said Governor and Company therefore, for and in consideration of Five Guineas per-cent to them paid, do assure, assume, and promise, that he the said Nicholas Bourne shall, by the Permission of Almighty God, live, and continue in this Natural Life, for and during the said Term and Space of Twelve Calendar Months, to commence as aforesaid. Or in Default thereof, that is to say, In case he the said Nicholas Bourne shall in or during the said time, and before the full End and Expiration thereof, happen to dye, or decease out of this world by any Ways or Means whatsoever, That then the abovesaid Governor and Company will well and truly satisfy, content, and pay unto the said Thomas Baldwin, his Executors, Administrators or Assigns, the Sum or Sums of Money by him Assured, and here underwritten, without any Allowance, Deduction, or Abatement whatsoever, to be be made out of the same, or any Part thereof, and without questioning why or wherefore this Assurance was or is made, and without any manner of Dispute, Plea, Pretence, or Allegation whatsoever, in Law or Equity, to the contrary: Hereby promising and binding themselves and their Successors to the Assured, his Executors, Administrators, and Assigns, for the true Performance of the Premisses, confessing themselves paid the Consideration due unto them for this Assurance by y^{e} Assured.

Provided always, And it is hereby declared to be the true Intent and Meaning of this Assurance, and this Policy is accepted by the said Thomas Baldwin upon condition that the same shall be utterly void and of no effect, in case the said Nicholas Bourne shall voluntarily go to Sea, or into the Wars, by Sea or Land, without License first had or obtained for his so doing, in Writing, under the Seal of the said Governor and Company. Any Thing in these Presents to the contrary hereof in anywise notwithstanding.

In Witness whereof the said Governor and Company have caused their Common Seal to be hereunto affixed, and the Sum or Sums by them Assured, to be here under-written, at their Office in London, This Twenty-fifth day of November, in the Eighth Year of the Reign of Our Sovereign Lord George by the Grace of God, of Great Britain, France, and Ireland, King, Defender of the Faith, &c.

Annoq. Domini One Thousand Seven Hundred and Twenty-one.

£100.

The said Governor and Company are content with this Assurance for One Hundred Pounds. prem^{m} rec^{d}.

Witness,

J. W. WILSON, *Sec^{y}.*

L. S.

On the Reserve that should be made for Policies on Recently Assured Lives. By Richard Teece, *of the Australian Mutual Provident Society.*

[Read before the Institute on 26 April 1880.]

THE question of the reserve which a life assurance society should hold, in order to meet the claims under the policies issued by it as they mature, is one of such vital importance that I am led to hope that, notwithstanding the many and valuable papers on the subject which have from time to time appeared in the *Journal of the Institute of Actuaries,* the following reflections may not be without interest, even though they may not throw much fresh light on the subject. I may premise by stating that the conclusions arrived at, and hereafter referred to, are the result of an investigation undertaken as a matter of curiosity, and carried on in the midst of laborious and urgent official duty; consequently the scheme propounded may appear somewhat crude and ill-defined, and doubtless lacks the completeness which a more mature consideration of the point taken up would possibly have developed. Further, the results given here were arrived at before Mr. Sprague's paper "On the Construction of Select Mortality Tables" (*Journal of the Institute,* vol. xxi, p. 229) appeared in print; otherwise I should probably not have undertaken the investigation.

What I have aimed at is the discovery of some method of arriving at the requisite reserves for new policies, which shall be somewhat further removed from the realm of pure theory than some others which have been suggested; a method which can be readily applied in practice; and, what is of more importance, one which can be made intelligible to the ordinary man of business. With all respect for the opinion of so eminent an authority and so indefatigable a labourer as Mr. Sprague, I cannot think that the tables prepared by him, and appended to his paper just referred to, can ever be made practically available for the ordinary valuations of a large office; for the use of these tables would, for policies less than five years in force, necessitate either an individual valuation or a separate classification of the policies for each year of assurance; while, I presume, the pure premiums, if such were valued, would require to be deduced from these tables, or from some other, such as those of Mr. Geo. King. The question of the pure premiums alone would, I fear, be a serious objection in practice. Offices, as a rule, will not go to the trouble and expense of continually writing-up new classification books in order to embody pure premiums deduced from such tables as those of Mr. Sprague or Mr. King. In practice, I think it will be found (speaking of pure premium valuations) that offices which have changed their basis and adopted the H^M Tables will now retain them (subject, possibly, to certain adjustments); while those which contemplate a change would be strongly inclined to follow their example.

Here I may advert to a point which must ever appear somewhat anomalous to the non-professional mind. The H^M Table, being the expression of the actual experience of assured lives for various ages at entry and for various periods of duration, would naturally be held to be the safest guide for the valuation of an ordinary life office. It is found, however, that this table does not require a sufficient reserve for policies of more than five years' duration, but that the $H^{M(5)}$ Table, which necessitates a higher reserve, is more reliable for the valuation of such policies. As, then, policies more than five years old require more than the average (H^M) reserve, it might be inferred that policies of less than five years of age should require less than the average (H^M) reserve. It now appears, however, that the H^M Table does not require a sufficiently high reserve either for old or young policies. It seems to me that this was the light in which the late Professor Pell viewed this table when he said (*Journal of the Institute,* vol. xxi, p. 137) "It appears to me, however, that the H^M Table does not represent any existing rate of mortality whatever." Notwithstanding this apparent anomaly, I think there can be little doubt that the H^M Table will continue to form the basis of valuation in the majority of our most important and successful life offices. What we require to know is what adjustments or modifications are necessary to render it more applicable to the practical labour of valuations.

I assume that there is a tolerably fair consensus of opinion as to the propriety of employing the $H^{M(5)}$ Tables of annuities and reversions with H^M pure premiums, in the valuation of policies which have been in force for five years and upwards. These tables are used by Mr. Valentine, in his paper, published in the *Journal,* vol. xviii, p. 229; they are recommended by Mr. Bailey (*Journal,* vol. xxi, p. 125); Mr. Berridge considers the combination "to be practically the most perfect measure of mortality which we have" (*vide Journal,* vol. xxi, p. 152); even Mr. Sprague appears to imply (*vide Journal,* vol. xxi, pp. 229 and 230) that in the light of existing knowledge the use of this combination is safe and prudent. I may here remark, in passing, that it appears to me to be utterly misleading to use the $H^{M(5)}$ annuities and reversions except in combination with the H^{M} pure premiums, and for the following reason:—When it is stated that an office has employed the $H^{M(5)}$ tables in a valuation the inference intended to be conveyed, and which is certainly drawn, is that such office has aimed at maintaining a very strong reserve for its liabilities—a reserve of a more stringent character than would be necessitated by the use of the H^{M} Tables. But if anything more than the H^{M} pure premiums has been valued, if credit has been taken for a considerable portion of the loading, it may be that the reserve will be smaller than would be produced by a pure H^{M} valuation. In such event, will anyone maintain that such a use of the $H^{M(5)}$ Table is admissible—not to use a stronger term? I have not lost sight of the fact that if full information be given concerning the premium valued, the professional actuary can readily form an opinion as to the sufficiency or otherwise of the reserve; but what I desire to maintain is that the policyholders—the general public—should be informed as to the nature of the reserve made to meet their claims whenever they may mature.

I shall assume it conceded, then, that it is desirable for an office making a valuation with a view to a distribution of profits, to employ the $H^{M(5)}$ annuities and reversions with the original H^{M} pure premiums in respect of policies in force for five years and upwards. Having reached this point, then, let us ask how is an office to place itself in a position to stand a valuation of this stringent character? And this brings me at once to the consideration of the question I have proposed. I start, then, with the case of a young office doing a considerable amount of new business, which has just arrived at the period of its first quinquennial investigation, and which is anxious in the future to maintain the standard of the $H^{M(5)}$ Tables for all its risks which have been in force for five years and upwards. On what principles is such an office to conduct its first valuation? (Of course, what is true of such a valuation will be true, as far as new policies are concerned, at any subsequent one.) On page 252 of vol. xxi of the *Journal*, Mr. Bailey is reported to have said, "In a valuation it is not very material what reserve is made for recently-effected policies, so long as they are not treated as assets." By "recently-effected", I presume he meant, speaking generally, policies less than five years in force. That statement, I opine, would be qualified by Mr. Bailey, if the valuation had reference to new offices transacting a large business. The following figures relate to the business of various offices whose published reports have come under my observation, and will serve to illustrate my arguments. The distinctive characteristic of these offices, the rapid increase of new business, is common to all offices in these Australasian Colonies, and is not altogether unknown in England.

TABLE A.

Designation of Office.	At End of Years.	Business in Force.		Increase.	
		No. of Policies.	Sums Assured.	No. of Policies.	Sums Assured.
A	20	8,888	3,966,191	...	...
	25	17,511	6,977,795	8,623	3,011,604
	30	35,218	13,392,121	17,707	6,414,326
B	5	4,989	1,833,819	...	...
	9	10,314	3,744,997	5,325	1,911,178
C	5	1,327	426,771	...	...
	10	5,209	1,666,150	3,882	1,239,379
D	5	973	319,713	...	...
	8	2,772	765,103	1,799	445,390
E	5	6,964	1,849,825	„	„

From these figures it will be seen that Office A, at the age of 25 years, had in force twice the business which it had at 20 years; and that its business was again doubled in the succeeding five years of its history. Office B, at 9 years of age, was more than twice as large as it was at 5 years of age. Office C was nearly four times as large at the end of 10 years as it was at the end of 5 years, and the same principle holds throughout. Now, bearing in mind that it is desirable that the liability in respect of policies of five years of age and upwards in these offices should be valued by the $H^{M(5)}$ Tables, it seems to me a matter of the very greatest moment "what reserve is made for recently-effected policies."

In vol. xv of the *Journal,* Mr. Sprague has propounded a scheme for the valuation of these recently-effected policies, which appears to me to be open to grave objections. Mr. Sprague says (p. 426), "If the expenditure in obtaining new business has been such that the current claims and expenses have together absorbed the whole of the first year's premiums, then a simple method of valuation, which, in my opinion, is perfectly justifiable, will be to reserve for policies in their first year only sufficient to meet the unexpired current risk, and to consider that all other policies have been effected at the next higher age." Mr. Sprague goes on to say, that "If the expenses attendant on procuring new business have been still heavier, so that together with the current risk they have absorbed the whole of the premiums for the two first years, then for policies of less than two years' standing a reserve may be made for the unexpired current risk, and for policies of more than two years' standing a reserve equal to

$$A_{x+n}-(1+a_{x+n})\pi_{x+2}=(1+a_{x+n})(\pi_{x+n}-\pi_{x+2})\text{."}$$

In other words, the more extravagant has been the management, the less is the reserve necessary to meet the claims in the future. Theories of this specious and ingenious character may be all very well when they are advanced in such a place as the Institute of Actuaries, where they are sure to be subjected to the lynx-eyed criticism of able and experienced professional gentlemen; but when they are resorted to by struggling young offices, with the view of displaying an attractive, though unearned profit, with which to gratify expectant and to allure intending policyholders; and when the proceeding is justified and defended on the high authority of Mr. Sprague, the case assumes a different aspect. I confess that I am utterly at a loss to see any possible connection between the cost at which the business of an office has been obtained, and the reserve which that office should at any given time have on hand in respect of that business. For example, if an office has secured a policy for £100 at the cost of two years' premiums, Mr. Sprague, if I interpret him aright, would maintain that, at the end of two years, it is not necessary for the office to hold anything in reserve for such policy; while if nothing more than the loading of a single year had been spent, a reserve would be necessary. I fail to see why any distinction whatever should be made in the two cases, and it seems to me perfectly clear that the risk which an office incurs when it issues a policy is in no way affected by the cost at which such policy is obtained.

The great point to bear in mind is, that when any expedient, such as that just referred to, or such as valuing a portion of the loading, is resorted to, the advantage gained is merely temporary, and becomes an incubus on the office ever after, rendering the attainment of the $H^{M(5)}$ standard, before alluded to, an extremely difficult if not impossible matter. If it be allowable in certain cases to take credit for a portion of the loading in a valuation, it seems to me that the question to consider is, not how much loading may be so included, but rather how much should be excluded or reserved; and I do not think it a healthy sign when, as is sometimes the case, we see an office at a valuation reserving for future expenses and *profits* a percentage of the premiums, which has been lamentably insufficient to defray even the bare expenses in the past. This state of things is apparently due to the fact that the premiums charged are too low—an example of the desire which, in the existing state of high pressure competition, every new office manifests to do business at lower rates than those of its competitors. Now my contention is, that in view of the effort which every office should make to attain to the standard of an $H^{M(5)}$ valuation for such of its policies as may be five years or more in force, the distribution of their surplus by these offices, whose reserved loading is insufficient to defray their management expenses, is a proceeding which is not only imprudent, but which is utterly indefensible, and which cannot fail to seriously impair their resources.

With these preliminary observations, I proceed to describe the method of arriving at the reserve which should be made by an office in respect of policies less than five years in force, before it is justified in making any distribution of its surplus among its members. I have before alluded to the fact that there is a growing opinion, if it be not indeed generally admitted, that the H^M Table does not require a sufficiently high reserve for such policies. The truth is, that in valuing such policies we must have regard to the mortality which has been experienced in the past, no less than to that which is to be expected in the future. If the actual mortality of an office five years old has been equal to the expectation according to the H^M Table, a valuation by that table would seem to be sufficient, but for the fact that such an exceptionally heavy death rate would point to the conclusion that undesirable risks had been taken. If, then, it be the case (as it should be in every prudently managed office), that the mortality among policies under five years in force, is considerably less than would be expected according to the H^M Table, it seems to me that the necessities of the case will be met by increasing the reserve required according to the H^M Table by the amount of claims apparently saved. In order to enable me to apply this method to an actual case, I compiled the table, marked B, by extracting from the Institute Tables (H^M) the numbers exposed to risk, and the numbers dying for the years of assurance 0 to 4. The numbers from the age 85 onwards

TABLE B.

Age x	Exposed.	Died.	l_x	d_x	q_x
0	21	...	...	...	...
1	47·5	...	...	...	...
2	65	1	...	...	...
3	79	...	...	...	...
4	93·5	...	...	...	...
5	95	1	...	...	...
6	109·5	1	...	...	...
7	140·5	...	...	...	...
8	188·5	...	...	...	...
9	247	1	...	...	...
10	298	3	100,000	479	·004790
11	332·5	...	99,521	380	·003818
12	352·5	1	99,141	310	·003127
13	399	1	98,831	267	·002702
14	501	1	98,564	251	·002547
15	648	2	98,313	262	·002665
16	836·5	...	98,051	300	·003060
17	1090	5	97,751	365	·003734
18	1405·5	8	97,386	457	·004693
19	1916·5	12	96,929	540	·005571
20	2707	16	96,389	581	·006028
21	3908·5	26	95,808	599	·006252
22	5584·5	31	95,209	605	·006354
23	7537	49	94,604	585	·006184
24	9632	69	94,019	564	·005999
25	11832·5	51	93,455	555	·005939
26	13657·5	83	92,900	562	·006050
27	15335·5	83	92,338	566	·006130
28	16729	129	91,772	598	·006516
29	17857	118	91,174	615	·006745
30	18702·5	136	90,559	646	·007134
31	19025	122	89,913	653	·007263
32	19135	159	89,260	665	·007450
33	19052·5	150	88,595	666	·007517
34	18597·5	135	87,929	666	·007574
35	18339·5	143	87,263	658	·007540
36	17805·5	121	86,605	670	·007736
37	17052	130	85,935	692	·008053
38	16234·5	142	85,243	694	·008141
39	15494	150	84,549	716	·008468
40	14739·5	109	83,833	738	·008803
41	14049	138	83,095	742	·008930
42	12995·5	121	82,353	749	·009095
43	12157	106	81,604	785	·009620
44	11101·5	104	80,819	815	·010084
45	10325	111	80,004	840	·010499
46	9582	117	79,164	883	·011154
47	8867·5	97	78,281	917	·011714
48	8251·5	108	77,364	915	·011827
49	7567	92	76,449	912	·011930

TABLE B—(*continued*).

Age x	Exposed.	Died.	l_x	d_x	q_x
50	6977·5	82	75,537	930	·012312
51	6438	76	74,607	939	·012586
52	5913	78	73,668	947	·012855
53	5329	76	72,721	1,015	·013957
54	4762	55	71,706	1,082	·015089
55	4280	88	70,624	1,139	·016128
56	3787	69	69,485	1,218	·017529
57	3364	58	68,267	1,314	·019248
58	3024	59	66,953	1,399	·020895
59	2695	59	65,554	1,490	·022729
60	2442·5	67	64,064	1,602	·025006
61	2147·5	59	62,462	1,675	·026816
62	1880	55	60,787	1,740	·028625
63	1586	43	59,047	1,777	·030095
64	1308	48	57,270	1,829	·031936
65	1140	36	55,441	1,860	·033549
66	963	36	53,581	1,977	·036897
67	812	26	51,604	2,062	·039958
68	654·5	27	49,542	2,158	·043559
69	502	31	47,384	2,326	·049088
70	390	21	45,058	2,472	·054863
71	302	18	42,586	2,493	·058540
72	226	13	40,093	2,515	·062729
73	181	6	37,578	2,619	·069695
74	142	16	34,959	2,404	·068766
75	105·5	12	32,555	1,999	·061404
76	68	1	30,556	1,783	·058352
77	55·5	2	28,773	1,713	·059535
78	33·5	...	27,060	1,488	·054989
79	27·5	5	25,572	1,655	·064719
80	20·5	...	23,917	2,507	·104821
81	15	...	21,410	2,948	·137692
82	11	4	18,462	3,295	·178474
83	6·5	1	15,167	3,505	·231094
84	4	2	11,662	3,415	·292832
85	2	...	8,247	2,515	·304960
86	2	...	5,732	1,962	·342289
87	1·5	...	3,770	1,208	·320425
88	1·5	...	2,562	747	·291570
89	1·5	...	1,815	396	·218182
90	·5	...	1,419	370	·260747
91	...	...	1,049	252	·240229
92	...	...	797	281	·352572
93	·5	...	516	215	·416666
94	1	...	301	156	·518274
95	·5	...	145	91	·627586
96	...	...	54	54	1·000000
		4,412		100,000	

being too small to be reliable, I substituted the figures of the H^M Table in the unadjusted p_x column; the unadjusted l_x and d_x columns were then deduced and graduated by Mr. Woolhouse's method, the formula employed being the very convenient one given by Mr. Geo. King in the *Journal of the Institute,* vol. xxi, p. 52. The adjusted q_x column was then deduced in the ordinary way. Some rather striking irregularities still appear in this column at the higher ages, but, for reasons which I need not explain, I did not think it necessary to apply any further graduation. For obvious reasons, the l_x column commences with a radix of 100,000 at age 10. The use to which this table was put will be seen presently.

I have taken for illustration the ordinary whole-life business of an office which has published the sums assured it had in force at each age of less than five years' duration, the net premiums and corresponding liability being also furnished. The business comprised 11,694 policies for assurances, amounting to £4,739,600, and was acquired, it is believed, at a tolerably uniform rate during the five years; the formula used was $A_{x+n}-\pi_x\ (\cdot 5+a_{x+n})$, π_x being the pure H^M premium for the age at entry; the rate of interest used was 4 per cent. The liability by the H^M Table was reported to be £160,072; but this reserve was increased by the actuary of the society from whose report the above figures have been taken. The justification for an increased reserve, I hold, lies in the fact that while the expected claims in respect of the policies at risk during the five years were, according to the H^M Table, £113,002, according to my Table B they were only £96,457, and the difference of £16,545, being payments deferred only, and not avoided, should be added to the reserve above, making £176,617 as the sum which should be held before any division of profits could fairly be made. I defend this method on the ground that a valuation by the H^M Table is only true if the claims which have arisen, have been in accordance with the H^M Table; that such claims never ought to reach the amount shown by that table, but, on the contrary, should not exceed that shown by Table B; and that the difference between these two sums should therefore be added to the pure H^M reserve. Had the $H^{M(5)}$ Table been used, the liability would have been £223,540, being £63,468 in excess of the H^M reserve, and £46,923 in excess of the value by my method. The actual claims which fell in during the five years amounted to only £64,675, being less by £31,782 than the expectation according to Table B. It may be held by some that this amount too should be added to the reserve; such a course would have the effect of increasing the reserve to £208,399, or within £15,141 of the value by the $H^{M(5)}$ Table. I hold, however, that the sum of £31,782 is legitimate and real profit, and as such should be divided among the members, for the reason that the Table B shows the mortality which will be experienced among policies of less than five years' duration if, as we must assume, the Institute of Actuaries' data are to be relied on; and therefore, when due provision has been made for the claims to be met in the future, payment of which has been temporarily deferred, and has not been provided for by the H^M valuation, any further advantage, from a still more favourable mortality, may be fairly assumed to be permanent. In other words, the result of this valuation may be taken as a very strong indication of the fact—that the office to which this business belongs, is surrounded by such circumstances that the vitality of its members will, throughout, be permanently greater than the average of the 20 British Companies whose experience has formed the basis for this valuation. From these figures it will be seen how grievously a young office may err, and to what an extent it may misrepresent the real state of things, by comparing (as is frequently done) the actual mortality with that expected by the H^M Table, any difference in favour of the office being loudly proclaimed as an evidence of its very favourable mortality experience; whereas I have shown that the H^M Table does not afford a true indication of the deaths to be expected. In the case under consideration, the H^M Table would indicate claims to the amount of £113,002, whereas the amount actually expected was only £96,457; had the claims in this office, therefore, reached the larger figure, that fact would have been a fair indication that due caution had not been exercised in the selection of the lives. The method which I have described, while it does not pretend to minute accuracy, is one which can be applied with great facility, as, if the classification books are kept posted up, the work of ascertaining the expected claims by both H^M and B Tables can be performed in a few hours; it appears to me, further, to be a method which is intelligible to the ordinary business man; it will materially assist the office adopting it to pass to the $H^{M(5)}$ valuation when the policies shall have been more than five years in force; and it has the merit of providing against the distribution of any profit "that has not actually been realized out of the profits of the past"—a point which Mr. Sprague (vol. xv, p. 428) somewhat inconsistently, it appears to me, maintains should be kept in view by an office which declares "an unconditional cash bonus". Of course, I only contend for the application of my method when the object of the valuation is to ascertain the divisible surplus; it cannot be held to apply in cases of amalgamation, or in proof of the ability of an office to meet its engagements. I have not entered on the question of the rate of interest which should be employed; 3 and 3½ per-cent seem to be the rates usually adopted in England, and I think the circumstances of these colonies amply justify the use of a rate one per-cent higher than would be considered prudent in England.

DISCUSSION.

The PRESIDENT (Mr. A. H. Bailey)—Gentlemen, I think you will agree with me that it is some evidence of the growing influence of the Institute that a paper so carefully elaborated as the one before us should be sent to us by a gentleman so far off as the Australian Colonies. The subject which Mr. Teece has discussed is one which has been many times before us. My own views have been given both in writing and verbally, and I do not wish to trouble you with them to-night. In the main I agree with Mr. Teece. When I formerly said, rather hastily, that there was little importance to be attached to the recently effected policies, I had in view the circumstances of this country. Since the passing of the Act of 1870, there may be said to be no new companies here, and I feel sure that in all the existing companies when valuations are in hand the important matter to be considered is the policies of long standing. There is a strong disposition to take the new business into account directly or indirectly as an asset. The one thing in my mind was that that should not be done. You know that I am at issue with Mr. Sprague, and cannot accept his theories on this subject. I should say at once that the sums that are now paid for business ought not in my judgment to be paid. But if that expense is incurred, it ought not to be thrown on the future. You must pay your way as you go.

Mr. WALFORD—I saw Mr. Teece here some few years ago, and found he was much interested in the proceedings of the Institute, and in actuarial science generally, and I am not at all surprised to find him dealing with some of the questions which have been discussed here from time to time, and trying to apply scientific principles to the circumstances of companies which he comes most in contact with. There are two or three points on which I cannot quite agree with Mr. Teece. The first is the idea of treating the benefit resulting from a favourable mortality in any way as a divisible surplus. It may indeed happen that in process of time the Australian mortality will be found so favourable as to justify that course of action, but it has always appeared to me, as well from actual observation as from theory, that if the mortality is very light in the early years of an office, it must be correspondingly heavy in future years; and that all that can happen from a light mortality in early years, or at any period of an office's history, is the benefit of the premiums received by reason of persons living longer than the tabular expectation, and the interest on the money which would have been paid if they had become claims. The surplus resulting from light mortality does not seem to me to be in any way an available asset beyond those two considerations. Then, a young office may have a great deal too much business for its own financial convenience; and, at the cost at which business is now obtained, it may, instead of presenting flourishing results at the period of its early valuations, appear very much the other way. The one conclusion to be drawn from that state of things is the very obvious one, that no mutual office ever can start while the cost of business is so great. As to the proprietary offices, if they exceed what can be afforded to be spent from the life fund, the capital has to bear it. It does not seem to me very material which way it is met, so long as it is honestly met. Although I am one of those who have sympathized a good deal with young offices, and would do anything I could to help them forward, and I believe that is entirely the view Mr. Sprague has taken with regard to the theories he has advanced—still I do not think it is a help to young offices to encourage them in making undue expenditure. If, however, they do make that expenditure, it is entirely a shareholder's matter, and the policyholders must be kept safe at all hazards. But any management that unduly increases the rate of expenditure of an office runs the risk of debasing the character of the business, and that in my mind is the real evil which high expenditure is tending to. To pay a high commission and get inferior lives must end in disaster. I am interested to see if there be any foundation for the belief that, because these young offices have shown very favourable mortality, therefore the duration of life in Australia may be expected to be longer than it is here. That is a theory I do not believe in. On the contrary, as far as I am able to judge from careful observation, the mortality in the Australian Colonies is likely to be rather heavier than amongst ourselves, more particularly in the younger male lives up to 35, for the reason that I find the deaths from accidental causes at those ages are very much in excess of the deaths in this country. Indeed, they are in such marked excess that it seems to me it would be impossible for the life offices there to escape feeling it. Whatever tendency, on the one hand, there may be by reason of the climate or other causes, to favour life there, I think that this drawback, and some others which might be mentioned, will more than counteract that tendency.

Mr. A. Baden—I incline to think that Mr. Walford has misunderstood the tenour of Mr. Teece's remarks. As I understand, he was enforcing the very point which Mr. Walford has so admirably put before us. He does not advocate the application of a surplus obtained by favourable mortality in the first five years to increase the bonus, but the contrary. His arguments all tend to show that it must—if not entirely, at all events to a considerable degree—be added to the reserve that is brought out by the ordinary theoretical mode of valuation; for the very reason that the mortality, if it did not take place in the first years, would take place by way of addition to the mortality to be expected in subsequent years.

Mr. J. R. Grimes—Mr. Teece's argument appears to be based upon applying the $H^{M(5)}$ Table to policies which have been less than five years in force, and therefore seems to me faulty. When he applies the H^M Table, he says the whole of the value under that table is required as reserve, and it is to be kept intact. It is only when he makes the valuation by a table which does not apply at all to policies less than five years in force, that he allows of any profit. He says: "Had the $H^{M(5)}$ Table been used, the liability would have been £223,540, being £63,468 in excess of the H^M reserve, and £46,923 in excess of the value by my method. The actual claims which fell in during the five years amounted to only £64,675, being less by £31,782 than the expectation according to Table B."

The Chairman—I think the gist of Mr. Teece's paper is that the H^M Table shows an insufficient reserve for policies of under five years' duration.

Mr. Baden—He actually puts it, I think, in the form which I spoke of just now—"The liability by the H^M Table was found to be £160,072; but this reserve I hold to be insufficient, for the reason that while the expected claims in respect of the policies at risk during the five years were, according to the H^M table, £113,002, according to my Table B they were only £96,457, and the difference of £16,545, being payments deferred only, and not avoided, should be added to the reserve above, making £176,617 as the sum which should be held before any division of profits could fairly be made."

Mr. G. W. Berridge—I think, Sir, you will find that Mr. Teece has divided the difference between the expected mortality according to the H^M Table, and that which actually did happen in the office, into two portions, so much of the difference as is represented by the difference in the expected mortality between the H^M Table and this Table B, which is deduced from the years 0 to 4, he adds to the reserve; the rest of it he considers may be treated as available for distribution. Mr. Teece has attacked the problem which has been attacked by Mr. Sprague, Mr. King, and myself, before now, in rather a similar form, namely, to estimate what has been unhappily called suspended mortality. Very elaborate tables have been made for that purpose, the difficulty being that they are very cumbrous in use. Mr. Teece, I am bound to say, has put before us a method very simple of application, and I should like to draw the attention of the members to the comparative results. Mr. Teece makes an addition of about 10 per-cent to the H^M reserve in this particular case. Whereas the H^M reserve for the early policies would be £160,000, he adds, on account of the difference between the H^M mortality and that shown by Table B, £16,000; that is, he makes an addition of 10 per-cent to the reserve. I have looked at Mr. George King's tables, and I find that this addition is considerably less than he would say is correct. If we take the reserve according to the H^M 4 per-cent table as £1000, then the reserve according to the Analyzed Tables at 4 per-cent would be £1,232, which is an addition of 23 per-cent. That seems a large addition to the reserve, a very much larger addition than was brought out in my table—only 5 per-cent. I rather suspect that Mr. Teece has hit the happy medium. I may say that I find these Analyzed Tables very difficult to apply.

Mr. H. W. Manly—I have been a little puzzled at the mode in which Mr. Teece worked out his extra reserve. I cannot bring myself to agree that the extra reserve should be dependent on the difference between the mortality that has actually occurred, or might have occurred, by one table, and the expected deaths by some other table; in other words, I cannot accept the doctrine that the difference between the actual and the expected mortality, is the present value of the deferred mortality. That appears to me what Mr. Teece is aiming at in his method. The object of the paper seems to be to arrive by some easy method at the necessary reserve to be made for policies of less than five years' duration, and for which the H^M and $H^{M(5)}$ Tables combined are not applicable. If you desire to arrive at a close approximation to the result, you have Mr. King's Analyzed Tables, where the reserves are calculated for a model office, and are based upon what I conceive to be the true theory, that is on mortality tables deduced by tracing the lives in groups of ages from their entrance to the end. Now, if you want to make an approximation to the reserve required, it is really a very simple thing, and can be done in ten minutes by the use of the tables in Mr. King's papers. If you first make your valuation (say) by the H^M pure premium tables, or by the H^M and $H^{M(5)}$ combined, and apply the proportions which are given in Mr. King's table, you will get an approximation so close that the difference in the resulting surplus would be less than that caused by an accident occurring to a policy-holder of £200. With experience and the exercise of a little judgment and attention to the rules, you could obtain an exceedingly close approximation in ten minutes. There are the tables ready to hand, and if there be any difficulty arising from a rapid increase of new business, or the business has been irregular, you have the means there also of making the necessary corrections; and it appears to me that a process of this kind proposed by Mr. Teece is not only unscientific, but is such that no one with any idea of making approximations would ever adopt with Mr. King's tables beside him. I have failed to grasp the meaning of the remarks in the first part of Mr. Teece's paper, as to the principles upon which the valuation should be based. Of course the object of the valuations should be to ascertain the amount of reserve you ought to have to meet all your future claims. But, some of us want more than that, namely to get an equalized bonus over the whole period of life. To get that, you must, with the heavy commissions now being paid, spread your commissions and expenses over the whole period of the policy. I think the high rate of expenditure in Australia must arise in a great measure from the practice of paying commissions in a lump sum—brokerage commission. They pay 1 or 1½ or even 2 per cent on the sums assured, and no renewal commissions. If you do that, you certainly do relieve the future at the expense of the present; and if then you want to equalize the bonuses, you must spread these extraordinary expenses over the whole existence of the policies. [Mr. BADEN—It is not peculiar to Australia.] I think if Mr. Teece had given Mr. King's paper that attention which it really deserves, he would not have sought for the proper additional reserves in a process like this.

Mr. T. Y. STRACHAN—I believe there are, at least, 19 or 20 English offices whose percentage of expenses on premium receipts is larger than the proportion of office premiums reserved at the last valuation; and if the word "expenses" is taken to include what it usually does include, that part of the expenses which capital costs, that is the surplus beyond what the capital realizes, but which is divided to the proprietors, then I think I would add another six offices to the 19 or 20, and make the number 25 or 26.

The PRESIDENT—Mr. Teece has endeavoured to arrive at some mode, a rough one perhaps, of providing a remedy for what the history of life offices shows to have been the case in years gone by, that the bonus declared at a first valuation could not be maintained. We know now that it was because the mortality increases as the duration of the policies increases. I think, therefore, that this paper may be welcomed as a contribution which may take a place in our *Journal*, and be discussed at our Institute.

REMARKS BY MR. SPRAGUE.

Mr. Teece, in the foregoing paper, has referred so frequently and so pointedly to the opinions I have on various occasions expressed, that it appears necessary I should make some answer to his criticisms. Although I cannot say that I approve of the tone and style of Mr. Teece's remarks, I am always pleased when persons who differ with me on professional subjects state their reasons for objecting to my views, as I believe that the truth is most likely to be elicited and recognized as such by free and unreserved discussion.

Mr. Teece, after quoting a passage from my paper "On the Proper Method of Estimating the Liability of a Life Insurance Company under its Policies" (*J.I.A.* xv, 411), says:—"If an office has secured a policy for £100 at the cost of two years' premiums, Mr. Sprague, if I understand him aright, would maintain that at the end of two years it is not necessary for the office to hold anything in reserve for such policy; while, if nothing more than the loading of a single year has been spent, a reserve would be necessary." Those who have read and understood my paper above mentioned, will not require to be told that this is not a fair representation of my views. It would, of course, be absurd to say that a reserve is necessary in the latter of the two cases supposed by Mr. Teece, but not in the former; and I have never said anything of the kind. Mr. Teece's mistake has arisen from his using the word "necessary", which I think he will not find in my paper. Under the circumstances supposed, if an adequate rate of premium is charged, and there is no obligation to pay a surrender value, it is not "necessary" for either office to hold anything in reserve for the policy. Nevertheless, I hold that the second office should make a substantial reserve for the policy, but I should advise the former office to make no reserve at all. It is quite true, as Mr. Teece says, that the risk which an office incurs when it issues a policy, is in no way affected by the cost at which such policy is obtained; but the reserve to be made by an office in respect of a policy, is to be determined by other circumstances besides the risk incurred. If the first two years' premiums have been absorbed by the expenses and current risk, there is no fund available out of which a reserve can be legitimately made in respect of the policy; but if the office has only spent a portion of the first year's loading, the company will have in hand the greater part of two years' premiums; and if no reserve were made for the policy, this sum would fall into the profit fund, and would have a tendency to raise the rate of bonus to a level at which it could not be permanently maintained.

A numerical example will make this clearer. Let us suppose, for example, that two offices, which we will call A and B, each issue in a year 1000 policies for £500 each on the lives of persons of the age of 30 at the annual premium of £12. 10*s.*; and let us trace the growth of the fund arising from the accumulation of the premiums on these policies in the two offices, assuming that there are no lapses, that the rate of mortality is the same in the two offices, and that the expenses are such as Mr. Teece indicates. Taking the rates of mortality deduced in my paper (*J.I.A.* xxi, 246), the deaths will be 5 in the first year and 8 in the second; and the progress of the two offices during their first two years will be approximately as follows:—

	OFFICE A.		OFFICE B.	
First Year's Premiums on 1000 Policies	...	£12,500	...	£12,500
Deduct Claims . . .	£2,500	...	£2,500	...
Expenses . . .	10,000	12,500	2,000	4,500
Funds at end of 1st year . .	...	0	...	8,000
Add Interest	...	0	...	320
Second Year's Premiums on 995 Policies . . .	...	12,437·5	...	12,437·5
		12,437·5		20,757·5
Deduct Claims . . .	4,000	...	4,000	...
Expenses . . .	8,437·5	12,437·5	1,000	5,000
Funds at end of 2nd year . .	...	0	...	15,757·5

At the end of the two years there are 987 policies in force in each office, and A has accumulated no reserve in respect of these policies, while B has accumulated a sum of nearly £16 for each policy (£15·96). If now we value the liabilities by the Institute 4 per-cent Table, we find that for each policy the value of the £500 assured, is £158·2, and the value of the premium of £12. 10*s.* is £222·2. Looking at these figures, will any one be bold enough to say that any reserve whatever is "necessary" to meet the liability under the policy? Arranging the figures in the form of a valuation balance-sheet, we have

Valuation Balance-sheet of Office A.

LIABILITIES.		ASSETS.	
Value of Sums Assured under 987 Policies . . .	£156,141	Value of Premiums receivable	£219,282
Provision for future Expenses and Profits; being 28·8 % of the value of the Premiums	63,141	Funds in hand . . .	*nil*
Balance, being Surplus . .	*nil*		
	£219,282		£219,282

Here we have virtually valued the net premiums for age 32; that is to say, we have made only such a provision for future expenses and profits as will prevent the appearance of any negative policy-values, but this provision has amounted to no less than 28·8 per-cent of the future premiums. Is Mr. Teece prepared to say that this provision is too small? If so, why is it too small? If we had made a smaller provision, we should have brought out a fictitious surplus composed of negative policy-values; for instance, if we had deducted only 25 per-cent of the value of the premiums, we should have had a fictitious surplus of £8,321. This method of procedure would properly be open to the charge Mr. Teece makes of displaying an attractive, though unearned profit; but such a procedure I have always consistently denounced in the strongest language. Mr. Teece, therefore, by his suggestion that my "specious and ingenious theories" sanction such a procedure, has done me a grievous injustice, for which he is in honour bound to make me such reparation as may be in his power.

For B the figures are—

Valuation Balance-sheet of Office B.

LIABILITIES.		ASSETS.	
Value of Sums Assured. .	£156,141	Value of Premiums receivable	£219,282
Provision for future Expenses and Profits; being 33·3 % of the value of the Premiums	72,951	Funds in hand . . .	15,578
Balance, being Surplus. .	5,948		
	£235,040		£235,040

Here we have deducted the whole of the loading as commonly understood, being 33·3 per-cent of the premium payable; and the result is to show a surplus of £5,948, being 23·9 per-cent of the premiums received by the office—a rate of profit which may well be maintained or even increased in subsequent years. If we had made only the same reserve for the liability as in the case of Office A,—that is, had made no reserve at all,—the surplus would have been increased so as to be equal to the total funds in hand, £15,758, which is 63·2 per-cent of the premiums received—a rate of profit which the office could not possibly expect to maintain in the future. The study of these figures will, I trust, satisfy even Mr. Teece that there are good reasons for making a larger reserve in the case of Office B than in the case of Office A, and that there is a real "connection between the cost at which the business of an office has been obtained, and the reserve which that office should at any given time have on hand in respect of that business."

It is not necessary to trace the subsequent progress of the two offices year by year; but it may be useful to consider what reserve should be made for the liability under a policy at the expiration of five years. I am of opinion that we should then value by the $H^{M(5)}$ Table, because we have reason to believe that this table represents more correctly than any other the mortality that will prevail among the lives. Still adhering to the same rate of interest, 4 per-cent, the single premium at the age of 35 is ·347427, and the annuity is 15·9669. With the help of these figures, we find the value of the sum assured, £500, is £173·7, and the value of the premium is £212·1. The difference between these is £38·4, being 18·1 per-cent of the value of the premium. It follows that, in order to avoid negative values, at least 18·1 per-cent must be deducted from the value of the office premiums. When this is done, the value of the policy is zero; and if it is desired to make a positive reserve for the policy, a larger percentage of the value of the premium must be deducted. If, for instance, the office is pledged to give for the policy a surrender value equal to one premium, £12. 10*s*., it must have in hand at least this amount on account of the policy; and if it makes no larger reserve than this for the policy, it must deduct £50·9(=38·4+12·5) from the value of the premiums—that is to say, 24·0 per-cent. If the surrender value of the policy is the very common one of one-third of the premiums paid, or £20. 16*s*. 8*d*., this amount must be reserved for the policy, and a deduction of £59·2(=38·4+20·8), or 27·9 per-cent of the value of the premium, must be made.

It must, I think, be admitted that the provisions for future expenses and profits, resulting from my "theories" of valuation, are of a very substantial character; and my writings cannot be appealed to, as sanctioning either the dangerously small provisions for those purposes, which are unfortunately made by some British offices, or the inadequate provisions, resulting in negative policy-values, which are made by a few offices of high standing that might have been expected to adopt more prudent methods of valuation than they actually adopt.

Mr. Teece proceeds to say that when any expedient, such as valuing a portion of the loading, is resorted to, the advantage gained is merely temporary. This, in my opinion, is quite a mistake. If, for such reasons as are explained above, an office has at its first valuation thrown off for future expenses and profits, say, 28·8 per-cent of the office premiums, instead of the whole of the loading, 33·3 per-cent, it may, in my opinion, with perfect propriety and safety, continue to throw off the same percentage, 28·8, in its subsequent valuations.

Mr. Teece is of opinion that the use of the $H^{M(5)}$ Table is "utterly misleading" except in combination with the H^{M} pure premiums. I entirely dissent from this opinion. I advocate the use of the $H^{M(5)}$ annuities and reversions because, as I intimated above, I believe the $H^{M(5)}$ Table is the best exponent we at present have of the mortality which is likely to be experienced among lives that have been insured for five years or upwards; and this is a sufficient reason for continuing to use it, even though some people may draw the erroneous conclusion that the office using it "has aimed at maintaining a very strong reserve for its liabilities." It is true that, as I shall presently show, the reserve I sometimes bring out by using that table, is less than would be brought out by a "pure H^{M} valuation", but the question to be considered is whether my reserve is not fully sufficient. It is certainly desirable "that the policyholders—the general public—should be informed as to the nature of the reserve made to meet their claims whenever they may mature"; but the best way to do this would be to give a table showing the actual reserves made for policies which were taken out at certain ages, and which have been in force for certain periods. This information is now required of Friendly Societies, under "The Friendly Societies Act, 1875", § 14 (*f*), and it might with very great advantage be extended to life insurance offices. It would be a very wholesome thing for the British offices which take credit for large amounts of fictitious assets in the form of negative policy-values, to have the fact brought home to themselves, and displayed in all its extent to the wondering gaze of the public at home and abroad. I have not, as Mr. Walford supposes, been actuated by a desire to help forward young offices; but it has been my object to point out, to both old and young offices, the danger of taking credit for negative policy-values, and to show them that, when new business has been obtained at a heavy expense, the proper allowance can be made for this circumstance in the valuations, without having recourse to the dangerous expedient of treating their recent policies as assets.

The question of the proper reserve to be made for recently-effected policies, is one that will come naturally to be considered in the second part of my paper "On the Construction and Use of a Series of Select Mortality Tables, to be employed in combination with the Institute $H^{M(5)}$ Table", which I hope very soon to submit to the Institute; but meantime it may be useful to make a few preliminary remarks.

It has been clearly proved, and is now admitted on all hands, that the rate of mortality among recently insured lives is very different indeed from that represented by any ordinary mortality table; and the proper reserve to be made for recently-effected policies can therefore not be determined by the ordinary formulas. It has been proposed by some persons that to the reserve given by the ordinary formula there shall be added the difference between the actual and the expected claims; and Mr. Teece advocates a modification of the same idea. It is clear, however, that the theoretically correct reserve can only be determined by the help of tables of annuity-values for every age at entry. Such tables, of which I gave specimens in my above-mentioned paper, enable us to determine the proper amount of the reserve, not only for recently-effected policies, but for policies of all durations; and by their means we can, if desired, make the proper allowance for the heavy expense at which new business is sometimes obtained. There will be no difficulty in using them practically in the valuations of any large office, and it will not be absolutely necessary for this purpose to have a separate classification of the policies for each of the first five insurance years. Still less will it be necessary to make an individual valuation of the recent policies. The pure premium difficulty mentioned by Mr. Teece is easily got over by having another column added to the classification books. But even if my tables should not be used practically in life-office valuations, there can be no question that they will be of the greatest service in enabling us to test the correctness of any approximate method that may be adopted for the sake of simplicity.

The following table exhibits the value of a policy for 100 taken out at the age of 35, after being in force 1, 2, 3, ... 10 years, calculated at 4 per-cent interest according to different methods. Column 2 contains the net values by the H^M Table, column 3 those by the combined H^M and $H^{M(5)}$ Tables, and column 4 those according to my Select Tables combined, after 4 years, with the $H^{M(5)}$.

The full policy-values, as obtained by the ordinary net premium method of valuation, are suitable in the case of a prosperous and economically managed company, but not in all other cases. If the whole

Table of Policy-Values, calculated at 4 per-cent.—Sum Assured, 100;
Age at Entry, 35.

Years in force n	${}_nV_{35}$			${}_{n-1}V_{[35]+1}$		
	H^M	Combined $H^{M(5)}$ and		H^M	Combined $H^{M(5)}$ and	
		H^M	Select.		H^M	Select.
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	1·180	...	1·590	0·000	...	0·000
2	2·386	...	2·988	1·220	...	1·422
3	3·618	...	4·313	2·466	...	2·768
4	4·879	...	5·596	3·743	...	4·072
5	6·176	7·232	6·873	5·055	6·123	5·369
6	7·518	8·534	8·180	6·413	7·441	6·697
7	8·911	9·903	9·554	7·823	8·827	8·094
8	10·353	11·320	10·977	9·282	10·261	9·539
9	11·833	12·788	12·450	10·780	11·746	11·037
10	13·353	14·297	13·965	12·318	13·273	12·579

of the new premiums have been absorbed by the expenses of obtaining the business and the current risk, the proper net premium to value is that for an age one year older than the office age at entry, and in this way a sufficient and proper reserve will be made for the liabilities of the office. For the purpose of comparison, the policy-values thus obtained, are given in columns (5), (6), and (7).

It will be seen that, although the values by my tables are considerably in excess of the H^M values, they are in every instance less than those deduced from the combined H^M and $H^{M(5)}$ Tables; in other words, the policy-values for age at entry 35, as found from these combined tables, are too high. This arises from the fact that the true net premium for age 35 is higher than the H^M net premium.

The use of the H^M table for policies of less than five years' duration, and of the combined H^M and $H^{M(5)}$ tables for older policies, leads to a very awkward break in the series of policy-values, as shown in the first half of the following table:—

Age at Entry, 35; *Policy for* 100.

Years in force.	Table used.	Value.	Differences.	Table used.	Value.	Differences.
3	H^M	3·618	1·261	"Select"	4·313	1·283
4	Do.	4·879	2·353	Do.	5·596	1·277
5	H^M & $H^{M(5)}$	7·232	1·302	Do. & $H^{M(5)}$	6·873	1·307
6	Do.	8·534		Do.	8·180	

Mr. Teece, assuming without question the accuracy of the policy-values obtained by the use of the combined tables, argues that the break in the series of policy-values is to be got rid of by raising the reserves for the recent policies, so as to "prepare the company to make after five years the reserve required by the combined tables"; but the fact mentioned above, that the policy-values given by the combined tables are larger than is either necessary or proper, demolishes, at once and entirely, the whole foundation on which Mr. Teece's argument is based. For the purpose of comparison, I have inserted in the table policy-values as obtained from my Select Tables and the differences of these values; and it will be seen that no break exists in this series of values.

I observe that Mr. Bailey states that he cannot accept my theories on the subject of the valuation of recent policies. I am able, however, to accept the principles he lays down. When he says, "There is a strong disposition to take the new business into account directly or indirectly as an asset . . . . . That should not be done", I entirely and unreservedly concur with him. The only way in which the new business can be taken into account as an asset, is by taking credit for negative policy-values, either directly or indirectly; and that should under no circumstances be done. When he proceeds, "The sums that are now paid for business ought not to be paid", I think it probable we should not altogether agree; but as he does not state what sums are now paid for business, it is impossible to speak with certainty on the point. When he adds, however, "If that expense is incurred, it ought not to be thrown on the future. You must pay your way as you go", I again approve unreservedly of the principle he enunciates. The method of valuation which I have consistently advocated for the last ten years, and which I have acted upon when necessary, and am prepared to act upon again, requires that an office should pay its way as it goes, neither introducing into its balance-sheet the doubtful asset "Preliminary Expenses", or the like, nor taking credit for fictitious assets in the form of negative policy-values.

Explanation ov a New Formula for Interpolation. *By* T. B. SPRAGUE, M.A., *Manager ov the Scottish Equitable Life Assurance Society.*

WHEN a series ov quantitys proceed by a regular law, ther is no difficulty in interpolating between each ajacent two ov them any number ov terms with any desired degree ov accuracy. The methods to be adopted for this purpos hav been described in varios original and reprinted papers containd in the *Jurnl ov the Institute ov Actuarys*: (see, for instance, Mr. Woolhouse's paper *On Interpolation, Summation, etc.*, xi, 61, 301, and xii, 136; Mr. Gray's paper *On the Construction ov Tabls by the Method ov Differences,* xiii, 61, 149, 293, and xiv, 307; and the translations ov M. Maurice's Essay on Interpolation, xiv, 1; and ov Briggs's explanation ov his Method ov Quintisection, xiv, 73). We somtimes, however, hav to interpolate between quantitys which do not accuratly follo any law. For instance, we may hav calculated premiums for every quinquennial age, and wish to obtain the premiums for the intermediat ages by interpolation. In this case, it wil somtimes happen that the ordinary formulas ov interpolation do not giv satisfactory results, unles we take a very large number ov differences, and then the amount ov labor is offen more than the result is worth. In my paper *On the Value ov Anuitys payabl half-yearly, quarterly, etc.,* I indicated briefly a method ov interpolation which I tho't miht be employd with advantage in such cases (xiii, 322), and it is now my intention to work out the formulas resulting from that method, and giv a practical ilustration ov their use.

The quantitys between which we desire to interpolate may be represented by a series ov equidistant ordinats. If the quantitys follo a regular law, the extremitys ov the ordinats wil ly upon a regular curv; but in any case we can supose a curv drawn thro the extremitys of the ordinats, and the curv wil be more or les regular acording as the givn quantitys proceed with greater or les regularity. The problem ov interpolating any givn number ov terms between two ov the original quantitys, is thus seen to be precisely analogos to the problem ov determining intermediat points upon the abov-mentiond curv, ov which we kno nothing more than the points at the extremitys ov the givn ordinats. If the quantitys follo a knoen law, this wil furnish us with an equation between the values ov the absissas and the ordinats, and by means ov this equation the value ov the ordinat corresponding to any absissa can be calculated. But when we do not kno the law which the quantitys follo, we can only aproximate to the values ov the intermediat quantitys by som proces ov interpolation. Every such proces is equivalent to drawing a continuos curv between the points forming the extremitys ov each two ajacent ordinats, and it is obvios that an unlimited number ov such partial curvs may be drawn. We may, for instance, so draw them that each ov them shal be a portion ov a common parabola with its axis paralel to the y-axis, in which case the second differences ov the interpolated quantitys wil be constant. This method ov interpolation was very extensivly uzed by the late Mr. Neison in his Vital Statistics, but the results ar offen very far from satisfactory, as I shal presently sho. Or we may make the third differences in each partial curv constant, in which case the equation ov any one ov them wil be ov the form $y = a + bx + cx^2 + dx^3$. Or we may supose each ov them to be a parabolic curv with constant differences ov a hiher order; and the hiher the order ov differences which we consider to be constant, the larger wil be the number ov the conditions that we can make the partial curvs conform to, and the smoother wil be the junctions which we can efect between the different partial curvs. Every successiv order ov differences, however, that we deal with, increases greatly the labor ov computation; and it is desirabl therefore to limit the number to the smallest with which a satisfactory result can be obtaind.

As an illustration ov the method ov interpolation by constant second differences, I hav taken the folloing figurs from Mr. Neison's

Rate ov Mortality in the Medical Profession ("Vital Statistics", p. 117) :—

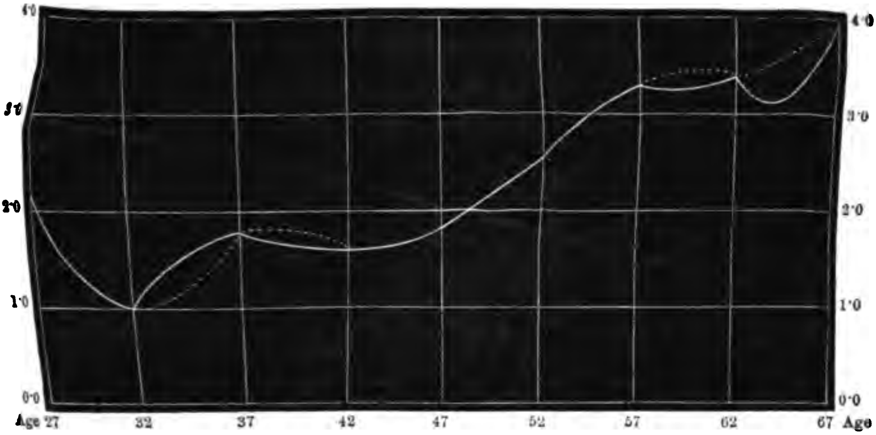
TABL IV.—*Marryd Members. Interpolation ov Mortality per-cent.*—(1.) *Original Quantitys.* Δ_1 *and* Δ_2=*1st and 2nd Differences.* $a = \cdot 2\Delta_1$, $b = \cdot 04\Delta_2$.

Age.	(1.)	Δ_1	Δ_2
27	2·174	−1·176	+1·926
		− ·2352 = a	+ ·07704 = b
32	·998	+ ·750	*
		*	*
37	1·748	*	*
*	*	*	*

TABL V.—*Interval* = $\frac{1}{5}$; (1.) = *Original Quantitys; 1st* $\delta_1 = a - 2b$, $\delta_2 = b$.

Age.	(1.)	δ_1	Age.	(1.)	δ_1	Age.	(1.)	δ_1	Age.	(1.)	δ_1
27	2·174	−·38928	42	1·594	+·00296	57	3·279	−·01148	72	8·458	−·57888
28	1·78472	·31224	43	1·59696	·02308	58	3·26752	+·00556	73	7·87912	−·08264
29	1·47248	·23520	44	1·62004	·04320	59	3·27308	·02260	74	7·79648	+·41360
30	1·23728	·15816	45	1·66324	·06332	60	3·29568	·03964	75	8·21008	·90984
31	1·07912	−·08112	46	1·72656	·08344	61	3·33532	+·05668	76	9·11992	1·40608
32	·998	+·22232	47	1·810	·14132	62	3·392	−·21124	77	10·526	1·90232
33	1·22032	·18616	48	1·95132	·14256	63	3·18076	−·05172	78	12·42832	2·39856
34	1·40648	·15000	49	2·09388	·14380	64	3·12904	+·10780	79	14·82688	2·89480
35	1·55648	·11384	50	2·23768	·14504	65	3·23684	·26732	80	17·72168	3·39104
36	1·67032	+·07768	51	2·38272	·14628	66	3·50416	·42684	81	21·11272	+3·88728
37	1·748	−·06040	52	2·529	·20096	67	3·931	1·10212	82	25·000	...
38	1·68760	·04560	53	2·72996	·17548	68	5·03312	1·00376	...	...	...
39	1·64200	·03080	54	2·90544	·15000	69	6·03688	·90540	...	...	...
40	1·61120	·01600	55	3·05544	·12452	70	6·94228	·80704	...	...	...
41	1·59520	−·00120	56	3·17996	+·09904	71	7·74932	+·70868	...	...	...
											...

Mr. Neison deduced from certain statistics the mortality per-cent at the ages 27, 32, 37, &c., and taking these without any ajustment as his starting-points, he interpolates by constant second differences the values for the intermediat ages. It is evident from the differences givn in the abov tabl that the interpolation is not a very succesful one, but this apears much more clearly from the anext diagram (p. 273). Mr. Neison's method ov interpolating between ages 27 and 32, is equivalent to drawing thro' the extremitys ov the ordinats representing the mortality per-cent at the ages 27, 32, and 37, a common parabola with its axis paralel to the y-axis; and his interpolated values ar represented by the ordinats ov the first haf ov this curv, indicated by a continuos line in the diagram, the second haf, which is represented by a dotted line, not being



uzed in the interpolation. The method ansers fairly wel between the ages ov 37 and 57, if we exclude from consideration the ages outside those limits; but the abrupt angls it givs at the ages 32 and 37 ar quite inadmissibl, and the fact that interpolation by constant second differences leads to them, shos that it is wholly unsuitabl in the present case. In fact, during a part ov the series, interpolation by constant first differences woud hav givn better results. If we difference out the givn values between which Mr. Neison interpolates, as shoen in the margin, we see that for the values from age 37 to 57 the three second differences, ·370, ·503, ·031, ar al ov the same sign and their values not very unequal, but the preceding and folloing second differences hav contrary signs and change rapidly in value. This not only indicates that interpolation by second differences can not be advantageosly employd, but sugests the idea that no satisfactory curv ov any description can be drawn thro' Mr. Neison's fixt points in this case;—an idea which is confirmd by a study ov the diagram. I believ that in order to get a regular progression in this case it woud be much better to take other starting-points, either by ajusting Mr. Neison's percentages as givn abov, or by having recours to his original statistics, which ar givn for singl years ov age, and endevoring to see whether som other grooping woud not giv more regular results than Mr. Neison's.

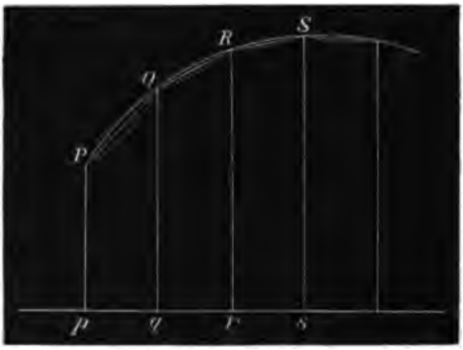
Age.	Mortality per-cent.			
27	2·174			
		− 1·176		
32	·998		+ 1·926	
		+ ·750		− 2·830
37	1·748		− ·904	
		− ·154		+ 1·274
42	1·594		+ ·370	
		+ ·216		+ ·133
47	1·810		+ ·503	
		·719		− ·472
52	2·529		+ ·031	
		·750		− ·668
57	3·279		− ·637	
		·113		+ 1·063
62	3·392		+ ·426	
		·539		+ 3·562
67	3·931		+ 3·988	
		4·527		− 6·447
72	8·458		− 2·459	
		2·068		+ 14·865
77	10·526		+ 12·406	
		14·474		
82	25·000			

The late Mr. Samuel Brown made extensiv use ov what he cals interpolation by even differences ov logarithms. Having got the probability ov dying in a year for a series ov quinquennial ages, n, $n+5$, $n+10$, &c., his formula for interpolating between any two consecutiv probabilitys, q_n, q_{n+5}, is

$$\log q_{n+x} = \log q_n + \tfrac{1}{5}x(\log q_{n+5} - \log q_n).$$

This is equivalent to $q_{n+x} = q_n\left(\frac{q_{n+5}}{q_n}\right)^{\frac{x}{5}}$,

so that the interpolated values of q form a geometrical progression with the original values. If we take the age as the absissa and the probability got from this formula as the ordinat, the curv passing thro' the extremitys ov the ordinats wil always be convex to the x-axis, or, in other words, the second differences ov the interpolated quantitys ar always positiv. This method ov interpolation, therefore, dos not giv satisfactory results when the series ov original values is such that the second differences ar negativ, in which case the curv thro' the extremitys ov the corresponding ordinats must be concave to the x-axis. Whenever the second differences ar negativ, interpolation by constant first differences wil giv more satisfactory results than Mr. Brown's. Let *Pp*, *Qq*, *Rr*, *Ss*, &c., in the anext figur, represent the probabilitys for a series ov equidistant ages; then we see that these probabilitys first increas and then decreas, and the second differences ar thro'out negativ. It is obvios that the interpolated values o't to giv us a curv concave to the x-axis, such as the continuos curv drawn thro' the points *P*, *Q*, *R*, *S*, &c. Mr. Brown's method ov interpolation, however, givs us a series ov curvs resembling the dotted curvs in the figur, *PQ*, *QR*, *RS*, each ov which is convex to the x-axis, while interpolation by constant first differences woud giv us strait lines *PQ*, *QR*, *RS*, &c.



In illustration ov this point, we may take the figurs givn by Mr. Brown in the tabl on page 236 ov his report on the Madras Military Fund, 1808–1858. He has determind from the statistics ov the Fund certain probabilitys for the ages 17, 22, 27, &c., which, together with their first and second differences, ar shoen in the folloing tabl:—

Age.	Probability.	First Differences.	Second Differences.
17	·0248	·0416	−·0100
22	·0664	·0316	−·0156
27	·0980	·0160	−·0235
32	·1140	−·0075	+·0481
37	·1065	+·0406	
42	·1471		

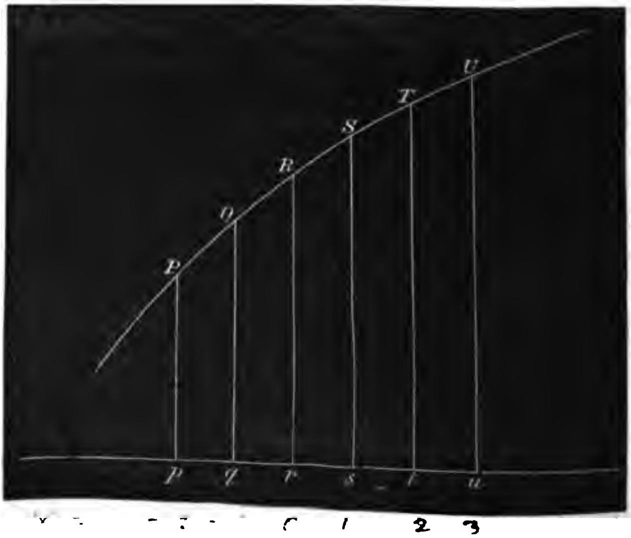
From these he deduced the interpolated values shoen in the folloing tabl, which also contains the logarithms ov the probabilitys, and the differences (1) ov the logarithms and (2) ov the probabilitys, and, lastly, the second differences ov the probabilitys.

Age.	Probability of Deth, Withdrawal, Marriage, and Retirement in a Year.	Log. of Col. 2.	Differences of the		Second Differences of the Probs.
			Logs.	Probs.	
(1)	(2)	(3)	(4)	(5)	(6)
17	·0284	$\bar{2}$·45332	·07377	53	
18	·0337	·52709	7377	62	9
19	·0399	·60086	7377	74	12
20	·0473	·67463	7377	87	13
21	·0560	·74840	7377	104	17
22	·0664	·82217	3381	54	−50
23	·0718	·85598	3381	58	4
24	·0776	·88979	3382	63	5
25	·0839	·92361	3381	68	5
26	·0907	·95742	3381	73	5
27	·0980	·99123	1313	30	−43
28	·1010	$\bar{1}$·00436	1314	31	1
29	·1041	·01750	1313	32	1
30	·1073	·03063	1313	33	1
31	·1106	·04376	1314	34	1
32	·1140	·05690	− 591	−15	−49
33	·1125	·05099	− 591	−16	− 1
34	·1109	·04508	− 591	−15	+ 1
35	·1094	·03917	− 591	−14	+ 1
36	·1080	·03326	− 591	−15	− 1
37	·1065	·02735	+2805	+71	+86
38	·1136	·05540	2806	76	5
39	·1212	·08346	2805	81	5
40	·1293	·11151	2805	86	5
41	·1379	·13956	2805	92	6
42	·1471	·16761			

Here it wil be observd that the second differences ov the original probabilitys from the ages 17 to 37 ar negativ, but that the second differences ov the interpolated probabilitys ar positiv, except at the points ov junction, at each ov which ther is a great break. In the case ov the interval 32 to 37, the second differences proceed irregularly in consequence ov a suficient number ov decimal places not having been taken to sho the law.

It may be remarkt that both Mr. Neison and Mr. Brown, after obtaining from their statistics data relating to each age, thro these together into quinquennial groops, and then make no further use ov the data at individual ages, but get from their groops values for quinquennial ages and simply interpolate for the other ages. It apears to me, however, that such a method ov procedure does not turn the observd facts to the best advantage. Insted ov taking the quinquennial values as fixt points ov departure, and simply interpolating, it is in my opinion better to uze som method ov ajustment—as, for instance, the grafic method—taking care to compare the ajusted with the original facts at each age. In this way we shal be abl to secure that our original facts ar not unduly alterd by the ajustment; but when we hav values only for quinquennial ages, we ar not so wel abl to juj how far it is alowabl to depart from our original values, and it is then safer to confine ourselvs to interpolation.

Mr. Woolhouse, in his paper abov referd to, describes a convenient practical method ov interpolation by means ov constant fourth differences, and remarks that hiher orders ov differences ar seldom employd, adding that perhaps in no case wil it be desirabl to proceed further than the sixth order ov differences. I find it convenient to employ constant fifth differences, for the folloing reason. Let six consecutiv quantitys, between which we hav to interpolate, be represented by the equidistant ordinats *Pp, Qq, Rr, Ss, Tt, Uu,* their values being $y_0, y_1, y_2, y_3, y_4, y_5$;



then, as abov explaind, the problem ov interpolating between y_2 and y_3 is the same as that ov drawing a curvd line between the points R and S, and, in order to get a satisfactory interpolation, it is necesary that this partial curv shoud join on smoothly to the ajacent partial curvs, namely, QR on the one side and ST on the other side. We shal secure this smoothnes ov junction, as regards QR and RS, if we so arange that these two curvs hav the same tangent and the same radius ov curvature at the point R; and this wil be the case if the values ov $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ ar the same for the two curvs at that point. Similarly, we may arange that at the point S the two curvs, RS, ST, shal hav the same tangent and the same radius ov curvature, or that the values ov $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, shal be the same for the two curvs. The values of the differential coeficients at the two points R, S, hav to be determind by means ov the givn quantitys, and the simplest way ov determining them for the point R, wil be to supose a parabola ov the fourth order (or a quartic parabola) drawn thro the points P, Q, R, S, T; to determin the values ov the differential coeficients at the point R in this parabola, and then make the values for the two partial curvs QR and RS equal to these. Similarly, we may make the values at the point S in the two partial curvs, RS, ST, equal to those at the same point in the quartic parabola passing thro the points Q, R, S, T, U. The curv RS must therefore, in adition to passing thro the point R, satisfy five other conditions; namely, it must pas thro the point S, and the values ov $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, at the extremitys, must hav givn values. We must therefore hav five disposabl constants in its equation. We may therefore asume it to be ov the form

$$y_{2+x}=y_2+ax+bx^2+cx^3+dx^4+ex^5 \quad . \quad . \quad . \quad . \quad (1)$$

Since the curv is to pas thro the point S, it follos that when $x=1$, the value ov y wil be y_3. This givs us the equation

$$y_3=y_2+a+b+c+d+e \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It wil be convenient to expres the values ov the differential coeficients in terms ov the differences ov the givn quantitys. For this purpos, let them be differenst out in the usual way, as shoen in the folloing sceme:—

$$\begin{array}{llllll}
y_0 & & & & & \\
 & \Delta y_0 & & & & \\
y_1 & & \Delta^2 y_0 & & & \\
 & \Delta y_1 & & \Delta^3 y_0 & & \\
y_2 & & \Delta^2 y_1 & & \Delta^4 y_0 & \\
 & \Delta y_2 & & \Delta^3 y_1 & & \Delta^5 y_0 \\
y_3 & & \Delta^2 y_2 & & \Delta^4 y_1 & \\
 & \Delta y_3 & & \Delta^3 y_2 & & \\
y_4 & & \Delta^2 y_3 & & & \\
 & \Delta y_4 & & & & \\
y_5 & & & & &
\end{array}$$

The equation to the curv ov constant fourth differences (or the quartic parabola) passing thro the points *P, Q, R, S, T,* is

$$y=y_0+x\Delta y_0+\frac{x.x-1}{1.2}\Delta^2 y_0+\frac{x.x-1.x-2}{6}\Delta^3 y_0+\frac{x.x-1.x-2.x-3}{24}\Delta^4 y_0 \quad . \; . \quad (3)$$

Differentiating this twice, we get

$$\frac{dy}{dx}=\Delta y_0+\frac{2x-1}{2}\Delta^2 y_0+\frac{3x^2-6x+2}{6}\Delta^3 y_0+\frac{4x^3-18x^2+22x-6}{24}\Delta^4 y_0 \quad . \; . \quad (4)$$

$$\frac{d^2y}{dx^2}=\Delta^2 y_0+\frac{6x-6}{6}\Delta^3 y_0+\frac{12x^2-36x+22}{24}\Delta^4 y_0 \quad . \; . \; . \quad (5)$$

Then, making $x=2$, we get

$$\frac{dy_2}{dx}=\Delta y_0+\tfrac{3}{2}\Delta^2 y_0+\tfrac{1}{3}\Delta^3 y_0-\tfrac{1}{12}\Delta^4 y_0 \quad . \; . \; . \quad (6)$$

$$\frac{d^2y_2}{dx^2}=\Delta^2 y_0+\Delta^3 y_0-\tfrac{1}{12}\Delta^4 y_0 \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

Now, let α, β, γ, δ, be the central differences ov the five quantitys $y_0 \cdots\cdots y_4$. Then we hav

$$\alpha=\tfrac{1}{2}(\Delta y_1+\Delta y_2)=\Delta y_0+\tfrac{3}{2}\Delta^2 y_0+\tfrac{1}{2}\Delta^3 y_0,$$
$$\beta=\Delta^2 y_1=\Delta^2 y_0+\Delta^3 y_0,$$
$$\gamma=\tfrac{1}{2}(\Delta^3 y_0+\Delta^3 y_1)=\Delta^3 y_0+\tfrac{1}{2}\Delta^4 y_0,$$
$$\delta=\Delta^4 y_0.$$

From these equations we find

$$\Delta y_0=\alpha-\tfrac{3}{2}\beta+\gamma-\frac{\delta}{2},\ \Delta^2 y_0=\beta-\gamma+\frac{\delta}{2},\ \Delta^3 y_0=\gamma-\frac{\delta}{2},\ \Delta^4 y_0=\delta.$$

By substituting in (6) and (7), we get

$$\frac{dy_2}{dx}=\alpha-\frac{\gamma}{6},\ \frac{d^2y_2}{dx^2}=\beta-\frac{\delta}{12}.$$

These ar the values ov the differential coeficients at the point *R* ov the quartic parabola passing thro the points *P, Q, R, S, T.* The equation (1) shos us that for the partial curv *RS*, $\frac{dy_2}{dx}=a$, and $\frac{d^2y_2}{dx^2}=2b$, and the condition that the differential coeficients shal hav the same value in the partial curv RS as in the quartic parabola, givs us the equations $a=\alpha-\frac{\gamma}{6}$, $2b=\beta-\frac{\delta}{12}$. Next let α', β', γ', δ', be the central differences ov the five quantitys $y_1 \ldots\ldots\ldots y_5$. Then for the quartic parabola $QRSTU$ we hav

$$\frac{dy_3}{dx}=\alpha'-\frac{\gamma'}{6}, \quad \frac{d^2y_3}{dx^2}=\beta'-\frac{\delta'}{12}.$$

Differentiating the equation (1) twice, and putting $x=1$, we get the values ov the differential coeficients at the point S for the curv RS; and the condition that these values shal be the same in the partial curv RS and in the quartic parabola, givs us the equations

$$a+2b+3c+4d+5e=\alpha'-\gamma'\div 6$$
$$2b+6c+12d+20e=\beta'-\delta'\div 12.$$

The problem is now to determin a, b, c, d, e, in terms ov the leading differences Δy_0, $\Delta^2 y_0$, &c. We first hav

$$\alpha'=\tfrac{1}{2}(\Delta y_2+\Delta y_3)=\Delta y_0+\tfrac{5}{2}\Delta^2 y_0+2\Delta^3 y_0+\tfrac{1}{2}\Delta^4 y_0,$$
$$\beta'=\Delta^2 y_2=\Delta^2 y_0+2\Delta^3 y_0+\Delta^4 y_0,$$
$$\gamma'=\tfrac{1}{2}(\Delta^3 y_1+\Delta^3 y_2)=\Delta^3 y_0+\tfrac{3}{2}\Delta^4 y_0+\tfrac{1}{2}\Delta^5 y_0,$$
$$\delta'=\Delta^4 y_1=\Delta^4 y_0+\Delta^5 y_0.$$

Now, uzing Δ, Δ_2, &c., as abreviations for Δy_0, $\Delta^2 y_0$, &c., we hav the five equations

$$a+b+c+d+e=y_3-y_2=\Delta+2\Delta_2+\Delta_3,$$
$$a=\alpha-\tfrac{1}{6}\gamma=\Delta+\tfrac{3}{2}\Delta_2+\tfrac{1}{3}\Delta_3-\tfrac{1}{12}\Delta_4,$$
$$b=\tfrac{1}{2}\beta-\tfrac{1}{24}\delta=\tfrac{1}{2}\Delta_2+\tfrac{1}{2}\Delta_3-\tfrac{1}{24}\Delta_4,$$
$$a+2b+3c+4d+5e=\alpha'-\tfrac{1}{6}\gamma'=\Delta+\tfrac{5}{2}\Delta_2+\tfrac{11}{6}\Delta_3+\tfrac{1}{4}\Delta_4-\tfrac{1}{12}\Delta_5,$$
$$2b+6c+12d+20e=\beta'-\tfrac{1}{12}\delta'=\Delta_2+2\Delta_3+\tfrac{11}{12}\Delta_4-\tfrac{1}{12}\Delta_5,$$

from which we find

$$\begin{aligned}
a&=\Delta+\tfrac{3}{2}\Delta_2+\tfrac{1}{3}\Delta_3-\tfrac{1}{12}\Delta_4,\\
b&=\tfrac{1}{2}\Delta_2+\tfrac{1}{2}\Delta_3-\tfrac{1}{24}\Delta_4,\\
c&=\tfrac{1}{6}\Delta_3+\tfrac{1}{12}\Delta_4+\tfrac{7}{24}\Delta_5,\\
d&=\tfrac{1}{24}\Delta_4-\tfrac{1}{2}\Delta_5,\\
e&=\tfrac{5}{24}\Delta_5,
\end{aligned}$$

and we thus get

$$\begin{aligned} y_{2+x}=y_2&+x(\Delta y_0+\tfrac{3}{2}\Delta^2y_0+\tfrac{1}{3}\Delta^3y_0-\tfrac{1}{12}\Delta^4y_0)\\ &+x^2(\tfrac{1}{2}\Delta^2y_0+\tfrac{1}{2}\Delta^3y_0-\tfrac{1}{24}\Delta^4y_0)\\ &+x^3(\tfrac{1}{6}\Delta^3y_0+\tfrac{1}{12}\Delta^4y_0+\tfrac{7}{24}\Delta^5y_0)\\ &+x^4(\tfrac{1}{24}\Delta^4y_0-\tfrac{1}{2}\Delta^5y_0)+x^5.\tfrac{5}{24}\Delta^5y_0 \quad . \quad . \quad . \quad (8)\end{aligned}$$

We hav thus completely determind the equation to the partial curv *RS*, and we can by means ov it calculate the values ov the ordinat at any intermediat point; that is to say, we can interpolate a value at any part ov the interval we choos between y_2 and y_3. It is convenient, however, in practis to calculate the leading differences ov the interpolated quantitys rather than the quantitys themselvs. Supose, for instance, that we wish to interpolate four terms between y_2 and y_3. We shal get them by making $x=\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, in the abov formula (8). Making this substitution, and differencing in the ordinary way, we find that the leading differences ov y_2 and the interpolated quantitys ar

$$\left.\begin{array}{rcl} \frac{a}{5}+\frac{b}{5^2}+\frac{c}{5^3}+\frac{d}{5^4}+\frac{e}{5^5} &=& \frac{\Delta}{5}+8\frac{\Delta_2}{5^2}+11\frac{\Delta_3}{5^3}-11\frac{\Delta_4}{5^4}+\frac{\Delta_5}{5^4}\\ \frac{2b}{5^2}+\frac{6c}{5^3}+\frac{14d}{5^4}+\frac{30e}{5^5} &=& \frac{\Delta_2}{5^2}+6\frac{\Delta_3}{5^3}+\frac{\Delta_4}{5^4}+3\frac{\Delta_5}{5^4}\\ \frac{6c}{5^3}+\frac{36d}{5^4}+\frac{150e}{5^5} &=& \frac{\Delta_3}{5^3}+4\frac{\Delta_4}{5^4}-3\frac{\Delta_5}{5^4}\\ \frac{24d}{5^4}+\frac{240e}{5^5} &=& \frac{\Delta_4}{5^4}-2\frac{\Delta_5}{5^4}\\ \frac{120e}{5^5} &=& 5\frac{\Delta_5}{5^4} \end{array}\right\} \quad (9)$$

The proces ov calculating these leading differences and obtaining by means ov them the interpolated values, is an extremely simpl one; but it may asist the understanding ov the proces if I giv a numerical illustration ov its aplication in practis. In my paper *On the Construction and Use ov a Series ov Select Mortality Tabls*, I hav givn a tabl ov the ajusted probabilitys ov deth in each ov the first five insurance years for quinquennial ages at entry (*J.I.A.* xxi, 246). The probabilitys in the fourth insurance year, or, in other words, the values ov $q_{[x]+3}$, ar givn in the second colum ov the folloing tabl. But it wil be notist that in my paper abov-mentiond, $q_{[35]+3}$ is incorectly printed ·010077 insted ov ·010177.

TABL A.

x	$q_{[x]+3}$	Δ	Δ_2	Δ_3	Δ_4	Δ_5
20	·010709					
		− 1124				
25	·009585		+ 764			
		− 360		+ 548		
30	·009225		+ 1312		−1676	
		+ 952		−1128		+ 4126
35	·010177		+ 184		+2450	
		+ 1136		+1322		− 3828
40	·011313		+ 1506		−1378	
		+ 2642		− 56		+ 3039
45	·013955		+ 1450		+1661	
		+ 4092		+1605		− 920
50	·018047		+ 3055		+ 741	
		+ 7147		+2346		− 4329
55	·025194		+ 5401		−3588	
		+12548		−1242		+12995
60	·037742		+ 4159		+9407	
		+16707		+8165		−17778
65	·054449		+12324		−8371	
		+29031		− 206		
70	·083480		+12118			
		+41149				
75	·124629					

The first step in the proces ov interpolation is to rite down the first, second, third, fourth, and fifth differences ov the quantitys; and the second, to calculate what may be cald the divided differences, $\frac{\Delta}{5}, \frac{\Delta_2}{5^2}, \frac{\Delta_3}{5^3}, \frac{\Delta_4}{5^4}, \frac{\Delta_5}{5^4}$. These ar givn in the folloing tabl, being calculated to two decimal places more than we wish ultimatly to retain.

TABL B.

x	$\Delta \div 5$	$\Delta_2 : 5^2$	$\Delta_3 \div 5^3$	$\Delta_4 : 5^4$	$\Delta_5 \div 5^4$
20	− 22480				
		+ 3056			
25	− 7200		+ 438		
		+ 5248		− 268	
30	+ 19040		− 902		+ 660
		+ 736		+ 392	
35	+ 22720		+1058		− 613
		+ 6024		− 221	
40	+ 52840		− 45		+ 486
		+ 5800		+ 266	
45	+ 81840		+1284		− 147
		+12220		+ 119	
50	+142940		+1877		− 693
		+21604		− 574	
55	+250960		− 994		+2079
		+16636		+1505	
60	+334140		+6532		−2845
		+49296		−1339	
65	+580620		− 165		
		+48472			
70	+822980				

Inasmuch as our formula requires that we shoud kno six values, namely, the two that precede and the two that follo the two between which we interpolate, we must begin our work ov interpolation with the interval corresponding to the ages at entry 30 and 35, that is to say, we hav to interpolate between $q_{[30]+3}$ and $q_{[35]+3}$. I hav found it convenient to arange the work as shoen in the folloing sceme. The givn values, between which we ar to interpolate, ar set opposit the ages 30, 35, 40, &c., as shoen by the figurs in hevy antique type, two cyphers being placed at the end ov each; and in the same line to the left ar set down the values ov the divided differences in revers order.

Tabl C.

Interpolation ov $q_{[x]+3}$ for Ages 30 to 35 and 35 to 40 by New Formula.

$\Delta_5 : 5^4$	$\Delta_4 \div 5^4$	$\Delta_3 \div 5^3$	$\Delta_2 \div 5^2$	$\Delta \div 5$		$q_{[x]+3}$	x
+ 660	− 268	+ 438	+ 3056	−22480	...	·00922500	30
+ 660	+2948	+4818	+24448	−22480	+10394	·00932894	31
+1980	− 268	+2628	+ 3056	+ 7396	+17790	·00950684	2
−1980	−1072	+ 438	− 2614	+ 4782	+22572	·00973256	3
−1320	− 268	−1588	− 4202	+ 580	+23152	·00996408	4
+3300	+3300	+1712	− 2490	− 1910	+21242	·01017650	5
− 613	+ 392	− 902	+ 5248	− 7200	...	·01017700	35
− 613	−4312	−9922	+41984	− 7200	+19937	·01037637	36
−1839	+ 392	−5412	+ 5248	− 1611	+18326	·01055963	7
+1839	+1568	− 902	+ 2505	+ 894	+19220	·01075183	8
+1226	+ 392	−1618	+ 4123	+ 5017	+24237	·01099420	9
−3065	−3065	−1447	+ 2676	+ 7693	+31930	·01131350	40

The multiplyers containd in the formulas (9) ar so smal that al the multiplications may be performd mentally, and the products ar placed down to the left ov the hevy line. As the multiplyers do not follo any obvios law, it is convenient to rite them on a separat slip ov paper in the folloing way:

1, −11, 11, 8, 1
3 1 6 1
−3 4 1
−2 1
5

Then adition ov the quantitys to the left ov the hevy line wil giv us the leading differences. Thus, taking the interval 30 to 35, we hav $\frac{\Delta}{5} = -22480, 8\frac{\Delta_2}{5^2} = 24448, 11\frac{\Delta_3}{5^3} = 4818, -11\frac{\Delta_4}{5^4} = 2948,$ $\frac{\Delta_5}{5^4} = 660$, and the sum ov these five quantitys is 10394, which is the first leading difference. The other leading differences ar 7396, −2614, −1588, and +3300. The consecutiv adition ov these givs us the interpolated values required. In the present instance we get as the value corresponding to the age at entry 35, ·01017650, which agrees with the original value in the figurs that we ar to retain.

For the purpos ov comparison I wil now giv the interpolation between the same two values by means ov Mr. Woolhouse's formula (*J.I.A.* xi, 69) which is based upon constant fourth differences.

TABL D.

Interpolation ov $q_{[x]+3}$ for Ages 30–40 by Woolhouse's Method.

b_0	c_1	(d)	$\frac{1}{8}a_1$		$q_{[x]+3}$	x
+13120	−11280	+3870	+1904	...	·0092250	30
− 1050	+ 360	+ 55	− 635	+1269	·0093519	31
+ 525	− 90	− 24	+ 411	+1680	·0095199	2
+ 525	− 180	− 31	+ 314	+1994	·0097193	3
+ 525	− 270	− 31	+ 224	+2218	·0099411	4
+ 525	− 360	− 24	+ 141	+2359	·0101770	5
+ 1840	+13220	+5360	+2272	...	·0101770	35
− 147	− 422	+ 77	− 492	+1780	·0103550	36
+ 73	+ 105	− 34	+ 144	+1924	·0105474	7
+ 74	+ 211	− 43	+ 242	+2166	·0107640	8
+ 74	+ 317	− 43	+ 348	+2514	·0110154	9
+ 73	+ 423	− 34	+ 462	+2976	·0113130	40

In order to compare the results ov the two methods ov interpolation, perhaps the best plan is to difference out the original and the interpolated values as follos:—

TABL E.

SPRAGUE.

9225				
	+104			
9329		+74		
	+178		−26	
9507		+48		−17
	+226		−43	
9738		+ 5		+20
	+231		−23	
9964		−18		+27
	+213		+ 4	
10177		−14		− 5
	+199		− 1	
10376		−15		+24
	+184		+23	
10560		+ 8		+19
	+192		+42	
10752		+50		−15
	+242		+27	
10994		+77		
	+319			
11313				

WOOLHOUSE.

9225				
	+127			
9352		+41		
	+168		−10	
9520		+31		+ 2
	+199		− 8	
9719		+23		− 1
	+222		− 9	
9941		+14		− 63
	+236		−72	
10177		−58		+144
	+178		+72	
10355		+14		− 61
	+192		+11	
10547		+25		− 2
	+217		+ 9	
10764		+34		+ 4
	+251		+13	
11015		+47		
	+298			
11313				

It wil be observd that, while Mr. Woolhouse's interpolated values proceed with regularity thro'out each separat interval, yet ther is a somwhat abrupt break in the series ov second differences at the point ov junction between the two intervals, whereas in my series ov values ther is no such abrupt junction.

In employing either Mr. Woolhouse's method or mine, it must not be suposed that the exact reproduction ov one ov the original values is a necesary proof ov the accuracy ov the proces. In my method it wil be found that if, in interpolating between y_2 and y_3, any arbitrary values be givn to Δ_4 and Δ_5, and these arbitrary values ar uzed in the proces, then the value ov y_3 wil be unafected. That this must be the case apears from the fact that, if we put $x=1$ in the formula (8), we get

$$y_3=y_2+\Delta y_0+2\Delta^2 y_0+\Delta^3 y_0.$$

This shos that the only condition that is necesary in order that the proces shal reproduce the original value ov y_3, is that such values shal be givn to Δy_0, $\Delta^2 y_0$, and $\Delta^3 y_0$, that the sum, $\Delta y_0+2\Delta^2 y_0+\Delta^3 y_0$, shal be equal to the difference y_3-y_2. In employing Mr. Woolhouse's formula, stil greater divergencys ar consistent with a reproduction ov the original values. In fact, any arbitrary values may be givn to his b_0, c_1, and (d); and provided the value ov a_1 be corect, the proces wil reproduce the original values. That this shoud be the case, is clear from the formula on which his method is based, (*J.I.A.* xi, 69)—

$$V_x=V_0+xa_1-\frac{x(1-x)}{2}b_0-\frac{x(1-x^2)}{2.3}c_1+\frac{x(1-x^2)(2-x)}{2.3.4}(d).$$

Here, if we make $x=1$, we get $V_1=V_0+a_1$, whatever may be the values ov b_0, c_1, or (d).

The method abov employd is not practicabl in the two first and the two last intervals ov the series, namely, from ages 20–30 and 65–75. For these we must proceed in a different way. Referring bak to the figur on page 276, if we supose *Pp* to represent the value for age 20, *Qq* for 30, *Rr* for 40, and so on, we hav determind the direction and curvature ov the curv *RS* at the point *R*, by making them the same as those ov the curv ov constant fourth differences that passes thro' the five points *P, Q, R, S, T*. Al we hav to do therefore is to take the portion *PQR* ov that curv, and it wil join on smoothly to the curv *RS*. In other words, we hav to interpolate by constant fourth differences deduced from the first five givn values, namely, those for ages 20, 25, 30, 35, 40. The formulas to be uzed in this interpolation ar givn and demonstrated by Mr. Gray (*J.I.A.* xiii, 161), and by M. Maurice (*J.I.A.* xiv, 23), and the work is containd in the folloing tabl, with the exception ov the actual multiplications, which either hav to be don on a separat piece of paper or to be taken from tabls such as those givn by Mr. Woolhouse in his paper alreddy several times referd to.

TABL F.

Interpolation ov $q_{[x]+3}$ for Ages 20–30 by Constant Fourth Differences.

$\Delta_4 \div 5^4$	$\Delta_3 \div 5^3$	$\Delta_2 \div 5^2$	$\Delta \div 5$		$q_{[x]+3}$	x
− 268	+ 438	+3056	−22480	...	·01070900	20
+5628	+2628	−6112	−22480	−20336	·01050564	21
−4288	−1752	+3056	− 2984	−23320	·01027244	22
+1608	+ 438	+2046	− 938	−24258	·01002986	23
− 268	− 268	+1778	+ 840	−23418	·00979568	24
	− 268	+1510	+ 2350	−21068	·00958500	25
	− 268	+1242	+ 3592	−17476	·00941024	26
	− 268	+ 974	+ 4566	−12910	·00928114	27
	− 268	+ 706	+ 5272	− 7638	·00920476	28
	− 268	+ 438	+ 5710	− 1928	·00918548	29
	− 268	+ 170	+ 5880	+ 3952	·00922500	30

At the other end ov the series, namely, for ages 65–75, a precisely similar proces has to be adopted; but in order to prevent any confusion as to the signs ov the terms, it is convenient to arange the last five quantitys in invers order, and difference them out as follos:—

x	$q_{[x]+3}$	Δ	Δ_2	Δ_3	Δ_4
75	·124629				
		−41149			
70	·083480		+12118		
		−29031		+ 206	
65	·054449		+12324		−8371
		−16707		−8165	
60	·037742		+ 4159		
		−12548			
55	·025194				

Bonuses—how Earned and how Distributed. By Archibald Hewat, F.F.A., A.I.A., *Hon. Memb. Inst. of Accountants and Actuaries; Secretary at Glasgow to The Edinburgh Life Assurance Company.*

[An Abridgment of a Paper read before the Actuarial Society of Edinburgh on 8 January 1880.]

I PREFER to take a practical view of my subject, hoping that such may prove at once more useful and more interesting than any mere theoretical handling of it by me. I have gone over all the Board of Trade Returns of the Life Assurance Companies of this country, as published in the Blue-books issued since the passing of the "Life Assurance Companies Act, 1870", and have procured most, if not all, of those issued by the offices since the publication of the 1879 Blue-book.

From the returns relating to seventy-seven offices, I find that £2,285,000 of hard cash is, on the average, annually divided *among the policyholders* by way of bonuses.*

Loading.—From my examination of the Returns, I think I am safe in estimating a margin of 9 per-cent of the annual premium income as legitimate profit for distribution arising from this source. The average annual premiums of these 77 offices amounting, in round numbers, to £9,882,000, the annual profit from "loading" may be put down at £890,000.

Interest.—The average rate of interest "expected" to be realized on the bulk of the business, I find to be £3. 8*s*. 8*d*. per-cent per annum; while the rate returned as actually realized averages £4. 10*s*., as shown in the following table:—

Average Rates of Interest realized by British Life Assurance Companies during the Thirteen Years 1866-78 *inclusive.*

Year.	Number of Offices.	Rate per-cent.	Year.	Number of Offices.	Rate per-cent.
		£ *s*. *d*.			£ *s*. *d*.
1866	23	4 10 5	1873	67	4 10 11
1867	32	4 9 11	1874	63	4 11 4
1868	47	4 9 3	1875	50	4 11 2
1869	56	4 9 0	1876	39	4 9 9
1870	61	4 10 2	1877	27	4 9 2
1871	65	4 9 9	1878	10	4 8 11
1872	65	4 10 2	Average	...	4 10 0

---

* This being equal to 23 per-cent of the premiums, all the "loading" is, practically, refunded.

But as there seem to be various methods of ascertaining the rates obtained by the different offices,—some excluding unproductive bank balances, income-tax, &c.; some giving actual rates, and others little better than mere estimates or averages,—I therefore assume the realized rate to be £4. 8*s*. 8*d*. This will give an increase over the average "expected" rate of £1 per-cent, which, on accumulated funds amounting to, say, £104,000,000, shows an annual profit from interest of £1,040,000.

PROFIT ON INVESTMENTS.—From the Returns, I find that a sum of not less than £110,000 is, on the average, annually realized from this source. There may be much more,—as, doubtless, there is;—but it is not directly stated in the Returns.

SUNDRIES.—Under this general head, I include direct receipts,* and indirect profits. The former is made up of fines for reviving forfeited policies; fees for recording notices of assignations under the "Policies of Assurance Act, 1867"; and some other petty receipts. These average £4,600 per annum. The sources of indirect profit are such as these:—Premiums being charged as for age next birthday, instead of for real age; the proportion of premium received for unearned risk on the large number of assurances effected after the nominal "close of the books" (thus—a year's premium may be received for only eleven months' risk); first premiums being seldom paid till near the end of the days of grace (*e.g.*—if the year's new premiums are, say, £52,000, a week's delay in making the first payment of premium would give £1000 from unearned risk). These items may bring up the annual profit from SUNDRIES to at least £8,000.

There is another source of profit to which I merely refer here, without attempting to estimate its probable fruitfulness; namely, bonuses allocated to policies which have ceased to exist before the expiry of the period required to transfer them from contingent to vested Bonuses. These forfeited bonuses fall in to the surplus fund, for accumulation and subsequent redistribution.

POLICIES LAPSED AND SURRENDERED.—We cannot learn the amount of profit from this source by a study of the Board of Trade Returns. I will not venture a guess at the probable annual sum so realized. I think that it cannot amount to much, seeing that it is usually policies on *good lives* that are surrendered or allowed to lapse, and that *bad lives* are left to increase the rate of mortality experienced by the office.

---

* These are sometimes specified in the Returns, but, no doubt, they are frequently placed to the credit of "Charges"; and so do not appear.

To summarize:—I have shown that the average annual cash profit *distributed among the policyholders* is £2,285,000, and that towards this there is

Contributed from "Loading" . .	£890,000	=39·00	p. ct.
" " Interest . .	1,040,000	=45·50	"
" " Profit on Investments	110,000	= 4·75	"
" " Sundries . .	8,000	= ·35	"
" " Balance, say, from dropped and surrendered policies and favourable mortality experience	237,000	=10·40	"
	£2,285,000		

Having thus shown how the profits are earned, I shall now endeavour to indicate how they are distributed among the policyholders.

The methods adopted for distributing realized profit, in the shape of bonuses, among the policyholders may be classed, generally, as follows:—

1.	In proportion to accumulated "loading" .	2 offices.
2.	In proportion to accumulated premiums .	20 offices.
3.	In proportion to the difference between the accumulated premiums and the value of the policy	8 offices.
4.	In proportion to the value of the policy .	3 offices.
5.	In proportion to the sum assured by a reversionary addition of a uniform percentage thereon for each premium paid .	30 offices.
6.	By reduction of premium; but this is merely an *application* of vested bonuses, which nearly all companies allow, so I do not further refer to it	8 offices.

Three offices have each a special system; and of 3 it is not stated precisely how the bonuses are allocated.

I shall refer briefly to each of these, and give average specimen bonuses taken from the Returns.

1.—*In proportion to "Loading."*

This system has the practical objection that decreasing reversionary bonuses are given under it. I cannot very well give average specimens; but those under the next head may serve the purpose.

2.—*In proportion to Accumulated Premiums.*

This system is really much the same as the last, and may be considered, on the whole, equitable, on the assumption that the premiums are "loaded" equally throughout by a percentage.

11 OFFICES.		AVERAGE SPECIMENS OF REVERSIONARY BONUSES, P. CENT. P. ANN.				
Age at Entry.	Average Premium.	Years in Force.				
		5	10	15	20	25
	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*
20	1 18 2	1 0 6	0 19 2	0 18 4	0 17 3	0 16 2
30	2 9 1	1 2 6	1 0 7	0 19 10	0 19 2	0 18 1
40	3 4 10	1 5 3	1 3 3	1 2 5	1 1 3	1 0 6
50	4 10 4	1 9 5	1 7 7	1 7 0	1 6 0	1 6 0

3.—*In proportion to the difference between the Accumulated Premiums and the Value of the Policy.*

At a first division of profits this proportion is simply the difference between the premiums paid since the date of the policy till the end of the bonus period, accumulated at interest, and the value of the policy at the end of the period. At second and future divisions a new policy is assumed to be taken out at the increased age, the premiums on which are accumulated, and the value of the hypothetical policy ascertained at the end of the period.

This system has, at least, the merit of meeting the popular demand for increasing bonuses. It may be briefly said that, under it, the more one pays in the more one will get out. In other words, if one lives to pay many comparatively small premiums by entering at a young age, or pays fewer but larger premiums on account of entering later in life, he will receive respectively more smaller bonuses or fewer larger ones.

There is sometimes a modification (not traceable in the Returns) in favour of the older policies by assuming the sum assured at the beginning of the period to be not only the sum originally in the policy, but also all the vested bonuses thereon. This gives bonus on bonus.

7 OFFICES.		AVERAGE SPECIMENS OF REVERSIONARY BONUSES, PER-CENT. PER ANN.					
Age at Entry.	Average Premium.	Years in Force.					
		5	10	15	20	25	30
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	1 18 6	1 0 11	1 2 1	1 3 0	1 4 9	1 6 1	1 7 11
30	2 8 6	1 2 0	1 3 10	1 5 1	1 6 10	1 9 7	1 14 8
40	3 3 7	1 3 9	1 5 11	1 8 8	1 13 5	2 0 10	2 12 3
50	4 8 3	1 7 4	1 12 6	1 19 10	2 9 6	3 4 3	4 13 7

4.—*In proportion to the Value of the Policy.*

This system is not much in vogue. There are some varieties of it. Such as these:—

(*a*). At a first division, bonuses are allocated in proportion to the value of the policy at the date of the valuation of the Company's affairs; and at future divisions in proportion to the *increase,* merely, on that value.

(*b*). In proportion to the value of the policy at the end of last period, improved at the higher rate of interest realized over that "expected".

(*c*). In proportion to the value of the policy at the end of last period, improved at compound interest, together with the accumulated gross premiums since paid, less the value of the policy at the end of the period.

I give specimen bonuses, taken from the returns of one office, to show the curious result under this (*c*) system:—

Age at Entry.	Years in Force.					
	5	10	15	20	25	
	£ *s.* *d.*	£ *s.* *d.*	£ s. *d.*	£ *s.* *d.*	£ *s.* *d.*	
20	1 7 7	1 7 11	1 9 10	1 8 8	1 10 11	p.ct. p.an.
30	1 13 5	1 12 1	1 14 6	1 11 8	1 8 10	
40	1 14 2	1 16 3	1 13 1	1 15 10	2 3 9	
50	2 1 0	2 3 3	2 12 6	2 10 9	2 15 10	

5.—*In proportion to the Sum Assured by Reversionary Addition of Uniform Percentage thereon for each Premium paid.*

16 OFFICES.		AVERAGE SPECIMENS OF REVERSIONARY BONUSES, P. CENT. P. ANN.				
Age at Entry.	Average Premium.	Years in Force.				
		5	10	15	20	25
	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*
20	1 18 10	1 7 9	1 7 9	1 7 9	1 7 9	1 7 9
30	2 8 8		(Same	for all	Ages.)	
40	3 4 7					
50	4 9 10					

With a view to give an additional benefit to older *policies* (not necessarily older *lives*), a modification of this system is practised, whereby compound bonuses (*i.e.* bonus on bonus) are given. This secures the double popular advantage of an increasing cash bonus, and an increasing reversionary bonus.

7 OFFICES.		AVERAGE SPECIMENS OF REVERSIONARY BONUSES, P. CENT. P. ANN.				
Age at Entry.	Average Premium.	Years in Force.				
		5	10	15	20	25
	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*	£ *s.* *d.*
20	2 0 1	1 5 0	1 8 0	1 9 5	1 11 10	1 14 4
30	2 10 4		(Same for	all Ages at	Entry).	
40	3 5 8					
50	4 10 8					

Having thus referred to the various methods of distributing profits, and having given specimens under each, I now show what bonuses the premiums paid should yield. For this purpose I have taken the ordinary whole-life participating rates of the 41 offices whose specimen bonuses I have given above, and found the increasing assurances which the average premium of these would provide. Calling such increase of assurance a *bonus,* the following are found to be the rates.*

Age at Entry.	Average Premium.	Annual Increment, or Bonus.
	£ s. d.	£ s. d.
20	1 18 11	1 10 0 per-cent
30	2 9 2	1 9 0 „
40	3 4 8	1 8 0 „
50	4 9 9	1 6 0 „

In order to compare these with the bonuses actually given, I add the following table, which contains the averages of all the specimens given throughout the paper.

41 OFFICES.		AVERAGE SPECIMENS OF REVERSIONARY BONUSES, P. CENT. P. ANN.				
		Years in Force.				
Age at Entry.	Average Premium.	5	10	15	20	25
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.
20	1 18 11	1 2 7	1 3 0	1 3 4	1 3 11	1 4 5
30	2 9 2	1 3 7	1 4 1	1 4 6	1 5 3	1 6 3
40	3 4 8	1 5 1	1 5 8	1 6 7	1 8 2	1 10 10
50	4 9 9	1 7 8	1 9 3	1 11 10	1 15 1	2 0 5

---

* H^M 3½; arrived at thus:—*Hardy's Valuation Tables,* Table XII, p. 36, and Table XIX, p. 63—Age 20:—

$$P_{20} = 1{\cdot}33$$

$$\begin{array}{lrlr} \pi_{20}\ \text{(Inc. Assurance)} = & {\cdot}408 & \text{per} & £1\ \ 0\ \ 0 \\ & {\cdot}204 & ,, & 0\ 10\ \ 0 \\ \hline & 1{\cdot}942 & & £1\ 10\ \ 0 \end{array}$$

Prem. = £1. 18*s*. 10*d*.

Actuarial Notes. *By* GEORGE KING, F.I.A.

No. 1.—*On a General Expression for the Value of an Annuity on the last* r *Survivors of* m *Lives*—$\left(a_{\overline{wxyz\ldots(m)}}^{\,r}\right)$.

LET ${}_np_{\overline{wxyz\ldots(m)}}^{\,r}$ represent the probability that at least r lives out of m lives will survive n years—the letter (m) in brackets relating to the number of lives involved, and not to the age of a life. The investigation for the present being perfectly general, and not depending on the value of n, we may in the meantime write the probability in question $p_{\overline{wxyz\ldots(m)}}^{\,r}$.

To find $p_{\overline{wxyz\ldots(m)}}^{\,r}$.

The required probability is the sum of the probabilities that exactly r lives will survive, that exactly $r+1$ lives will survive, and so on, up to and including the probability that all the m lives will survive. It may therefore be represented by the following scheme :—

$$
\begin{array}{ll}
 & \left\{\begin{array}{llll}
 p_{wx\ldots(r)}(1-p_y)(1-p_z) & \ldots\ldots & (\text{to } m-r \text{ factors}) & \\
 +p_{wy\ldots(r)}(1-p_x)(1-p_z) & \ldots\ldots & (\quad\text{do.}\qquad\text{do.}\quad) & \\
 +p_{wz\ldots(r)}(1-p_x)(1-p_y) & \ldots\ldots & (\quad\text{do.}\qquad\text{do.}\quad) & \\
 +\quad\&c.\qquad\&c. & \&c. & \&c. &
 \end{array}\right\} \\[2ex]
 + & \left\{\begin{array}{llll}
 p_{wx\ldots(r+1)}(1-p_y)(1-p_z) & \ldots & (\text{to } m-r-1 \text{ factors}) & \\
 +p_{wy\ldots(r+1)}(1-p_x)(1-p_z) & \ldots & (\quad\text{do.}\qquad\text{do.}\quad) & \\
 +p_{vz\ldots(r+1)}(1-p_x)(1-p_y) & \ldots & (\quad\text{do.}\qquad\text{do.}\quad) & \\
 +\quad\&c.\qquad\&c. & \&c. & \&c. &
 \end{array}\right\} \\[2ex]
 + & \qquad\&c.\qquad\&c.\qquad\&c.\qquad\&c.
\end{array}
$$

for exactly $r+2$, $r+3$, &c., survivors.

If by Σ^r we represent the sum of the probabilities of r joint lives surviving for all the combinations of r lives which can be made out of m lives, so that, for example, when there are four lives,

$$\Sigma^2=(p_{wx}+p_{wy}+p_{wz}+p_{xy}+p_{xz}+p_{yz}),$$

and if by (m, r) we denote the number of combinations which can be made of m things taken r together, when of course $(m, 0)$ becomes a symbol for unity, then the following expression will be seen to be identical with the foregoing scheme :—

$$\text{(I)}\quad p_{\overline{wxyz\ldots}\,\frac{r}{(m)}} = .\ \Sigma^{r}$$

$$-\Sigma^{r+1}\times\frac{(m-r,1)(m,r)-(m-r-1,0)(m,r+1)}{(m,r+1)}$$

$$+\Sigma^{r+2}\times\frac{(m-r,2)(m,r)-(m-r-1,1)(m,r+1)+(m-r-2,0)(m,r+2)}{(m,r+2)}$$

$$-\Sigma^{r+3}\times\frac{(m-r,3)(m,r)-\ldots\ldots-(m-r-3,0)(m,r+3)}{(m,r+3)}$$

$$+\quad \&c. \qquad\qquad \&c. \qquad\qquad \&c.$$

the law of the terms being sufficiently manifest.

It is evident that expression (I) correctly represents the scheme. Σ^{r} can arise only from the first section of the scheme, and appears only once. Σ^{r+1} arises from the first and second sections. The first section consists of (m, r) lines, and the number of probabilities in each line, each probability including $r+1$ lives, is the same as the number of bracket factors, that is, $(m-r, 1)$. The total number then, in the first section, of probabilities each including $r+1$ lives, is $(m,r)(m-r, 1)$. Among these every possible probability of $r+1$ lives is to be found, but each of such distinct probabilities occurs more than once. To find how often each distinct probability is repeated, we must divide the total number of probabilities just found, $(m,r)(m-r, 1)$, by the number of distinct probabilities, that is, $(m,r+1)$, and we have $\frac{(m,r)(m-r,1)}{(m,r+1)}$ as the number of times that Σ^{r+1} appears negatively in the first section. It also appears once, or, what is the same thing, $\frac{(m-r-1,0)(m,r+1)}{(m,r+1)}$, positively in the second section, and if we add the two results together we get the coefficient of Σ^{r+1} in expression (I).

A precisely similar argument holds for the coefficients of Σ^{r+2}, Σ^{r+3}, &c., and therefore the equation in (I) is established. The expression admits of great reduction, but the process, being comparatively easily carried out, may be omitted here. The result is the equation

$$\text{(II)}\quad p_{\overline{wxyz\ldots}\,\frac{r}{(m)}}=\Sigma^{r}-r\Sigma^{r+1}+\frac{r(r+1)}{2}\Sigma^{r+2}-\frac{r(r+1)(r+2)}{2.3}\Sigma^{r+3}+\&c.$$

The expression on the right of equation (II) follows the law of the binomial theorem; and if we take a liberty, and consider Σ to be a symbol of operation, we can write symbolically

$$\text{(III)}\quad p_{\overline{wxyz\ldots(m)}}^{\;r}=\Sigma^r(1+\Sigma)^{-r},$$

a very compact expression, easily remembered, and consequently very useful. It must be observed that all terms must be omitted after that containing Σ^{r+t} where $r+t=m$, because a higher index than m, for Σ is, by the nature of the case, impossible.

Since $a_{\overline{wxyz\ldots(m)}}^{\;r}$ is merely the sum for all values of n of $v^n{}_np_{\overline{wxyz\ldots(m)}}^{\;r}$, to obtain $a_{\overline{wxyz\ldots(m)}}^{\;r}$, we have only to substitute annuities for probabilities under the symbol Σ. If then we write Z^r for the sum of the values of the annuities on r joint lives for all the combinations of r lives that can be made out of m lives, we have

$$\text{(IV)}\quad a_{\overline{wxyz\ldots(m)}}^{\;r}=Z^r-rZ^{r+1}+\frac{r(r+1)}{1.2}Z^{r+2}-\text{\&c.},$$

and $$\text{(V)}\quad a_{\overline{wxyz\ldots(m)}}^{\;r}=Z^r(1+Z)^{-r}.$$

As an example, let it be required to determine, when $m=4$, the value of $a_{\overline{wxyz}}^{\;2}$.

$$\text{By (IV)}\qquad a_{\overline{wxyz}}^{\;2}=Z^2-2Z^3+3Z^4$$
$$=(a_{wx}+a_{wy}+a_{wz}+a_{xy}+a_{xz}+a_{yz})$$
$$-2(a_{wxy}+a_{wxz}+a_{wyz}+a_{xyz})+3a_{wxyz}.$$

This result we can easily verify independently.

The problem does not appear to have been treated very satisfactorily by any author. *Mr. Milne,* pp. 41 *et seq.*, and Problem II, p. 124, gives the solution of an analogous problem, but his solution seems to me to be laboured, and the notation he adopts tends rather to increase than to alleviate obscurity. *Mr. Baily,* Prob. 3, page 58, solves the problem for the special case where $m=3$, and $r=2$, and he then gives the general formula, but without proof. It should be noted that in the spurious edition of *Baily,* as well as in the original edition, the passage appears correctly, but that in the edition of *Baily,* edited by the late Mr. Filipowski, an error occurs in the general formula. This last-named edition is almost worthless on account of the numerous mistakes, as well of principle as typographical. Mr. Jones diplomatically avoids the subject, but in such a way as to cause confusion to students. He tabulates, pages 110, 113, and 114, the symbols for the probability and the annuity, but nowhere assigns to them a value, except, pages 112 and 132, in the special case where $r=1$; while yet, page 154, he finds the assurance corresponding to the still unknown annuity.

No. 2.—*On Reversionary Annuities.*

THE theory of Reversionary Annuities may be elegantly put into a very general form. In trying to do so, we shall find it convenient to begin by demonstrating four formulas for the value of an annuity on a compound status.

Employing the letters a, b, c, to denote the lives belonging to one status, and x, y, z, to denote those belonging to another status, we have

$$\alpha. \quad a_{(abc)(xyz)} = a_{abcxyz} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

$$\beta. \quad a_{\overline{xyz}} = a_x + a_y + a_z - a_{xy} - a_{xz} - a_{yz} + a_{xyz}.$$

If now the receipt of the annuity in β be further conditioned on the joint existence of other lives, a, b, c, each probability involved in each payment of the annuity $a_{\overline{xyz}}$ must be multiplied by the corresponding p_{abc}. We therefore have

$$\begin{aligned} a_{abc.\overline{xyz}} &= \Sigma v^n {}_np_{abc}\,{}_np_{\overline{xyz}} \\ &= a_{abcx} + a_{abcy} + a_{abcz} - a_{abcxy} - a_{abcxz} - a_{abcyz} \\ &\qquad + a_{abcxyz} \quad . \ . \ . \ . \quad (2) \end{aligned}$$

$$\begin{aligned} \gamma. \quad a_{\overline{abc}.\overline{xyz}} &= \Sigma v^n {}_np_{\overline{abc}}\,{}_np_{\overline{xyz}} \\ &= \Sigma v^n \{1-(1-{}_np_a)(1-{}_np_b)(1-{}_np_c)\}\{1-(1-{}_np_x)(1-{}_np_y)(1-{}_np_z)\} \\ &= \Sigma v^n [\{1-(1-{}_np_a)(1-{}_np_b)(1-{}_np_c)\} + \{1-(1-{}_np_x)(1-{}_np_y)(1-{}_np_z)\} \\ &\qquad - \{1-(1-{}_np_a)(1-{}_np_b)(1-{}_np_c)(1-{}_np_x)(1-{}_np_y)(1-{}_np_z)\}] \\ &= a_{\overline{abc}} + a_{\overline{xyz}} - a_{\overline{abcxyz}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (3) \end{aligned}$$

This last formula is evidently true, for if 1 be receivable annually as long as one of the lives a, b, c, survives, and another 1 be receivable as long as one of the lives x, y, z, survives, and if at the same time 1 be payable as long as any of all the lives a, b, c, x, y, z, survives, there remains an annuity of 1 receivable as long as one of the lives a, b, c, survives jointly with one of the lives x, y, z.

Similarly,

$$\delta. \quad a_{(\overline{abc})(\overline{xyz})} = a_{\overline{abc}} + a_{xyz} - a_{\overline{abcxyz}} \quad . \ . \ . \ . \ . \ . \quad (4)$$

All the above four formulas are quite general. Any number of lives from one upwards may be included in each status, and a term certain may be substituted for a life.

By means of the formulas, we can conveniently pursue our further investigations.

If s be one status and z another status, required $a_{z|s}$, the value of an annuity to continue during the existence of the status s, but the first payment of which is not to be made till the end of the year in which the status z fails.

Evidently $$a_{z|s}=a_s-a_{sz} \quad \text{. . . . . . .} \quad (5)$$

For the benefit required is an annuity payable as long as the status s subsists, less an annuity payable as long as the status s subsists jointly with the status z.

The formula is absolutely general, and, with the help of the four previous formulas, can be easily applied in all cases. To take a few examples:—

$$a_{y|x}=a_x-a_{xy}$$ *Jones*, p. 139.

$$a_{xyz|abc}=a_{abc}-a_{xyzabc}$$ *Baily*, Prob. 4.

$$a_{\overline{n}|\,|x}=a_x-a_{x\overline{n}|}=a_x-{}_{|n}a_x={}_{n|}a_x$$

$$a_{x|\overline{n}|}=a_{\overline{n}|}-{}_{|n}a_x$$

$$\begin{aligned}a_{\overline{xyz}|abc}&=a_{abc}-a_{abc.\overline{xyz}}\\&=a_{abc}-a_{abcx}-a_{abcy}-a_{abcz}+a_{abcxy}\\&\quad+a_{abcxz}+a_{abcyz}-a_{abcxyz}\end{aligned}$$ *Baily*, Prob. 5.

$$\begin{aligned}a_{xyz|\overline{abc}}&=a_{\overline{abc}}-a_{\overline{abc}.xyz}\\&=a_{\overline{abc}}-a_{xyza}-a_{xyzb}-a_{xyzc}+a_{xyzab}\\&\quad+a_{xyzac}+a_{xyzbc}-a_{xyzabc}\end{aligned}$$ *Baily*, Prob. 6.

$$a_{\overline{xyz}|\overline{abc}}=a_{\overline{abc}}-a_{\overline{abc}.\overline{xyz}}=a_{\overline{abcxyz}}-a_{\overline{xyz}}$$ *Baily*, Prob. 7.

In all the above cases, if the annuity is not to continue during the whole duration of the given lives, but is to be temporary or deferred, we must make the component annuities temporary or deferred also. That is

$${}_{|n}a_{z|s}={}_{|n}a_s-{}_{|n}a_{sz}\,; \quad {}_{n|}a_{z|s}={}_{n|}a_s-{}_{n|}a_{sz}\,.$$

The five formulas given above enable us, by inspection, to write down the answers to Mr. Baily's problems 8 to 18, and to the examples following Problem 3 in Mr. Milne's work, without having, in each case, to analyze *de novo* the probabilities. A single illustration will suffice.

Baily, Problem 10.—A, B, and C, (aged x, y, and z) agree to purchase an annuity on the longest of their lives, to be divided amongst them in the following manner: A and B are to enjoy it equally during their joint lives; if A dies first, then B and C are to enjoy it equally during their joint lives, and the survivor of them is to have the whole; but if B dies first, then A is to enjoy the whole during his life, and after his decease it is to devolve wholly to C. To find the value of their respective shares.

$$\begin{aligned}\text{Share of A} &= \tfrac{1}{2}a_{xy} + a_{y|x} \\ &= a_x - \tfrac{1}{2}a_{xy}.\end{aligned}$$

$$\begin{aligned}\text{Share of B} &= \tfrac{1}{2}a_{xy} + \tfrac{1}{2}a_{x|yz} + a_{\overline{xz}|y} \\ &= a_y - \tfrac{1}{2}a_{xy} - \tfrac{1}{2}a_{yz} + \tfrac{1}{2}a_{xyz}.\end{aligned}$$

$$\begin{aligned}\text{Share of C} &= \tfrac{1}{2}a_{x|yz} + a_{\overline{xy}|z} \\ &= a_z - \tfrac{1}{2}a_{yz} - a_{xz} + \tfrac{1}{2}a_{xyz}.\end{aligned}$$

The sum of these is, as it ought to be, $a_{\overline{xyz}}$.

The foregoing note was first written out some years ago, before I had had the opportunity of perusing Mr. Sang's treatise on Life Contingencies. He gives an analysis of the same character, but there is possibly sufficient dissimilarity to justify me in carrying out the original idea of publishing the note.

INSTITUTE OF ACTUARIES.

PRELIMINARY EXAMINATION, 1880.

Examiners.—J. HERON DUNCAN, Esq.; W. VAUGHAN, Esq.; F. B. WYATT, Esq.; and T. E. YOUNG, B.A.

I.

1. Define precisely the principle of Double Entry Book-keeping.

2. In what accounts should the following transactions be entered, and how?

(*a*) Paid £150 for Office Furniture, $\frac{1}{10}$th of which is to be charged against the Current Year's Income.

(β) Shares to be forfeited on which £350 have been paid up.

3. Express as decimals $\frac{1}{17}$, $\frac{2}{17}$ . . . . $\frac{16}{17}$, and explain any methods of obtaining one value from another, and for shortening the work.

4. State and prove the rule for converting a recurring decimal into a vulgar fraction.

Find the value of $\cdot\dot{0}7692\dot{3}$ of $\cdot\dot{1}4285\dot{7}$ of £4. 11*s*.

5. Multiply ·001 by ·01, and divide the product by ·00001 × ·0001.

6. Solve:—

$$x+\sqrt{3x+13}=9,$$

and explain why one of the results obtained does not satisfy the equation.

7. A railway train, running from one station to another, meets on the way with an accident, which causes it to diminish its speed by $\frac{1}{n}$th of what it was before, and it is in consequence a hours late. If the accident had happened b miles further on, the train would have been c hours late. Find the rate at which the train was going before the accident happened.

8. Construct the equation whose roots are $\frac{1}{m}$ and $\frac{1}{n}$.

9. There are p arithmetical progressions each beginning with 1, and the common differences are respectively 1, 2, 3, ... p. Find the sum of the nth terms.

10. Sum to n terms

$$1+3+7+15+31+\ldots\ldots$$

and

$$1+5+13+29+61+\ldots\ldots.$$

11. State shortly the advantages of the Common system of Logarithms, and prove that the Logarithms to the base 10 of all numbers which consist of the same figures, but differ merely by the position of the decimal point, have the same mantissæ.

12. The Captain of a ship has on board 8 trained sailors, 10 trained engineers, and 32 other men who can help in either department. In how many ways can he select for duty 12 men working as sailors, and 13 as engineers, there being 4 trained sailors and 5 trained engineers only in each arrangement?

13. In a bag there are 4 black and 6 white balls. A is to draw, and then B. The prize (20*s.*) is to be won by the person who first draws a white ball. Find the value of B's first chance, (I) replacing, (II) not replacing.

II.

14. How many different numbers of 3 digits can be formed out of the numerals 0, 1, 2, 3, 4, and 5, and what is the sum of all these numbers?

15. Determine a formula for the sum of the squares of the terms composing a Geometrical Progression.

16. Explain in detail the formation and use of the Differences and Proportional Parts in Tables of Logarithms, and from the Tables before you write down

(*a*) The logarithm of ·0456789.

(*β*) The number corresponding to Log $\bar{2}\cdot3456789$.

17. Solve *any two* of the equations

(*a*) $x^4-96x=80$.

(*β*) $\sqrt[m]{(1+x)^2}-\sqrt[m]{(1-x)^2}=\sqrt[m]{1-x^2}$.

(*γ*) $\left.\begin{aligned} x^2+y^2&=5 \\ x^6+y^6&=35 \end{aligned}\right\}.$

18. Write down the first 3 terms of the expansion of

$$\frac{x}{(1+x)^i-1}.$$

19. Expand $\sqrt{(\sqrt{a}-\sqrt{x})}$ by the Binomial Theorem.

20. A certain number is composed of n digits, all different, and no cyphers; and the first two digits are 1 and 2. If a person be informed of these facts only, what is the probability that he will guess the number correctly?

21. A die is thrown four times. Express the *chances* in favour of a six being thrown (1) once only, (2) twice only, and (3) once at least.

22. A and B each cast a column of figures, and make the same total. If A casts correctly 9 times out of 10, and B 19 times out of 20, and the chances of their agreeing in error be 1 in 50,000, what is the probability that the real total has been obtained?

23. Having given the values of Annuities at the following rates of interest, namely, at

3 $=15·863$
3½ $=14·941$
4 $=14·105$
4½ $=13·343$
5 $=12·648$.

Find the value at 4·328 per-cent.

24. If $u_0=100{,}000$, $u_7=97{,}624$, $u_8=97{,}245$, and $u_9=96{,}779$, find u_1 to u_6 inclusive by means of constant *third* differences.

25. If u_x be a function of x of the form $u_x=b_1x+b_2x^2+\&c.$ *ad inf.*, show that it can also be expressed in the form

$$u_x=\frac{b_1x}{1-x}+\frac{\Delta b_1x^2}{(1-x)^2}+\frac{\Delta^2b_1x^3}{(1-x)^3}+\ldots\ldots$$

(3 orders of differences will suffice).

INTERMEDIATE EXAMINATION, 1880.

Examiners.—T. H. Cooke, Esq.; C. D. Higham, Esq.; W. Manly, Esq.; and J. Whitcher, Esq.

I.

1. Find the following probabilities, namely:—That of two lives (x) and (y)—

(α) both will not survive n years;
(β) neither will survive n years;
(γ) either or both will survive n years;
(δ) x will die in the nth year, y being alive at the moment of x's death.

2. What is the probability that out of seven individuals of a given age, four at least will die in a given time?

3. An amount is advanced for n years certain. What sum must be paid annually in order that the lender may realize i per 1 on the

capital, and that the amount set apart yearly for reproducing capital may accumulate at j per 1 to the sum advanced by the end of the term?

4. Four persons, A, B, C, and D, contribute equal sums towards the purchase of a perpetuity. Find the number of years that A, B, and C may successively enjoy it, D having the absolute reversion, so that all four may benefit equally.

5. Given the time in which a debt bearing interest is discharged by equal annual payments, show in how much less time the debt will be discharged if m equal instalments are paid each year.

6. Give a short account of the Government Annuity Tables of 1825 and 1860.

7. Show that by De Moivre's hypothesis—

(α) the average duration of life at any age equals the probable after-lifetime;

(β) that at any age the probability of dying in each of the subsequent years is the same.

8. Determine by argument the single premium for an assurance of 1; and afterwards put your argument into symbols.

9. Find the value of an annuity on a single life, and show how the annuity at any age may be given in terms of that at the next higher age.

10. Show how to construct the D column by the continuous method, and point out how the results can be verified at any stage.

11. Show how the C, M, and R columns may be derived from the D, N, and S columns.

12. Show how Orchard's tables can be employed in finding the single and annual premiums for an endowment assurance.

13. The value of all the policies in an office at the beginning of any year, and the premiums for that year, both accumulated at interest for one year, are equal to the claims of the year and the value of the policies at the end of the year. Under what conditions is this true? and show what modification would be required on the assumption that the premiums are payable throughout the year.

II.

14. Explain what is meant by a Table of Mortality, and describe its most usual and most convenient form.

15. Given a table of annual premiums, show how to construct a corresponding Table of Mortality.

16. Find the value of a policy by the retrospective method.

17. Find the single premium for an assurance payable at the instant of death.

18. If the Premiums for a deferred annuity are returnable with compound interest if the life fail before the time of entering into possession, show that

$$P_{t|}a_x = \frac{vi \, . \, a_{x+t}}{(1+i)^t - 1} \, .$$

19. Find the single premium for an assurance payable in the event of x dying in the lifetime of y or within t years after the death of y.

20. Show that the value of 1 to be received on the failure of the successive lives x and y is A_xA_y. When is the second life supposed to be nominated?

21. Investigate a formula for the market value of a life interest in a terminable annuity.

22. Determine the annual premium for an annuity of s to continue during the lifetime of B aged y, after the death of A aged x; with the provision that should A survive the age of $x+n$ a sum of t is to be at once paid to B instead of the annuity.

23. Investigate a formula for applying a bonus towards the payment of a policy during the lifetime of the assured.

24. If the law of sickness be such that at any age two are constantly sick for one that dies, show that the single premium for a sick allowance of 10*s*. a week at age x, to cease at age 65, is

$$£52{\cdot}18\left\{\frac{M_x-M_{65}}{D_x}\right\}.$$

25. Draft endorsements for the following:—

(*a*) For reduction of premium in lieu of reversionary bonus;
(*β*) For whole-world license.

FINAL EXAMINATION, 1880.

Examiners.—G. W. BERRIDGE, Esq.; G. FINLAISON, Esq.; G. HUMPHREYS, M.A.; and T. B. SPRAGUE, M.A.

I.

1. Classify the various methods of graduation proposed by English writers on the subject according to their fundamental differences, and state any reasons you may have for preferring one to another.

2. Describe the method of determining, from the Registers of an Assurance Company, the relative numbers of living and dying at each year of age.

3. Give reasons why adjustment should not be applied to monetary values rather than to the elementary results of a mortality table.

4. What facts would you consider necessary to be ascertained previously to determining the rates of mortality at every age in a given community?

5. What may be considered the comparative advantages and disadvantages of the methods of graduation proposed by Messrs. Gompertz and Woolhouse?

6. What is the relative amount of reserve for ordinary whole-life policies brought out by the use of the following tables?—

a. H^{M}.
b. $H^{M(5)}$.
c. Carlisle.
d. English Life Table No. 3.

7. Describe briefly the various systems of distribution of profit in use among life offices, and the advantages incidental to each system.

8. Supposing a valuation to be required of a large insurance company, on a gross premium basis, with a percentage deduction for future profits and expenses less than the loading, how would you proceed so as to be able to eliminate negative values?

9. State the conditions which must obtain in order that the reserve at a valuation may be sufficient and no more than sufficient.

10. Explain briefly the way in which you would determine the net liability of an assurance company.

11. When a company makes a loan on one of its policies, what provisions and powers should the mortgage-deed contain?

12. Explain the term "Equity of Redemption."

13. Are there any circumstances under which an assurance company may be compelled to pay interest on the sum assured after the death of the life assured?

II.

14. Supposing an assurance company discovers that an assurance has been effected fraudulently, can any immediate relief be obtained, and how?

15. Mention the penalties under the Stamp Act for issuing an unstamped Policy. How could such an omission be remedied? Is there any limit as to time?

16. What is the meaning of the term "acceptances" in the accounts of joint-stock banking companies? Are there any analogous items in the accounts of the Bank of England?

17. How should the calculation of the average yearly rate of interest obtained on the investments of a life insurance society be affected by reversions purchased?—

a. When no amount of principal has been received in the year.

b. When money has been received on account of reversions fallen in.

18. State the privileges enjoyed by the Bank of England. In what respects does it differ from other joint-stock banks.

19. Supposing it were required to show the quarterly or monthly Balance (of the accounts of a life insurance society) with quickness and correctness, what system of book-keeping would you propose should be èmployed?

20. Mention the leading objects of the Bank Act of 1844. Do you consider they have been attained by the working of the Act?

21. What items in the account of a life assurance company would you select as particularly illustrating the condition of the company? Give reasons.

22. An estate has been devised to A for life with remainder to B for life, with remainder to B's issue in tail male. B has a son, C, who has attained 21. State the necessary arrangements as to assurance, so as to indicate the terms on which, during A's tenancy, money can be raised by B and C separately and together.

23. A landed estate is the property of A, a male aged 35; subject to the life interest of B, a female aged 60, in one-third of the annual income; and the further life interest of C, a male aged 59, to another one-third of the annual income, and in the event of his surviving B to a further one-sixth of the annual income. What are the proportionate values of the interests of A, B, and C?
Give your answer in symbols, but state what tables and rate of interest you would employ.

24. A, aged 70, has a life interest in a Fund of £10,000 invested at 4 per-cent, which Fund comes to B, or his representatives, absolutely on the death of A. State in what proportions you would be disposed to divide the £10,000 between A and B at the present time, under the three following circumstances:—

At the request of A;
Do. do. B;
By mutual consent.

Give reasons.

25. A is entitled to a life interest in an estate on the death of B, his uncle, provided B dies without leaving issue male surviving him, and is desirous of raising money by way of a mortgage on it. Upon what principles would you determine the relation between the sum to be advanced and the redemption money and annuity payable on the death of the uncle? Give the wording generally of the Issue Policy.

PROCEEDINGS OF THE INSTITUTE.—SESSION 1879-80.

First Ordinary Meeting, Monday, 24 *November* 1879.

The President in the Chair.

The following gentlemen were elected members of the Institute, namely—

Associates.

Colenso, Francis Ernest.
Lawrence, Alfred.
Pewtress, Stephen Leslie.
Sunderland, Arthur Wellesley, M.A.
Wornum, Thornton Selden.

A paper, by Mr. T. B. Sprague, "On the Rate of Remarriage among Widowers", was read by the Hon. Sec., Mr. G. W. Berridge.

The following gentlemen took part in the discussion upon this paper:—The President, Messrs. Baden, Berridge, G. King, Neison, Sutton, Walford, and Whitcher.

Mr. Cornelius Walford briefly described the Collection of Mortality Experience in the United States.

Second Ordinary Meeting, 29 *December* 1879.

The President in the Chair.

The following gentleman was elected a member of the Institute—

Associate.

Hutchinson, George Ross.

The Hon. Sec., Mr. G. W. Berridge, read a paper, prepared by Mr. T. B. Sprague, "On the Probability that there will hereafter be Issue of a Marriage hitherto Childless."

The following gentlemen took part in the subsequent discussion:—The President, Messrs. Berridge, Brown, A. J. Finlaison, Higham, and Walford.

Third Ordinary Meeting, 26 *January* 1880.

The President in the Chair.

The following gentlemen were elected members of the Institute—

Associates.

Calderon, Henry Philip.
Hawkins, Percy Ingleby.

Mr. G. King read a paper, prepared by himself and Mr. G. F. Hardy, entitled "Notes on the Practical Application of Mr. Makeham's Formula to the Graduation of Mortality Tables."

The discussion which followed was conducted by the President, Messrs. Berridge, G. F. Hardy, Manly, Sutton, and Walford.

Fourth Ordinary Meeting, 23 *February* 1880.

The President in the Chair.

Mr. Cornelius Walford read a paper on "The Position of Life and other Insurance Associations in relation to their Local Medical Examiners—Is it satisfactory?"

The following gentlemen took part in the discussion upon this paper:—The President, Messrs. Adler, Baden, Capern (a visitor), R. P. Hardy, Humphreys, Newbatt, and Dr. Ogle (a visitor).

Fifth Ordinary Meeting, 5 *April* 1880.

The President in the Chair.

The following gentlemen were elected members of the Institute—

Associates.

Da Costà, Louis Edward Martius.
Manwaring, Henry.
Stock, Edward James.

Mr. J. Sorley read a paper on "Observations on the Graduation of Mortality Tables, with special reference to the Conditions under which certain Methods are to be preferred."

In the subsequent discussion, the following gentlemen took part:—The President, Messrs. A. J. Finlaison, R. P. Hardy, G. F. Hardy, G. King, Cornelius Walford, and J. Terry.

Sixth Ordinary Meeting, 26 *April* 1880.

The President in the Chair.

The following gentlemen were elected members of the Institute:—

Fellow.

Thomson, Robert.

Associates.

Ogilvie, Alexander.
Povah, Charles.
Ryan, Gerald H.
Straker, Edward Robert.
Thompson, Jacob.

A paper by Mr. R. Teece, on "The Reserve that should be made for Policies on Recently-Assured Lives", was read by the Hon. Sec., Mr. G. W. Berridge.

In the discussion which followed, the President, Messrs. Ackland, Baden, Berridge, Grimes, Manly, Strachan, and Walford, took part.

The Thirty-First Annual General Meeting, Saturday, 5 *June* 1880.

The President (Mr. A. H. Bailey) in the Chair.

Mr. Humphreys (Hon. Sec.) read the circular convening the meeting, the minutes of the ordinary meeting held in April, which were signed as correct by the President, and the following Report of the Council and Statement of Accounts:—

"The Council have great pleasure in meeting the members at the end of the thirty-first session of the Institute.

"The total number of members on the 31st March 1879 was 374, since when 8 new members have been elected, whilst 26 have resigned or otherwise left the Institute, leaving 356 members at the close of this session.

"The income during the year was £881. 9*s*. 10*d*., and the expenditure £571. 10*s*. 10*d*.; leaving the funds of the Institute now £3,127. 11*s*. 7*d*., as against £2,803. 11*s*. 11*d*. last year.

"Papers on the following subjects have been read at the ordinary meetings:—

"*First*—'On the Rate of Remarriage among Widowers'—by Mr. T. B. Sprague.

"*Second*—'On the Probability that there will hereafter be Issue of a Marriage hitherto Childless'—by Mr. T. B. Sprague.

"*Third*—'Notes on the Practical Application of Mr. Makeham's Formula to the Graduation of Mortality Tables'—by Messrs. G. King and G. F. Hardy.

"*Fourth*—'The Position of Life and other Insurance Associations in relation to their Local Medical Examiners—Is it satisfactory?'—by Mr. Cornelius Walford.

"*Fifth*—'Observations on the Graduation of Mortality Tables, with special reference to the Conditions under which certain Methods are to be preferred'—by Mr. J. Sorley.

"*Sixth*—'The Reserve that should be made for Policies on Recently-Assured Lives'—by Mr. R. Teece.

"EXAMINATIONS.

"The report of the examiners for the examinations of the Institute, held on the 23rd and 24th April last, gives as the results for the

"*Preliminary Examination.*

"That 28 candidates presented themselves for this examination, of whom 13 passed, in the following order of merit:—

1.—A. Carter.
2.—F. M. Ashton.
3.—E. R. Straker.
4.—J. H. Barnes.
5.—H. P. Calderon.
6.—A. H. Bridgman.
7.—J. D. Elliott.
8.—S. N. Moon.
9.—S. Day.
10.—S. L. Pewtress.
11.—T. J. Pound.
12.—W. R. Hopkins.
13.—G. H. Ryan.

"*Intermediate Examination.*

"Fifteen gentlemen presented themselves for examination, and 8 passed, in the following order of merit:—

1.—A. W. Sunderland.
2.—L. M. Simon.
3.—H. E. Wilson.
4.—J. MacGowan.
5.—A. F. Burridge,
E. A. Colquhoun, } eq.
7.—H. W. Andras,
A. J. Hume, } eq.

"*Final Examination.*

"Three candidates presented themselves, and they passed in the following order:—

1.—G. F. Hardy.
2.—F. W. White,
W. J. H. Whittall, } eq.

"CLASSES.

"The Council being desirous of assisting as far as possible the younger members in their studies, have arranged for the continuance of the classes during the next session. The time at which the classes will meet will be duly announced, and the subjects of the lectures will be those of the preliminary and intermediate examinations.

" TEXT-BOOK.

" The Council regret to be informed by Mr. Sutton that his official and professional engagements for some time past have prevented rapid progress with the text-book. The necessity for accuracy and compendious method would under any circumstances extend the time required for such a work, and neither the Council nor the editor desire to sacrifice the quality of the work to expedition in its publication.

" LIBRARY.

" A number of books and pamphlets have been added to the library, and authority has been given for the purchase of others. Considerable progress has been made in the preparation of the catalogue, and the whole of the manuscript is expected to be shortly ready for the printer.

[*Statement of Accounts, p.* 308.]

The President moved and Mr. A. Baden seconded the adoption of the Report and the Accounts for the year, which was agreed to unanimously.

The following gentlemen were unanimously elected as President, Vice-Presidents, Council, and Officers, for the ensuing year:—

President.

Arthur H. Bailey.

Vice-Presidents.

Andrew Baden.
Charles John Bunyon, M.A.
George Cutcliffe.
Ralph Price Hardy.

Council.

*Marcus N. Adler, M.A.
Andrew Baden.
Arthur H. Bailey.
George William Berridge.
Thomas G. C. Browne.
Charles John Bunyon, M.A.
George Cutcliffe.
Archibald Day.
David Deuchar.
John Duncan.
Alexander John Finlaison.
John Ralph Grimes.
Ralph Price Hardy.
Stewart Helder.
Augustus Hendriks.
William Barwick Hodge.
George Humphreys, M.A.
Frank McGedy.
Henry William Manly.
James Meikle.
Edward A. Newton, M.A.
*Arthur Pearson.
Henry William Porter, B.A.
H. Ambrose Smith.
Thomas Bond Sprague, M.A.
William Sutton, B.A.
Spencer Campbell Thomson, B.A.
James Valentine.
John Hill Williams.
Thomas Emley Young, B.A.

Note.—Those marked * are new members.

Treasurer.

John Ralph Grimes.

Honorary Secretaries.

George William Berridge.
George Humphreys, M.A.

A vote of thanks was given to the retiring Auditors, Messrs. Gage, Searle, and White; and Messrs. Colquhoun, Searle, and Windett, were elected Auditors during the ensuing year.

It was unanimously resolved that a vote of thanks be given to the President, Vice-Presidents, Council, Editor, Officers, and Examiners, for their services during the past year.

INSTITUTE OF ACTUARIES.

Income and Expenditure for the Year ending 31 *March* 1880.

Dr.

1879-80.	£	s.	d.	£	s.	d.
Amount of Funds on 31 March 1879—						
Messenger Legacy Fund	262	15	0			
Brown Prize Fund	260	1	1			
General Fund	2,280	15	10			
				2,803	11	11
Annual Subscriptions—						
Town Fellows	245	14	0			
Country "	73	10	0			
Town Associates	294	0	0			
Country "	101	17	0			
				715	1	0
Mortality Experience Sales	78	18	3			
Log. Card Sales	0	4	0			
				79	2	3
Examination Fees				14	14	0
Dividends				86	18	3
				£3,699	2	5

Cr.

1879-80.	£	s.	d.	£	s.	d.
Rent	81	5	0			
Salaries	178	19	0			
Journal (less Sales £101)	117	0	0			
Stationery	58	17	5			
Ordinary Meetings	13	18	1			
House Expenses	25	8	6			
Postage	11	1	8			
				486	9	8
Tutor				31	10	0
Examination Expenses				22	0	0
Library				25	15	4
Miscellaneous Expenses				5	15	10
Amount of Funds on 31 March 1880—						
Messenger Legacy Fund	268	15	0			
Brown Prize Fund	267	17	9			
General Fund	2,590	14	10			
				3,127	11	7
				£3,699	2	5

Balance Sheet, 31 *March* 1880.

Dr.

	£	s.	d.	£	s.	d.
Messenger Legacy Fund (£211. 1*s*. 10*d*. Consols), cost	203	17	8			
Unappropriated Dividends	65	1	4			
				268	19	0
Brown Prize Fund (£200 Metropolitan Railway 4 per-cent Debenture Stock)	200	0	0			
Unappropriated Dividends	67	17	9			
				267	17	9
General Fund				2,590	14	10
				£3,127	11	7

Cr.

	£	s.	d.	£	s.	d.
Consols, £1,600 Stock, cost				1,498	19	8
Metropolitan Railway 4 per-cent Debenture Stock, £1,200	1,225	17	3			
Cash at London and Westminster Bank	392	2	0			
Cash in hand	1	4	1			
				393	6	1
Arrears of Subscriptions:—						
Country Fellows	4	4	0			
Town Associates	2	2	0			
Country "	3	3	0			
				9	9	0
				£3,127	11	7

Examined and found correct, 4 *May* 1880.

(Signed) { U. W. Gage, Thomas J. Searle, Francis W. White, } *Auditors.*

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

Observations on the Graduation of Mortality Tables; with special reference to the Conditions under which certain Methods are to be preferred. By James Sorley, *of the Life Association of Scotland, Edinburgh, Fellow of the Institute of Actuaries.*

[Read before the Institute, 5 April 1880.]

THE subject of Graduation has recently been brought somewhat prominently before the profession in connection with the exposition of various methods that have been proposed. While the bases of these individual methods have been set forth in a way leaving nothing to be desired, the precise relation of each to the entire problem of the adjustment of probabilities derived from statistics, does not appear to have been so clearly marked out; and therefore, in the following observations, it is proposed to keep prominently in view the conditions under which one method is to be preferred to another.

An intelligent outsider, desirous of becoming acquainted with the actuarial theory of graduation, might, if he overlooked the necessarily limited application of particular systems, derive from his researches an impression of disagreement among the various authorities, which would have a rather bewildering effect. For example, he would find that:—

Mr. Woolhouse has stated (*J.I.A.* xxi, 38), "Every reconsideration of the subject has only confirmed my settled conviction that, from all points of view, the theoretical basis of my method is the true one, and that it must eventually meet with universal acceptance";

That Professor McCay (*J.I.A.* xxii, 24), after developing an entirely new method, and applying it to graduate the Institute H^{MF} observations, compares the resulting table, by means of five criteria of a good adjustment, with the corresponding graduation of Mr. Woolhouse, and concludes (p. 37) that "in all these five particulars this adjustment is better than the excellent table" of that gentleman.

Again, that Mr. Sprague states (*J.I.A.* xxii, 95) he believes "the graphic method of adjustment . . . . to be generally the best that can be adopted";

While Mr. Walford informs the members of the Institute (*Insurance Record,* 5 Dec. 1879) that, in the combined experience of the American offices, "Mr. Makeham's formula had proved applicable all the way through, while Mr. Woolhouse's has not been found to work so well."

And, lastly, he would probably have his attention directed to the well-known remarks of Professor De Morgan (*Essay on Probabilities,* p. 162), "The events of single years are subject to considerable error, and generally present such varieties of fluctuation, that it has become usual to take some arbitrary and purely hypothetical mode of introducing regularity. This practice cannot be too strongly condemned, since the tables thereby lose some of their value as representations of physical facts, without any advantage ultimately gained."

But, if the outsider alluded to pursued his researches far igh, he would discover that the divergencies of opinion are not so great as at first appeared, because the areas occupied by the various methods are by no means conterminous—a considerable portion of the ground, it is true, being common, but large tracts being foreign to certain of the methods. And thus he would come to understand that no one scheme supersedes the others; that each one has its distinctive excellence, rendering it in certain circumstances preferable to the others; and that consequently no actuary can get rid of the responsibility of using his own judgment in the choice of a method for each individual case as it arises.

1.—*Graduation in general.*

The adjustment of mortality tables framed from the records of actual observations, may be held to be undertaken mainly for two reasons :—

(1). To correct errors in the records themselves—mis-statements, either wilful or clerical—and which may be called Personal Errors;

(2). To smooth down such irregularities, arising from paucity of numbers, as we have good reason to believe should not be expected to recur were a similar class of lives again similarly exposed. These, for the want of a better name, may be called Natural Errors.

The working of the first influence is well illustrated in national vital statistics; and of the second in assured-life experience, where the first has little or no place. It may be observed as a rather important fact, that we seldom require to deal with Personal and Natural Errors at the same time. Thus, in the reports of the Registrar-General, there is no lack of numbers to form an average, but unfortunately these numbers are not trustworthy. Besides accidental errors, which will probably neutralize each other, there are serious mis-statements influencing the enumerations in particular directions, which are known of beforehand as certain to occur, and which therefore require to be dealt with before a satisfactory table can be framed. To such—as for instance, the recording of ages in decades instead of exactly—the mathematical Law of Error does not apply. Before that law becomes applicable, it is necessary that allowance be made for all *known* disturbing causes, as it relates solely to residual errors.

Against some correction of Personal Errors, however, no reasonable objection can be brought. The opinion of De Morgan, quoted above, is favourable rather than otherwise, for the correction is applied with the view of getting the exact physical facts—not of altering them. And as regards the other instance, that of assured-life statistics, we submit that De Morgan's words do not, when fairly interpreted, tell against a scientific adjustment. It should not be overlooked that they were written in 1838, when the theory was still in its infancy, and before any of the methods I have alluded to had been developed; and they clearly refer to arbitrary and unscientific graduations, than which he probably then knew no other. Coupling this with his admission that the fluctuations of single years are "errors", we surely get a legitimate warrant for eliminating them, if that can possibly be done. This being granted, the controversy can only be conducted on the more restricted ground—Whether the process of graduation generates more errors than it expels? In other words, opponents of graduation may fairly demand a reason for believing that the graduated results correspond better with the law of life than the unadjusted facts.

Now the process of graduation is simply the alteration of the expression for the law of mortality of the community observed, from an irregular rectilinear form to a curve—a line, it may be, presenting frequent flexures, but no angles. For this alteration two good reasons may be given:—First, the principle of Continuity, founded upon the reign of Law existing throughout all nature—expressed in the maxim, *Natura non agit per saltum*—teaching the inherent improbability that sudden fluctuations up and down in the death-rate as we pass from age to age, are likely to recur at corresponding ages, were a similar body of lives again observed. Second, Experience, derived from inspection of other observations in which we find that like fluctuations do not exist, or possibly that the deviations from the average or curve line are in opposite directions; and which has also clearly established the fact, that as numbers are increased irregularities are in large measure diminished. Hence Experience, applied, of course, with judgment, is a most important factor, enabling us, for instance, to avoid eliminating as accidental deviations any well-established physical fact, such as the maximum point in the curve of the probability of dying, demonstrated by Mr. Bailey as occurring in the period of early manhood.

It should be particularly noticed that the mere theory of graduation involves no assumption as to what the law of life is; it simply takes for granted that life, like everything else, is under law; that the action of Nature is, on the whole, regular and harmonious, not erratic and fickle;

> " So careful of the type she seems,
> So careless of the single life."

So long as these modest limits are not exceeded, graduation may be successfully defended. But to justify it upon the ground, not of the orderly character of results, but of some definite theory as to their causes—or, in other words, to assign the forces of mortality, of whose secret springs we know so little, and thence to graduate by a formula in which the number living is expressed as a definite function of the age, is a generalization far transcending the present, and, in the opinion of those most capable of judging, the possible, attainments of science. Any such formula must always be purely empirical, and can have no claim on our acceptance when it would give results differing to any extent from those of an adjustment made independently of its assumptions.

2.—*Mr. Woolhouse's Method.*

Probably no method of graduation has met with such general acceptance as that devised by Mr. Woolhouse for the adjustment of the Institute observations. The comparison challenged by Professor McCay becomes thus all the more interesting. That Mr. Woolhouse's formula is open to the objection, that by it "no greater influence is given to the large number of observations at one age than to the small number at others", cannot be denied. A graduation of the same facts, therefore, "with reference to the weight of observation", and by a process so carefully elaborated as Professor McCay's, has a special value, not only as a means of putting Mr. Woolhouse's system to the test, but also of establishing confidence in the Institute Tables. While it might have been preferable to have had the H^{M} observations submitted to the treatment bestowed upon the H^{MF}, still there is so much similarity between the two bodies of facts, that little difficulty will be felt in

Table A.—*Institute* H^{MF} *Observations—Showing the Progression in the Rate of Mortality according to the Graduations of Mr. Woolhouse and Prof. McCay.*

Age x	Δq_x		Age x	Δq_x	
	Woolhouse.	McCay.		Woolhouse.	McCay.
10	−·00033	·00009	55	·00131	·00132
11	−·00021	·00021	56	·00140	·00147
12	−·00007	·00030	57	·00153	·00161
13	·00004	·00036	58	·00180	·00179
14	·00019	·00041	59	·00204	·00199
15	·00032	·00043	60	·00231	·00220
16	·00047	·00043	61	·00262	·00243
17	·00060	·00040	62	·00281	·00265
18	·00061	·00035	63	·00290	·00283
19	·00045	·00029	64	·00296	·00304
20	·00030	·00020	65	·00310	·00325
21	·00013	·00013	66	·00324	·00352
22	·00003	·00013	67	·00337	·00388
23	·00000	·00013	68	·00395	·00433
24	·00005	·00013	69	·00497	·00487
25	·00010	·00016	70	·00590	·00556
26	·00023	·00018	71	·00682	·00621
27	·00026	·00018	72	·00786	·00689
28	·00024	·00019	73	·00850	·00744
29	·00023	·00018	74	·00795	·00791
30	·00016	·00019	75	·00782	·00813
31	·00015	·00019	76	·00741	·00847
32	·00018	·00021	77	·00788	·00867
33	·00021	·00022	78	·00828	·00918
34	·00024	·00024	79	·00930	·00982
35	·00033	·00026	80	·01039	·01082
36	·00035	·00028	81	·01161	·01141
37	·00032	·00027	82	·01358	·01261
38	·00027	·00028	83	·01431	·01366
39	·00023	·00029	84	·01410	·01556
40	·00019	·00030	85	·01465	·01662
41	·00023	·00033	86	·01516	·01800
42	·00036	·00038	87	·01333	·01916
43	·00045	·00044	88	·01341	·02044
44	·00059	·00053	89	·01856	·02176
45	·00069	·00056	90	·01930	·02309
46	·00071	·00063	91	·01361	·02444
47	·00070	·00067	92	·01960	·02577
48	·00069	·00071	93	·02666	·02707
49	·00067	·00075	94	·00746	·02833
50	·00073	·00082	95	·00892	·02952
51	·00081	·00088	96	·09476	·03060
52	·00100	·00097	97	·19191	·03155
53	·00113	·00108	98	·34000	·03235
54	·00120	·00119	99	...	...

allowing any conclusions arrived at regarding the one to apply equally to the other.*

Mr. McCay's five criteria of a good adjustment are :—1st, the general regularity in the rates; 2nd, the near agreement in the whole number of deaths; 3rd, the equality of the positive and negative deviations; 4th, the smallness of the accumulated deviations; 5th, the frequency in the changes in their signs from positive to negative.

Now as regards the first criterion—the general regularity of the rates—an examination of Table A, which gives the values of the function Δq_x for the two adjustments, will show that on the whole the advantage probably lies with the new graduation, and that the progression of Mr. Woolhouse's rates might, at some points, have been made a little smoother without lessening the trustworthiness of the table, or in any way detracting from its other merits. But it should not be forgotten that Mr. Woolhouse has himself suggested that, when greater regularity is desired than can at first be obtained, it may be produced by a subsequent process for the correction of isolated errors. While it is perhaps to be regretted that some additional finishing touches were not given here and there, especially at those periods where the observations are few—for example, at age 91, and at ages 94 and 95,—still the want of such a process does not tell against the formula of graduation itself, nor even against the particular application of it, which must be judged mainly on other grounds. †

For the purposes of further comparison, Table B, contrasting the expected deaths according to the two adjustments with the actual mortality, has been constructed.

---

* Since this was written Mr. Sutton has submitted to the Institute a Table showing that the H^{M} graduation retains the same close agreement with the raw observations as is here found to be so marked a feature of the H^{MF} graduation.

† I may perhaps be allowed to suggest, however, that Mr. Woolhouse's method of adjusting the numbers living at the first seven ages in the table, to which his formula cannot be applied, should be reconsidered. In the H^{MF} observations, as will be observed, that method has led to the somewhat anomalous circumstance of the rate of mortality decreasing after age 10, and being less at age 16 than at the former age.

TABLE B.—*Institute* H^{MF} *Observations—Comparison of the Graduations of Mr. Woolhouse and Prof. McCay by means of the Actual and the Expected Deaths.*

Age.	Actual Deaths.	EXPECTED DEATHS.		Sum of the Actual Deaths.	SUM OF THE EXPECTED DEATHS.		ERRORS.	
		Woolhouse.	McCay.		Woolhouse.	McCay.	Woolhouse.	McCay.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
10	6	3	2	6	3	2	− 3	− 4
11	...	3	2	6	6	4	0	− 2
12	2	3	3	8	9	7	+ 1	− 1
13	2	3	3	10	12	10	+ 2	0
14	7	4	4	17	16	14	− 1	− 3
15	7	5	6	24	21	20	− 3	− 4
16	3	7	8	27	28	28	+ 1	+ 1
17	7	9	10	34	37	38	+ 3	+ 4
18	14	13	14	48	50	52	+ 2	+ 4
19	25	19	19	73	69	71	− 4	− 2
20	25	27	26	98	96	97	− 2	− 1
21	37	37	36	135	133	133	− 2	− 2
22	45	49	49	180	182	182	+ 2	+ 2
23	76	67	66	256	249	248	− 7	− 8
24	92	85	85	348	334	333	−14	−15
25	84	106	107	432	440	440	+ 8	+ 8
26	131	127	130	563	567	570	+ 4	+ 7
27	148	154	156	711	721	726	+10	+15
28	192	181	182	903	902	908	− 1	+ 5
29	211	210	209	1,114	1,112	1,117	− 2	+ 3
30	248	238	236	1,362	1,350	1,353	−12	− 9
31	243	261	260	1,605	1,611	1,613	+ 6	+ 8
32	298	284	284	1,903	1,895	1,897	− 8	− 6
33	299	306	307	2,202	2,201	2,204	− 1	+ 2
34	335	328	330	2,537	2,529	2,534	− 8	− 3
35	331	350	351	2,868	2,879	2,885	+11	+17
36	368	373	372	3,236	3,252	3,257	+16	+21
37	394	393	389	3,630	3,645	3,646	+15	+16
38	427	412	405	4,057	4,057	4,051	0	− 6
39	448	427	421	4,505	4,484	4,472	−21	−33
40	421	438	434	4,926	4,922	4,906	− 4	−20
41	445	442	444	5,371	5,364	5,350	− 7	−21
42	443	446	451	5,814	5,810	5,801	− 4	−13
43	439	453	459	6,253	6,263	6,260	+10	+ 7
44	461	461	467	6,714	6,724	6,727	+10	+13
45	473	473	474	7,187	7,197	7,201	+10	+14
46	482	485	483	7,669	7,682	7,684	+13	+15
47	512	494	490	8,181	8,176	8,174	− 5	− 7
48	489	501	496	8,670	8,677	8,670	+ 7	0
49	501	508	503	9,171	9,185	9,173	+14	+ 2
50	532	512	510	9,703	9,697	9,683	− 6	−20
51	521	513	513	10,224	10,210	10,196	−14	−28
52	518	514	517	10,742	10,724	10,713	−18	−29
53	472	517	519	11,214	11,241	11,232	+27	+18
54	500	521	521	11,714	11,762	11,753	+48	+39
55	575	526	526	12,289	12,288	12,279	− 1	−10
56	544	526	527	12,833	12,814	12,806	−19	−27
57	516	526	528	13,349	13,340	13,334	− 9	−15
58	532	528	531	13,881	13,868	13,865	−13	−16
59	484	531	534	14,365	14,399	14,399	+34	+34

TABLE B—(*continued*).

Age.	Actual Deaths.	EXPECTED DEATHS.		Sum of the Actual Deaths.	SUM OF THE EXPECTED DEATHS.		ERRORS.	
		Wool-house.	McCay.		Wool-house.	McCay.	Wool-house.	McCay.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
60	559	533	535	14,924	14,932	14,934	+ 8	+10
61	524	532	532	15,448	15,464	15,466	+16	+18
62	541	535	532	15,989	15,999	15,998	+10	+ 9
63	542	533	528	16,531	16,532	16,526	+ 1	− 5
64	519	525	519	17,050	17,057	17,045	+ 7	− 5
65	520	515	511	17,570	17,572	17,556	+ 2	−14
66	492	502	499	18,062	18,074	18,055	+12	− 7
67	480	485	486	18,542	18,559	18,541	+17	− 1
68	485	466	471	19,027	19,025	19,012	− 2	−15
69	473	444	452	19,500	19,469	19,464	−31	−36
70	391	429	435	19,891	19,898	19,899	+ 7	+ 8
71	380	418	421	20,271	20,316	20,320	+45	+49
72	421	409	409	20,692	20,725	20,729	+33	+37
73	390	393	388	21,082	21,118	21,117	+36	+35
74	427	373	364	21,509	21,491	21,481	−18	−28
75	320	342	334	21,829	21,833	21,815	+ 4	−14
76	313	311	306	22,142	22,144	22,121	+ 2	−21
77	259	276	274	22,401	22,420	22,395	+19	− 6
78	263	247	247	22,664	22,667	22,642	+ 3	−22
79	217	214	215	22,881	22,881	22,857	0	−24
80	175	183	185	23,056	23,064	23,042	+ 8	−14
81	162	157	159	23,218	23,221	23,201	+ 3	−17
82	126	134	136	23,344	23,355	23,337	+11	− 7
83	121	114	115	23,465	23,469	23,452	+ 4	−13
84	90	93	93	23,555	23,562	23,545	+ 7	−10
85	82	76	77	23,637	23,638	23,622	+ 1	−15
86	59	59	60	23,696	23,697	23,682	+ 1	−14
87	42	44	45	23,738	23,741	23,727	+ 3	−11
88	34	32	34	23,772	23,773	23,761	+ 1	−11
89	20	22	24	23,792	23,795	23,785	+ 3	− 7
90	16	17	18	23,808	23,812	23,803	+ 4	− 5
91	12	12	14	23,820	23,824	23,817	+ 4	− 3
92	11	9	10	23,831	23,833	23,827	+ 2	− 4
93	8	6	7	23,839	23,839	23,834	0	− 5
94	...	4	4	23,839	23,843	23,838	+ 4	− 1
95	1	3	4	23,840	23,846	23,842	+ 6	+ 2
96	5	3	4	23,845	23,849	23,846	+ 4	+ 1
97	1	1	1	23,846	23,850	23,847	+ 4	+ 1
98	1	1	1	23,847	23,851	23,848	+ 4	+ 1
						Total	+550 −240	+426 −640

The conclusions to be derived from this table may be most clearly set forth by quoting the figures instanced by Mr. McCay in proof of the success of his own method, and placing beside them the corresponding results in Mr. Woolhouse's case as now obtained.

	By Mr. McCay.	By Mr. Woolhouse.
(*a*) The total number of deaths . . .	1 too many.	4 too many.
(*b*) The positive accumulated errors compared with the negative .	2 : 3[*] nearly.	2 : 1 nearly.
(*c*) The largest deviation .	+ 49 and only two removes from a − 36 before it and three from a − 28 after it.	+ 48 and only two removes from a − 18 before it and next a − 1 after it.
(*d*) The accumulated deviations over 30 are .	7[†].	6.
(*e*) The sign of the deviation changes so often as	23 times.	23 times.

To these we may add that the accumulated deviations are zero four times in Mr. Woolhouse's table, and only twice in Mr. McCay's.

There will, doubtless, be no difference of opinion as to the satisfactory manner in which the principle of graduation adopted by the Institute has stood the somewhat stringent tests, to which, at the suggestion of its learned critic, it has just been subjected. In addition to the foregoing features, the entire absence of anything like sudden breaks in the curve, and the easy way in which it lends itself to the subsequent smoothing down of remaining irregularities, renders the method undoubtedly the best that has yet been practically used in similar cases involving numerous observations. Such a view is supported by an examination of what would have been the result had Mr. Makeham's modification of Gompertz's formula been used in the preparation of the New Experience tables. For in the sixteenth volume of the *Journal* (p. 428) the then president of the Institute (Mr. Samuel Brown) has given a complete graduation of the H^{MF} observations by this formula, and from the eminence of its author, it may fairly be taken as showing the best that could be done for the facts, when they had to be bent to such a hypothesis. Table C compares the expected deaths by Mr. Brown's table with the actual deaths in quinquennial periods, and these are very far from showing so satisfactory an agreement as in the case of the two other graduations we have examined.

---

[*] Mr. McCay, through some mistake, states this ratio as 4 : 5.

[†] Mr. McCay says 6: he adds that none of these are one-tenth of the annual deaths at the corresponding ages; but such does not appear to be the case.

TABLE C.—*Mr. Samuel Brown's Graduation of the Institute* H^{MF} *Observations by Mr. Makeham's Formula—Comparison of the Expected and the Actual Deaths.*

Ages.	Expected Deaths.	Actual Deaths.	Ages.	Expected Deaths.	Actual Deaths.
10–14	28	17	60–64	2,651	2,685
15–19	72	56	65–69	2,426	2,450
20–24	282	275	70–74	1,997	2,009
25–29	806	766	75–79	1,346	1,372
30–34	1,400	1,423	80–84	697	674
35–39	1,885	1,968	85–89	247	237
40–44	2,219	2,209	90–94	54	47
45–49	2,416	2,457	95–99	10	8
50–54	2,574	2,543			
55–59	2,669	2,651			
			Total...	23,779	23,847

When the observations are few, it is probable that no process of smoothing—not even a second graduation—will suffice to remove the irregularities left by Mr. Woolhouse's method. But even in that event, the work will not be wasted, for—no assumption having been made as to the final form that the curve will assume—the preliminary graduation will greatly facilitate the application of other methods.

Besides paucity of numbers, there are two other conditions under which, as I venture to think, Mr. Woolhouse's method will not suit; and which, therefore, serve to show that, many as are its advantages, its range is not sufficiently wide to include all the problems that arise in practice; and, consequently, that, even independently of rivalry, there is abundance of room left for other systems. The first of these conditions is, when the rate of mortality is undergoing rapid changes, as during infancy and childhood. This fact is specially alluded to by Mr. Charles Ansell, junior, in his very valuable work, *Statistics of Families, &c.*, who found that in adjusting the mortality during the early years of life, it was necessary to avoid using so great a range as fifteen years to determine the rate at each age. The second condition is, when the facts are affected to any considerable extent by Personal Errors, as in the case of the national statistical records. What in other circumstances is the chief merit of Mr. Woolhouse's method becomes here its disadvantage. From their very nature we require, not only to tone down, but entirely to eliminate all Personal Errors; and for this purpose, Mr. Woolhouse's method, reflecting as it does every marked feature in the raw material, fails. Suggestions will be made further on as to the best means of meeting these difficulties.

3.—*The Graphical Method.*

Several recent graduations by Mr. Sprague, in which a graphical process is employed with singular success in unusually trying circumstances, suggest the suspicion that a most powerful instrument has been, to a considerable extent, neglected by actuaries. A like indifference does not, however, appear to have obtained in other branches of science. Professor Stanley Jevons, in his *Principles of Science,* a treatise devoted to an exposition of the methods of scientific investigation, has a small section entitled "The Graphical Method", and in it the elements of the process are clearly set forth. The fact of such a system of adjusting the raw results of observations having already vindicated for itself a well recognized position elsewhere, will be considered by actuaries as a good *primâ facie* ground on which to base a claim for increased attention at their hands; all the more, when some of its most successful applications are found to have been made under conditions similar to those occurring in the graduation of mortality tables.

Besides the exhibiting of numerical results in an easily intelligible form, there are two distinct purposes for which a graphical projection may be employed:—(1) To obtain intermediate values of a function, when the values at fixed intervals only have been determined; (2) To generalize the results of observation or experiment, so as to eliminate their imperfections, and, if possible, obtain from them a mathematical expression of the law connecting the two variables that the diagram has been constructed to represent. In the former case, all points already marked on the chart remain unaltered, and the process, being thus simply one of ordinary interpolation, is fully as familiar to actuaries as to physical investigators; although actuaries may generally prefer to use some formula in the calculus of finite differences. It is to the latter case that attention is now being more especially directed. In it no position laid down on the chart is considered unalterable, and this is precisely the point at which the actuarial graduations most frequently begin to diverge. In these, by assuming the average of the rates for a series of consecutive ages to apply to the middle age, or by holding (arbitrarily or otherwise) that the numbers at certain ages are to remain unchanged, or in some other like way, the problem is generally removed from case (2) to case (1): Mr. Woolhouse's method is an exception: in theory it corresponds partly to the one case, and partly to the other. The construction of five complete tables from the ungraduated figures split up into five different sets of ages, each set giving the numbers living at intervals of five years, is a simple interpolation. But if we may suppose the values of the function for each age laid down on a chart, then it may be said that the final table is got by drawing a curve among the various points, so that, at each age, the deviation from the whole of the observations at that age may be as little as possible. This latter process is like case (2), but differs from it in that each final number is determined solely from the various values of the function at the one age, without reference to the continuity of the series from age to age.

The two applications of the graphical method are so clearly distinct as regards their object, that it may be of advantage to refer to them by the respective names of Graphical Interpolation and Graphical Graduation;—names which bear their own interpretation. To get fuller illustrations of the method, and especially of the second case, it is necessary to travel beyond the limits of strictly actuarial subjects. Before introducing the following outside examples, however, I wish to prevent possible misconception by frankly stating that they are simply the fruit of an examination of a number of scientific works, made with the sole object of obtaining information regarding the graphical method.

So far as I have been able to discover, the earliest and most complete exposition of the theory of a graphical adjustment, is contained in a paper by the late Sir John Herschel describing his investigations regarding the orbits of revolving double stars.* The data afforded by observations of these stars are remarkable for their uncertainty and irregularity; so much so, that, when (as was formerly attempted) the orbits were deduced analytically from four selected observations, assumed to be accurate, it was nearly always possible, by substituting other observations, apparently equally trustworthy, to produce results completely antagonistic;—to get in the one case a highly eccentric, and in the other a nearly circular, orbit.

---

* "On the Investigation of the Orbits of revolving Double Stars", by Sir J. F. W. Herschel; *Memoirs of the Royal Astronomical Society*, vol. v, p. 171. This paper was read before the Society in 1832, and so far as can be judged by internal evidence, the author devised independently what has been called a *graphical graduation*. Such an opinion is supported by a passage in Whewell's *History of the Inductive Sciences*, third edition, vol ii, p. 204, where he, referring to the laws of double stars, mentions Sir J. Herschel as "*inventing* a method

It was under these difficult conditions that Herschel made his brilliant investigations—recognized by the award of a Royal Society medal—wherein he appears to have been gradually led on to the substitution at every stage, of graphical for analytical methods,—a procedure the theoretical basis of which, he, in the memoir referred to, vindicates with a clearness and precision leaving nothing to be desired. A recent writer thus alludes to his labours*:—"The father [Sir William Herschel] had been the original discoverer of the extension of gravitation to the remotest boundaries of the visible universe, and the son had now put the crowning stone to this edifice of discovery by the invention of a graphical method whereby the eye could, as it were, see the two component stars of the binary system revolving before it after the regularity of the Newtonian law."

If Sir John Herschel's procedure had merely been to describe an ellipse through or near certain observations recorded upon a chart, it would not have presented the same striking analogy to an actuarial graduation which it actually does. But the fact, that, of the two component parts of each observation—the angle of position of the line joining the two stars, and their distance from each other—Herschel felt compelled to discard the latter as hopelessly untrustworthy, rendered a direct deduction of the elliptic orbit impossible. He accordingly commenced by recording the angles of position by themselves as ordinates upon a chart, of which the abscissa represented the date of observation; and graduated them graphically by drawing the best curve the circumstances would admit of among the extremities of these ordinates. Next he deduced from this curve—also graphically, by simply drawing tangents—a table of the angular velocities, and

---

which depended upon the whole body of observations, and not upon selected ones only." It is quite likely, however, that an analogous process may have been used by previous investigators in other fields. At the beginning of his paper Herschel states that, with permission of the Council, he had suppressed a "brief and imperfect account of the principle of this method" contained in a former communication to the Society.

* *Encyclopædia Britannica*, ninth edition, article—"Herschel, Sir J. F. W."

thence obtained a series of distances independently altogether of direct measurement. From these distances and the previously-adjusted angles of position, the apparent path of the one star about the other was laid down. An orbit so obtained could scarcely be expected to prove a perfect ellipse, and therefore the remaining part of the problem was to transform the orbit into an elliptic form with as little violence as possible. This last process may be compared to the finding the constants of Mr. Makeham's function, which will most nearly represent a table already graduated by Mr. Sprague's method. The earlier is more like the graduation itself; and therefore I propose to quote at some length from Herschel's own description of that portion of the work.

After forcibly pointing out the unsatisfactory condition of the raw materials, Herschel goes on to say, that we must either wait for another half century till better are accumulated, or devise "some process by which the observations themselves shall be, as it were, purified in the very outset from the worst portion of their errors, and made to speak a language at least consistent with itself, and susceptible of intelligible interpretation: we must fall, in short, on some mode of treating the question, in which the peculiarities of individual data shall merge in a general consideration of their whole mass, and where not merely a few selected ones, but *all* the observations known of any binary star, shall be combined in a single operation. Thus, instead of deriving from so many different combinations of them so many conflicting or mutually-contradictory orbits, we shall rather educe from their whole assemblage taken together, and regarded as a single set of data, a single result, in whose favour they shall all so far conspire, as their actual, though unknown, errors will permit them to conspire, in favour of any orbit.

"The process by which I propose to accomplish this is one essentially graphical; by which term I understand, not a mere substitution of geometrical construction and measurement for numerical calculation, but one which has for its object to perform that which no system of calculation can possibly do, by bringing in the aid of the eye and hand to guide the judgment, in a case where judgment only, and not calculation, can be of any avail." In proceeding to give more detailed explanations, he describes the cross-ruled paper with which actuaries are familiar, and the laying down thereon of the points representing the observed angles of position, stating that the square subdivisions will, if neatly executed, enable the investigator to dispense entirely with measurements. He then continues:—"A series of points will thus be obtained on our chart, which, if the observations were exact, would necessarily admit of a regular curve being drawn through them, whose nature would, of course, be determined by the laws of elliptic motion, and one of whose essential characters is to have, within those limits of the abscissa which correspond to a whole period of revolution, (that is, to a difference of 360 units in the ordinates,) in some cases two, and in some four points of contrary flexure, but never more than the latter, nor fewer than the former number; and to have, moreover, in all its parts, a peculiarly graceful and flowing outline. Such a line, however, can never be drawn through all the points laid down: the errors of observation prevent it. Our next step, then, must be to draw, by the mere judgment of the eye, and with a free but careful hand, not *through*, but *among* them, a curve presenting as few and slight departures from them as possible, consistently with this character of large and graceful sinuosity, which must be maintained at all hazards.

"But, since equal reliance can probably not be placed on all the observations, we must take care to distinguish those points which correspond to observations entitled to the greatest confidence, such as those which appear to have been made under peculiarly favourable circumstances, or which rest upon the average of a very great number of individual measurements. . . . . . These should be marked on the chart, in some especial manner, not liable to be mistaken, and so as to strike the eye at a general glance; for example, by larger and darker points than the rest, or by one or more circles drawn round them, according to their presumed importance. And when we draw our curve, we must take care to make it pass either through or very near all those points which are thus most strongly distinguished, or at least to deviate from them with much more reluctance than from such as have no claim to our peculiar attention. . . . . . This curve once drawn, must represent, it is evident, the law of variation of the angle of position, with the time, not only for instants intermediate between the dates of observations, but even *at the moments of observation themselves*, much better than the individual *raw* observations can possibly (on an average) do. It is only requisite to try a case or two to be satisfied that, by substituting the curve for the points, we have made a nearer approach to nature, and in great measure eliminated errors of observation. In a large majority of cases, a series of lines joining the consecutive points, instead of uniting into a regular curve, offers only an uncouth approach to such a form, which cannot possibly represent reality. . . . . It is not meant to deny that occasional errors may still subsist; nay, even that, in some singular cases, a good observation may become charged with error by this proceeding, while a bad one may receive the honours due to perfect measurement; but this can never happen unless there *be* great error in some of the data, such, and so concealed, as would render any conclusion deduced from them precarious. In all ordinary cases, the effect of the process above described will be to impress a strong conviction on the mind that we have approached the truth, and that the points, if any, which deviate very widely from our curve, do really correspond to bad observations, and deserve no confidence."

After describing the reading off from the chart of the artificial series of interpolated observations destined to be employed, to the exclusion of the original observations, in the deduction of the orbit, Herschel continues:—"I cannot help remarking here, how much more satisfactory this, as a mere process of interpolation, appears to me than any of those in use which depend on 'passing a curve, of a parabolic nature,' through points given by direct measurement. Supposing even that the observations were perfect, there is nothing in such a curve to recommend it, except in the peculiar case when the intervals of observation are very small. . . . . . In every other case, not only a parabolic, but any species of algebraic or transcendental curve, may be used with *fully* as great a claim to probability; and, in fact, no curve but *the right* has any claim to preference at all. . . . . . . On the other hand, in following the graphical process above pointed out, we have a conviction, almost approaching to moral certainty, that we cannot be greatly misled; and that the errors of observation, if not altogether eliminated, are at least materially diminished."

In closing these extended quotations from Sir John Herschel's paper, I should perhaps explain, that in submitting them in the present form, I am influenced by the consideration that their application to the graduation of mortality tables is so obvious, that it seems better, by retaining a portion of the context, to allow the illustrious author to speak for himself. From his experience many useful hints may be gathered. Thus, his recommendation to distinguish in the projection the more important and weighty observations, appears to be one of very considerable practical value to actuaries; and, although Mr. Sprague gives no hint of a similar artifice, I cannot help thinking that the adoption of it would probably save various unsuccessful attempts, otherwise unavoidable, before a completely satisfactory result is attained. All the more is this likely, since Mr. Sprague requires his graduated probabilities to reproduce the actual events; for, obviously, the curve cannot, in that case, deviate far from the observation at any age where the events are numerous.

It will have been observed that the plotting down of the orbits of binary stars, is, like the construction of mortality tables from limited and fragmentary data, an extreme case, which, if not performed graphically, could only be effected by the assumption of some definite empirical law, more or less probable. Now, we have just seen that Herschel is opposed to all such assumptions. On a subsequent occasion, in his article, "Meteorology", contributed to the Eighth Edition of the *Encyclopædia Britannica,* he speaks more explicitly on this point; stating, regarding the law regulating the decrements of temperature in ascending into the atmosphere, that previous investigators, by commencing with the assumption of some hypothetical law and confining their observations to moderate heights, had simply wasted a vast amount of laborious research and calculation; seeing that almost any law of decrement not giving a density increasing upwards, would, if made to represent the temperature at the highest and lowest levels correctly, do so also at the intermediate points, when the ascents were of limited extent. A similar criticism is, I venture to think, equally applicable to hypothetical laws of mortality, *i.e.*, that almost any assumed law, involving an increasing death-rate, can be made to represent, with a fair degree of accuracy, a considerable section of most mortality tables. De Moivre's theorem is an example; and, may we not add Gompertz's? Mr. Makeham's modification of the latter, perhaps, should be placed in a different category; although, indeed, the difference is one only of degree.

The description of the manner in which the meteorological law in question was derived graphically from the observations without any assumptions, is extremely interesting; more especially as the history of the investigation of that problem presents several very striking analogies to the history of researches as to the law of human mortality. Too much time has, however, already been occupied by such illustrations; and, therefore, I content myself with thus referring to the article.

Allusion has been made above to the neglect of the graphical method by actuaries, but by that I did not, of course, mean to imply that they had overlooked it altogether. Both Mr. Davies and Mr. Milne are mentioned as having used a process of a graphical nature, although in neither case is there, I believe, a description of its precise character. It is quite possible that both these writers did not go beyond interpolation between fixed points —Graphical Interpolation. But Mr. Edward Sang, in his *Essays on Life Assurance* (1840), has, in dealing with the raw material in the First Report of the Registrar-General, used and described a Graphical Graduation, thus:—"Having drawn a number of equidistant parallel lines to represent the successive ages, I measured off on these, distances proportional to the number of deaths actually recorded, and then sketched a curve line through, among the terminal points, so as to make (over arbitrary intervals) the sum of the distances on the one side equal to the sum of the distances on the other side of the line; and then, by help of that curve line, I corrected the number of deaths, leaving, of course, the total number in each arbitrary interval unchanged" (pp. 2, 3). Little further progress seems, however, to have been made in the development of the system as an actuarial instrument, until Mr. Sprague took it up. He, at the same time, added the valuable practical test of requiring the graduated probabilities to reproduce the actual events in given moderate intervals; thereby preventing artistic considerations becoming more powerful than a desire that the curve should fairly represent the data. Thus, if the first adjustment should not be successful in this latter respect, a second, a third, or even more, must be tried. In making these second, &c., adjustments, it will often be found simpler to manipulate the actual figures, than to alter the curve and re-measure the ordinates. When the former plan is used, the computer will, of course, be careful to see that each of the sections which he may separately treat, flows naturally into its successor; so that, whether the corrections be made by differences or graphically, the continuity of the curve may not be impaired.

Now, as regards the conditions under which the graphical method may be employed, it is at once evident, from its very nature, that it is capable of universal application—for, indeed, it is graduation in its most elementary form, while the most perfect analytical systems of graduations ought to meet the requirements of a graphical adjustment. But from this it does not, of course, follow that it is the best that can be used in all circumstances.

The special difficulties which Sir John Herschel designed it to meet, were paucity and irregularity in the facts; and, if the quotations made from the writings of that great man prove anything, they certainly prove that it is more suitable than any other method for such a purpose. Mr. Sprague's applications show that it is a valuable instrument for dealing with curves subject to somewhat rapid flexures, such as rates of mortality frequently become when selection is operating powerfully side by side with the normal law. In graduating tables largely affected by Personal Errors, the graphical method, from its flexibility, will also prove of advantage —since these pre-eminently require the exercise of common sense, and the mere passing of the records through the actuarial mill of a rigid system, would fail to produce useful results.

But, when we come to deal with facts, accurately recorded, and involving a sufficient number of deaths distributed throughout the whole range of the table, the graphical method ceases to be of any great importance. It could certainly be employed in such a connection, but I venture to think that Mr. Woolhouse's system is likely to prove more satisfactory. With the latter, care should, however, always be taken that, if any isolated errors are left, they be removed; and for this, as well as other purposes, it might be convenient always to have the graduated probabilities plotted down as a curve.

4.—*Mr. Makeham's Method.*

Notwithstanding the number of eminent mathematicians who have treated of Gompertz's celebrated hypothesis, Prof. McCay, in his paper already alluded to, has approached the subject with a freshness which renders his discussion of it extremely interesting. While it is probable that subsequent writers, working along his lines, may make various practical modifications, still Mr. McCay has the honour of originating a new departure in this important subject. Apparently thinking out the question for himself in preference to obtaining a familiarity with what has been done by others, Mr. McCay has fallen, here and there, into slight inaccuracies. In particular, he has scarcely made so full acknowledgment of the great services rendered by Mr. Makeham in the development of the theory, as a more careful examination of that gentleman's writings would have led him to do. Further, the numerical portion of his investigation does not seem to have been worked out, at all points, with that precision to which actuaries are accustomed. In spite of these minor objections, the paper, as a theoretical discussion, is both important and seasonable. To the few persons who may still cling to a lingering belief that, in the theory, propounded by Mr. Gompertz, of all the forces of mortality being comprised under two classes of partial forces—the one constant and the other increasing in geometrical progression—there is not only an ingenious hypothesis, but also *the Law of Life*, in the strict sense of the term, the material submitted by Mr. McCay will be found difficult to harmonize with their existing opinions. Coming before the profession in this country at a time when the intelligence that the combined experience of American Life Offices has been graduated by the formula in question, and after the appearance of a similarly adjusted mortality table for the experience of the Mutual Life Office of New York, it will help to insure that these examples will not be followed too hastily, or without due consideration, by British actuaries. For in Prof. McCay, we have an eminent American actuary, acknowledging the practical assistance that may be derived from the theory, but objecting to it as giving an ultimate formula for final adjustment—thus showing that there is by no means a consensus of opinion, even in America, in favour of the course adopted. If Prof. Bartlett's graduation of the experience of the Mutual Life of New York, may be taken as an index of the measure of success attained in the graduation of the combined experience, caution seems all the more necessary. After examining Prof. Bartlett's table with considerable care, I must confess to entertaining grave doubt whether his adjustment is the best that could have been obtained; and, indeed, whether the future experience of the company will not prove the table to be seriously misleading both at the young and at the old ages, where, as yet, the exposures have not been numerous. It seems unnecessary to enter into numerical details, but the trustworthiness of this table being open to suspicion, forms an additional reason for suspending judgment regarding the combined table, until it has been thoroughly tested.

Since my remarks were first drafted, an additional contribution to the literature of this branch of the subject has appeared, in the shape of an interesting joint paper by Messrs. George King and George F. Hardy, read before the Institute of Actuaries in January last, and entitled "Notes on the Practical Application of Mr. Makeham's Formula to the Graduation of Mortality Tables." This paper contains the results of attempts to render the application of the formula more successful than it has hitherto been, by devising new methods for determining the constants. Mr. McCay has also devoted attention to this particular point, while there were already existing what may be called the ordinary method, as exemplified by Mr. Woolhouse (*J.I.A.* xv, 389) in connection with the Institute observations, and the noteworthy process of Mr. Makeham, (*J. I. A.* xvi, 344), where the actual numbers exposed to risk are made factors in determining the force of mortality. It may be well, therefore, before going further, to examine whether any of the new processes so increase the pliability of the formula, as to modify to any extent opinions previously formed.

In the outset it is to be observed that, when the formula is perfectly applicable, or in other words, when the table happens to be already of the exact form of Mr. Makeham's curve, the ordinary method, and the new methods of Mr. McCay, and of the joint authors, will produce exactly similar results: further, that precisely the same constants may be obtained from the probability of dying in (or of surviving) one year at three isolated ages taken alone, without reference to the rates at any other ages in the table. Thus, if $l_x = \kappa s^x g^{q^x}$, then $p_x = s g^{q^x(q-1)}$, and from the three values

$$\log p_x,\ \log p_{x+t},\ \log p_{x+2t},$$

we obtain $$\log q = \Delta \log \Delta \log p_x : t,$$

and the other constants by working backwards with this value of $\log q$. Under any other condition than that proposed, of a perfect, though unknown, graduation already existing, it would, however, be very unsafe to use only the rate of mortality at three isolated ages, since that function is liable to great fluctuations in individual years. In the ordinary process, the numbers living at four ages, separated by equal intervals of time, are employed; and since

$$\log l_{x+t} - \log l_x = \log p_x + \log p_{x+1} + \ldots\ldots + \log p_{x+t-1},$$

this insures that the probabilities at all the intermediate ages are taken into account. It thus appears that, while we may speak of isolated values of q_x or of p_x, it is scarcely accurate, in this connection, to use the expression *isolated* values of l_x. The same view of the matter seems to afford a practical reason for fixing upon the four ages so that they may embrace within them the whole period over which it is sought to make the curve extend; or at least so much of it as may be represented by trustworthy observations. Hitherto the difference between the extremes of the four ages has frequently been forty-five or sixty years only.

Mr. McCay is incorrect in stating that Mr. Woolhouse and Prof. Bartlett determined their constants from isolated *rates*, for a reference to their respective works shows that the function they employed for the purpose was the *number living*; and he probably was not aware, as the joint authors point out, that in his own method he might have employed $\log l_{x+t} - \log l_x$ instead of $\Sigma_x^{x+t-1} \log p_x$. But Messrs. King and G. Hardy are mistaken in supposing that his process is practically identical with Mr. Woolhouse's. After from *a priori* considerations deciding to use the observations between ages 20 and 92, Mr. McCay virtually finds $\log q$ from

$$\log l_{20},\ \log l_{44},\ \log l_{68},\ \log l_{92},$$

just as would be done by the ordinary method. But having got this value of $\log q$, he then discards the subdivision of the observations into three equal sections, and divides them into two equal sections represented by

$$\log l_{20},\ \log l_{56},\ \log l_{92},$$

and from these, with the assistance of the value of $\log q$ already obtained, finds the remaining constants. The result is, that instead of having an actual coincidence with the unadjusted numbers living at the four ages, 20, 44, 68, 92, he has it only at the three ages, 20, 56, 92; but the way in which $\log q$ is found probably encouraged the hope of an approximation to the unadjusted table at ages 44 and 68 as well.

Whether this modification of the ordinary process is likely to prove advantageous, only an extended series of trials can determine. That it does not do so in every case will be seen presently. One thing, however, may be said in its favour, namely, that a portion of the calculation being common to it and the ordinary process, when the latter has failed, a trial may be given to the former without losing the benefit of all previous work.

Messrs. King and G. Hardy propose two new ways of finding the constants, for which they suggest the respective names of the "Combined Method" and the "Aggregate Method". The former was developed in connection with an attempt to combine two (or more) curves, so as to get a more satisfactory final result than was afforded by either curve taken alone. In similar circumstances, Mr. Sutton had contented himself with simply taking a mean of the two values of $\log l_x$ at each age, he having no interest in preserving the property of uniform seniority. But this property being essential in their case, the joint authors endeavoured, by operating upon the constants, to obtain like results. While, strictly speaking, they have not succeeded in getting a "combined curve"—understanding by that a curve occupying a *mean*, or some other definite position in regard to the curves from whose constants it has been derived—still, by combining the constants in the way they propose, it is probable that in many cases the resulting table may be a considerable improvement on its predecessors.

The "Aggregate Method" is the one upon which they lay most stress, as a practical instrument. Indeed, it was devised with the view of rendering unnecessary the repeated trials, upon the results of which the "Combined Method" is built up. They thus introduce it:—"Hitherto we have considered the graduation of tables by means of constants formed from isolated values of l_x, and we think it appears that, by combining in a scientific manner different sets of constants, practically useful results may, in most cases, be arrived at. The process, however, is not a little laborious, at any rate, when many sets of constants are used, and skill is required in selecting them, and in deciding on the proportions to be taken of the several sets. It is necessary, moreover, that the table should be subjected to a preliminary graduation by some such method as Mr. Woolhouse's, before Mr. Makeham's formula can be applied with the best effects. Fortunately, by (so far as we are aware) a new method of finding the constants, all necessity for calculating and combining different sets is obviated, preliminary graduation also being dispensed with, and a final set, giving the average of the whole table, is at once obtained." Then follows a description of the plan, which consists in dividing a table of log l_x (unadjusted, and omitting the extreme ages) into four equal sections, and deriving the constants from the differences of the sum of the logarithms in each section. For example, the best "aggregate curve" obtained from the Carlisle table was got by differencing the four quantities

$$\Sigma_{15}^{32}\log l_x,\ \Sigma_{33}^{50}\log l_x,\ \Sigma_{51}^{68}\log l_x,\ \Sigma_{69}^{86}\log l_x.$$

But the first differences of these are

$$\Sigma_{15}^{32}\log {}_{18}p_x,\ \Sigma_{33}^{50}\log {}_{18}p_x,\ \Sigma_{51}^{68}\log {}_{18}p_x;$$

and with the latter we may compare the ordinary method which, using the same data, practically derives the constants from

$$\Sigma_{15}^{38}\log p_x,\ \Sigma_{39}^{62}\log p_x,\ \Sigma_{63}^{86}\log p_x.$$

In other words, the "Aggregate Method" employs the probability of living *eighteen* years at each age, from 15 to 68, the ordinary method the probability of living *one* year at each age, from 15 to 86.

Judging by appearances, it must be admitted that the "Aggregate Method" does not contrast very favourably with the one it is intended to supersede. Nor, indeed, can it claim *par excellence* the title of "aggregate", in using which name it is well to bear in mind that it is not a name at once distinctive and descriptive. To indulge a feeling of curiosity as to the actual numerical deviations of curves derived in different ways from the same data, the data used in obtaining the best of the four Carlisle "aggregate curves" (Table G of Messrs. King and Hardy) just described, was chosen, and the corresponding curves by the ordinary method and by Mr. McCay's modification (already explained) were calculated. The following table shows the deviations of each of these three curves from the actual numbers surviving at quinquennial intervals of age. To make the comparison more complete, it also contains like information regarding Messrs. King and Hardy's "Combined Curve", which is the one finally adopted by its authors, as being the best of their nine Carlisle curves. This "Combined Curve", it should be explained, is the result of a vast amount of labour, being constructed by combining the constants of four ordinary curves, themselves derived from the Carlisle table only after that table had been submitted to a double graduation by Mr. Woolhouse's process.

TABLE D.—*Comparison of the Results of Different Methods of applying Mr. Makeham's Formula to the Adjustment of the Carlisle Table.*

Age x	Actual l_x	DEVIATIONS IN THE ADJUSTED VALUES OF l_x			
		Ordinary Method.	Prof. McCay's Modification.	"Aggregate Method."	"Combined Method."
15	6,300	0	0	+ 91·7	+ 102·5
20	6,090	− 24·1	− 33·3	+ 31·1	+ 45·5
25	5,879	− 47·3	− 64·9	− 23·3	− 6·5
30	5,642	− 48·5	− 73·6	− 49·7	− 32·3
35	5,362	− 16·1	− 47·9	− 35·8	− 19·7
40	5,075	+ 6·2	− 31·1	− 24·4	− 11·5
45	4,727	+ 62·0	+ 20·6	+ 28·9	+ 36·1
50	4,397	+ 58·4	+ 14·4	+ 31·6	+ 31·0
55	4,073	− 9·9	− 54·2	− 22·4	− 32·2
60	3,643	− 49·5	− 91·5	− 42·9	− 61·5
65	3,018	+ 13·8	− 22·7	+ 38·0	+ 14·4
70	2,401	− 21·8	− 49·8	+ 9·2	− 11·6
75	1,675	− 4·6	− 22·2	+ 14·8	+ 6·8
80	953	+ 35·4	+ 27·6	+ 28·2	+ 38·4
85	445	+ 4·6	+ 3·3	− 23·7	− 2·6
90	142	− 5·4	− 4·7	− 30·4	− 14·5
95	30	− 7·7	− 7·3	− 16·4	− 11·5
100	9	− 7·6	− 7·5	− 8·5	− 8·1
Total Error...		422·9	576·6	551·0	486·7

Using Messrs. King and Hardy's own test—the "total error" —we find that, singularly enough, the ordinary curve comes out much better than either of the others framed from the same data, and with which, therefore, it is, so to speak, on all-fours—that is, better than the curve by Mr. McCay's method, and better than the "Aggregate Curve": further, that, although the ordinary curve is merely a first attempt, it gives a considerably less total error than even the "Combined Curve". Having tried only the one example, I do not seek to draw any general inference as to which system is the best for practical use. Indeed, I strongly suspect that the success of one rather than another in any given case, will depend largely on the chapter of accidents.

From Table D we may, however, derive two important practical conclusions:—(1), that the "Aggregate Method" does not necessarily give a superior curve; (2), that preliminary graduation by a formula of the nature of Mr. Woolhouse's is not indispensable to the success of the ordinary method, even with so irregular a table as the Carlisle. The explanation of this latter fact is to be found in the circumstance, already adverted to, that by employing the function log l_x we are in effect taking the sum of some twenty or twenty-five ungraduated rates, and, over so long a period, the errors of these may be expected to neutralize each other.

The constants of the curves by the ordinary method and by Mr. McCay's are given below. Those derived by the "Aggregate Method" are also reproduced, on account of their somewhat striking dissimilarity.

	Ordinary Method.	Prof. McCay's Method.	"Aggregate Method."
log q	·0373241	·0373241	·0408787
log g	− ·0006289	− ·0006235	− ·0003118
log s	− ·0030434	− ·0031774	− ·0036038
log k	3·8472742	3·8492644	3·8609564

From the preceding considerations, it will doubtless be at once admitted that the practical application of Mr. Makeham's formula has not much to hope from improvements in the methods of deriving the constants. The difficulties which stand in the way of its adoption are, unfortunately, of a much more radical character. The general understanding has been that it would not represent the actual mortality during infancy and youth—that in infancy it would give too low, and in youth too high, a rate—but that it would fairly cover the whole of adult life; at least, to this extent, that it would be impossible to say beforehand whether the deviation would be in excess or defect of the observed rate at any other given periods. This understanding is, I fear, a mistake; and my impression is that a careful analysis of existing curves would enable us to speak as definitely regarding the applicability of the formula to other periods of life, as we do at present about infancy and youth. For instance, from an examination of a large number of curves, derived in different ways from various observations—in fact, of all that are known to me—I have noticed the following so invariably to be the case that I might almost venture to announce it as a general law, namely:—*that for all ages under thirty* (*infancy alone excepted*) *Mr. Makeham's formula, if employed to graduate actual observations, gives the rate of mortality too high.* A good example will be found in the case of the H^{MF} observations. Table E shows the probability of dying in a year from age 10 to age 30, according to four different adjustments. The first two are based on Mr. Makeham's formula, being respectively Mr. Brown's, already made use of in Table C, and Professor McCay's—not, however, the adjustment by his method previously explained, but

TABLE E.—H^{MF} *Observations—Graduated Values of* q_x *by four different Methods.*

Age x	ASSUMING MAKEHAM'S FORMULA.		ASSUMING NO FORMULA.	
	Mr. Brown.	Prof. McCay.	Prof. McCay.	Mr. Woolhouse.
10	·00679	·00692	·00311	·00442
11	·00682	·00694	·00320	·00409
12	·00684	·00697	·00341	·00388
13	·00686	·00699	·00371	·00381
14	·00688	·00702	·00407	·00385
15	·00693	·00705	·00448	·00404
16	·00695	·00709	·00491	·00436
17	·00700	·00713	·00534	·00483
18	·00704	·00717	·00574	·00543
19	·00709	·00721	·00609	·00604
20	·00714	·00726	·00638	·00649
21	·00718	·00732	·00658	·00679
22	·00725	·00738	·00671	·00692
23	·00732	·00744	·00684	·00695
24	·00739	·00751	·00697	·00695
25	·00748	·00759	·00710	·00700
26	·00755	·00768	·00726	·00710
27	·00766	·00777	·00744	·00733
28	·00775	·00788	·00762	·00759
29	·00787	·00799	·00781	·00783
30	·00800	·00811	·00799	·00806

a second adjustment, in which the weights of the observations are allowed for, and which he professes to be the best that can be got from the formula; while the last two are the graduations of Professor McCay and Mr. Woolhouse, so fully examined in a preceding section, and illustrated in Tables A and B. All four are constructed in entirely different ways, but yet it will be noticed that the first two are alike in general character, and that the last two are alike in general character, but that the first two differ greatly from the last two, and show throughout a higher rate of mortality.

Further illustration of this point will be got by referring again to Table D. There it will be found that an excess in the number of deaths in each of the three quinquenniums, 15—20, 20—25, and 25—30, is given by all the four Carlisle curves. The same is the case with all the four H^M curves in Messrs. King and Hardy's Table H. Now, if there be one thing more than another for which a mortality table exists, it is surely to indicate the rate of mortality; and if we find a certain method of graduation uniformly over-stating that rate throughout an important section of human life, it becomes scarcely justifiable to employ it in any case where some other method would prove equally serviceable. In certain circumstances there may be advantages outweighing these objections; but it may safely be concluded that Mr. Makeham's formula is not satisfactory for the final adjustment of tables to be used for ordinary purposes.

In short, it would seem that any additional light since shed upon the subject only confirms the general accuracy of the very judicious remarks of Mr. Woolhouse (*J.I.A.* xv, 409), where he states his "general impression that the important and reliable aid to be derived from Mr. Makeham's formula will be found to be not so much, perhaps, in the construction of mortality and annuity tables, as in the useful application of the general properties evolved, of which the most valuable is the law of uniform seniority, to tables independently, and more accurately, deduced from experience." "It will at least", he goes on to say, "entirely obviate in all cases the necessity of forming joint-annuity tables on more than two lives, excepting for equal ages; and it is by no means impossible that future investigations, founded on Mr. Makeham's hypothesis, may reveal other important relations at present unknown." There can be little doubt, as Mr. Woolhouse here suggests, that even if Gompertz's formula should never come into general use in the actual graduation of tables, there are, nevertheless, wide, and to a large extent, unexplored fields of research, in which it must prove of great assistance to the actuary; and in considering its comparative merits, therefore, it seems right to devote some attention to these.

An ordinary comparison of two different mortality tables frequently takes the form of an examination of the respective probabilities of dying at each age. But it is no easy matter to form sound conclusions from such a process, for the mind has difficulty in generalizing accurately the principal effects that would be produced by eighty or a hundred varying rates. If, however, both tables could fairly be expressed by means of a definite function of the age, a comparison of the constants of the functions might suffice. This would be even preferable to a graphical representation of the two tables by curves; the curves would only show *where* they differ, while the constants would tell *why* they differ. For instance, one table might show an excess in the partial forces of mortality which are constant at all ages, or another, an unusual inability to withstand the diminution of the vital force accompanying old age.

Some time ago, I took the liberty of suggesting that investigations on the important and rather troublesome question of the relation of the rate of mortality to policy values, might also be facilitated if conducted on this basis; thus:—Herr Lazarus has found that an ordinary mortality table may be fairly represented throughout by a function which gives the force of mortality at any age $x = a\epsilon^{mx} + b + c\epsilon^{-nx}$, in which expression the first term represents Gompertz's hypothesis, the first two together represent Makeham's, and the last is added by Lazarus to make the formula include the period of childhood. Now, if the partial force of mortality, $c\epsilon^{-nx}$, alone existed, that is, if the death-rate diminished with the age, the values of policies would be negative. Again, if the partial force b operated alone, the column l_x would form a geometrical progression, and the values of policies would be zero; while the force $a\epsilon^{mx}$, if taken by itself, would give a much larger policy-value than in combination with either or both of the others. Leaving out of account $c\epsilon^{-nx}$, as inoperative in the period of adult life, and bearing in mind that it is a rapid increase in the rate of mortality, and not a mere heavy rate, that gives a high policy value,—we infer that, of tables which may be represented by Mr. Makeham's function, the mark of one requiring a large reserve is not the absolute magnitude of $a\epsilon^{mx} + b$, but the preponderance of $a\epsilon^{mx}$ in reference to b. From the elasticity of this method, it is probable that if such a line of investigation were pursued, various broad general principles of considerable importance might be established.

Regarding the calculation of annuities on three or more joint lives, I am not aware that Mr. Makeham's plan has, as yet, been practically employed to any extent by actuaries. When an *accurate* value is required, recourse is generally had to Mr. Woolhouse's own method, explained in his paper *On Summation* (*J.I.A.* xi, 301); and really, when a four-figure logarithm card is used, this calculation is by no means laborious. The former method does not, for ordinary mortality tables, profess to give more than an approximate result, and, as a calculation involving the constants of the formula is necessary in each individual case, it may seem too complicated for approximate purposes. The way in which a three-life annuity is most commonly ascertained is by means of Simpson's rule; but most of those who have used the rule, in circumstances where a fair degree of accuracy is necessary, have discovered that it sometimes proves a rather treacherous guide. Nor is this other than what we should expect from theoretical considerations. In assuming the rule, we assume, as De Morgan has shown, that the force of mortality increases in geometrical progression; or, in other words, that the table used conforms to Gompertz's hypothesis unmodified. But such conformity, as has frequently been proved, does not exist in any of our standard tables for a longer period than about thirty years on the average. Beginning with the limiting age, the curve will not, in the most favourable circumstances, reach a younger age than, say, 55, with reasonable accuracy. Hence, it follows that if one or more of the lives in the annuity is under that age, the approximation not only *may*, but *must*, be wrong. In individual cases the error may not be great, but in no case can it be foretold that it will keep within due bounds. Since, however, Mr. Makeham's modification has brought Gompertz's formula into much better accordance with actual mortality during the whole of adult life, it follows that a corresponding modification of Simpson's rule would have a like effect on the approximate three-life annuity values. This modification, as may easily be shown by a process similar to the investigation of Mr. Woolhouse (*J.I.A.* xv, 401), consists in substituting for two of the lives in the annuity two equivalent joint lives, instead of a single life—or, in other words, in making $a_{xyz}=a_{wwz}$, where $a_{ww}=a_{xy}$. Practically to employ the process, it is necessary to have three-life annuity tables for the various combinations of w and y. These would in extent be exactly equal to ordinary tables for two joint lives, and might with facility be framed along with such. For the purpose of calculating isolated values, however, it is not at all likely that these tables would be constructed, since, with a little labour, individual three-life annuities may be found by the formula from a table of annuities for three equal ages. But the present suggestion is thrown out under the impression that circumstances might occur in which it would be serviceable. For instance, a certain fund, in connection with the calculations of which three lives largely entered, might require to be valued by a single table of mortality, and at a single rate of interest. In such a valuation, Simpson's rule might be found scarcely sufficiently trustworthy, and the actuary would, therefore, require to choose between computing a table of a_{wwy}, once for all, or ascertaining each three-life annuity by the ordinary rule from Mr. Makeham's formula. In the latter case, each individual value would entail a special calculation, and of such a nature that he might hesitate to entrust it to unskilled assistants; while, in the former, the values would be got as simply and rapidly as by Simpson's rule.

In bringing these remarks to a conclusion, I may sum up the principal views which I have ventured to submit.

(1) That there can be no question as to the necessity of graduation when Personal Errors exist to any extent.
(2) That in all cases, a graduation that assumes no other law than that of Continuity, and which brings in the aid of Experience, may be successfully defended.
(3) That no logical defence can be made of the assumption of a definite law, except on the ground that the table would be substantially the same were it graduated without reference to that law.
(4) That Mr. Woolhouse's method is admirably suited to all graduations where the exposures are numerous, and where Personal Errors may be ignored; but that it is scarcely applicable in the following cases:—
    (*a*) When the observations are few and irregular;
    (*b*) When the rate of mortality is rapidly changing, as in infancy, or at other periods under the influence of special circumstances, such as Selection;
    (*c*) When the observations are largely affected by Personal Errors.

(5) That the Graphical method may be advantageously employed in the three cases where Mr. Woolhouse's method fails.

(6) That Mr. Makeham's formula is not suitable for final graduation; but that it is of considerable service in theoretical investigations.

Discussion.

The President (Mr. A. H. Bailey)—Gentlemen, I am sure you will welcome this paper of Mr. Sorley's, for two reasons; first, because it does something to remove the reproach that has been laid upon the gentlemen who have passed our examinations—that they have not done as much as we have a right to expect in adding to our stock of general knowledge; and, secondly, which is of more importance, because of the intrinsic merits of the paper itself, which are to my mind very great. Several papers have been read to us upon the subject of graduation, but none have treated, as Mr. Sorley has done this evening, of graduation as a whole. I think in connection with this paper might be read usefully Mr. Young's interesting letter which appeared in the last number of the *Journal*, in which he has discussed the question, whether we can arrive at a law of mortality. We have not arrived at the law of mortality, and all efforts in that direction seem at present to be unsuccessful. The first attempt of that kind was De Moivre's hypothesis, which, if it had been found to be true, would have been a very convenient law. The next was Gompertz's theory, which was more complicated, and of this there has been Mr. Makeham's modification, very ingenious but more complicated still, and therefore unlike physical laws, which generally admit of being simply expressed. In connection with this subject, Mr. Sorley's reference to Sir J. Herschel brought to my mind a passage which I will read from his *Discourse on the Study of Natural Philosophy* (p. 178, § 187):—"Laws thus derived, by the direct process of including in mathematical formulæ the results of a greater or less number of measurements, are called 'empirical laws.' A good example of such a law is that given by Dr. Young (*Phil. Trans.* 1826) for the decrement of life, or the law of mortality. Empirical laws in this state are evidently *unverified inductions*, and are to be received and reasoned on with the utmost reserve. No confidence can ever be placed in them beyond the limits of the data from which they are derived; and even within those limits they require a special and severe scrutiny to examine *how nearly* they do represent the observed facts; that is to say, whether, in the comparison of their results with the observed quantities, the differences are such as may fairly be attributed to error of observation. When so carefully examined, they become, however, most valuable; and frequently, when afterwards verified theoretically by a deductive process (as will be explained in our next chapter), turn out to be rigorous laws of nature, and afford the noblest and most convincing supports of which theories themselves are susceptible. The finest instances of this kind are the great laws of the planetary motions deduced by Kepler, entirely from a comparison of observations with each other, with no assistance from theory. These laws, namely, that the planets move in ellipses round the sun; that each describes about the sun's centre equal areas in equal times; and that in the orbits of different planets the squares of the periodical times are proportional to the cubes of the distances; were the results of inconceivable labour of calculation and comparison: but they amply repaid the labour bestowed on them, by affording afterwards the most conclusive and unanswerable proofs of the Newtonian system. On the other hand, when empirical laws are unduly relied on beyond the limits of the observations from which they were deduced, there is no more fertile source of fatal mistakes." There is, I believe, a prevailing impression that the process of graduation is always a correction of errors. This is not, I think, always the case. Still it is necessary that we should have a certain regularity in our calculations, and therefore graduation is unavoidable—a necessity more than an improvement. And it must be remembered there are many other subjects on which we have to work besides mortality. There is sickness. There is no doubt whatever that sickness varies very much in different employments, and no one has ever yet pretended to have discovered any general law of sickness. But, further, actuaries have had to work upon superannuation funds, such as the funds of the East India Company, where we have to discover the rate of promotion, the contributions being made by deductions from salaries. In other superannuation funds we have to get at rates of retirement. In all these we have to go through some process of graduation, and therefore I am gratified to see that Mr. Sorley has emphasized that no one system is applicable in all cases. It is a matter of judgment. I am also pleased to notice his reference to the graphical method, to which I am inclined myself. In my youthful days I was for a short time an engineer student, and was taught amongst other things to take levels, to plot the section of a line of country for the purpose of laying out a line of railway. When you had got a section of the line of country, the problem was if possible to draw a horizontal line, so that the railway should be perfectly level. This was seldom practicable, as there must be a variety of gradients, and these gradients must be so adjusted that the cubical contents of the ground to be excavated should be as nearly as possible equal to the cubical contents of the embankments. That is to my mind something like what we have to go through in what we call graduating tables. De Morgan, in a passage succeeding that which Mr. Sorley has quoted, says, I think, that he wishes the graduation was applied to money results. And so do I.

Mr. FINLAISON—If you graduate your money results, you would be merely in a secondary way graduating the original results, and throwing a cloud over your proceedings.

Mr. G. KING—Our great difficulty regarding the law of mortality is to get at the true rates of mortality. Until we know these, we cannot possibly get at any law. I am not sanguine of ever finding a law. I do not believe one exists; but if we cannot get at the law, the formulas resulting from our inquiries will be of great use to us in our further investigations. Looking superficially at the various tables which have been published, one is very well pleased with them. But in examining further into them doubts arise, and those doubts increase rather than diminish by further study. We have heard, for instance, a great deal about Experience tables lately, and they are objected to by some on the ground that they do not show the real rates of human mortality, because selection of one kind or another is involved, and we really are unable to unravel it. With regard to tables formed from statistics derived from other sources, equally great difficulties arise. I have lately given a great deal of attention to such tables, and the conclusions I have come to are—that they are not very satisfactory, and that some useful work may yet be done in preparing tables from general statistics in a more exact way. Seeing that is the case, I do not think we can at present get very much further in our investigations into the law of mortality; but it would be a great mistake to throw aside those formulas to which we have already attained. Mr. Sorley in his paper admits that, under certain circumstances, to take one example, Mr. Makeham's is a suitable formula, but he says at the end of his paper the formula is not suitable for final graduation, but that it "is of considerable service in theoretical investigations". Now I should like to add, that under certain circumstances, when the care that Sir John Herschel, in the very apposite extract read by the President, speaks of in testing results has been applied, Mr. Woolhouse's formula can be usefully adopted for final graduation. I have no doubt, when the tables that Mr. Hardy and myself had the honour to present at a late meeting are published, the profession will see that for certain mortality tables, at any rate, most excellent tables can be produced by Mr. Makeham's graduations; and by use of them, and by means of a most simple calculation, we can get almost absolutely correct joint-life annuity values too. Under these circumstances, I think when any new experience tables are published, it will be well to investigate whether Mr. Makeham's formula may be applied to save the great expense and labour of preparing joint-life annuity tables. Mr. Sorley hints more or less broadly that Mr. Hardy and I have fallen into several mistakes. In the first place, he says that we are "mistaken in supposing that Mr. McCay's process is practically identical with Mr. Woolhouse's." I will add a rider to that, that Mr. Sorley is mistaken in supposing that we suppose so. If he will look at our paper again, he will see we simply refer to the preliminary grouping of the data, and that we do not go into the question of the final operations. We speak merely of Mr. McCay's statement, that in taking the sum of $\log p_x$, he was doing differently from Mr. Woolhouse, who took isolated values of $\log l_x$. I do not think, looking at them fairly, our remarks can be construed in any other way. With regard to the different methods we adopted in getting out the constants, Mr. Sorley gives certain tables which, if glanced at superficially, condemn our work. But I do not think, if they are looked at more carefully, that condemnation will be found to stand. Mr. Sorley's curve of the Carlisle Table gives a total error of 422, as against the error of our combined curve of 486. In that respect his curve apparently is better than ours. But he will find that the chief part, almost 25 per-cent, of the error in our curve occurs at the age of 15, and we rejected that age, and introduced a fresh graduation. Therefore you must look at the curve only after the age of 15. If you strike off the age of 15, Mr. Sorley's error remains 422, ours is reduced to 384; so the only part of the table we use, is considerably more correct than the curve Mr. Sorley proposes to substitute for it. Again, it is one of the beauties both of the combined method and the aggregate method, that only exceptionally have we an error of zero at any point. Our curve runs among all the points, and not through specially selected points; and approaches a better average of them all, without necessarily cutting through any. But this very advantage tends to increase the apparent error in a comparison with such a curve as Mr. Sorley's. Further on Mr. Sorley gives the constants of the curves calculated by the different methods, and they are reproduced, he says, partly on account of their striking dissimilarity. There is one small mistake in them. He has not noticed that we have raised the radix of our table, which makes the characteristic of $\log \kappa$ 4 instead of 3, but that only changes the radix of the table. If reference be made to our own paper, read the other day, it will be found that our constants also differ at least as much from each other as Mr. Sorley's do from those he places beside them. I think that fact shows one of the objections to graduations from isolated values. On the other hand it is a remarkable fact, that if you take the aggregate method, and twist it about as you please, you get constants very much like each other; and more than that, you get constants much like the constants derived from a combination of simple curves, thus leading us to conclude that we have attained to a good average of the table. In conclusion, I would remark that I am somewhat surprised to see one of Mr. Sorley's sentences in which he refers to Mr. Milne's graduation of the Carlisle Table. He says, "Both Mr. Davies and Mr. Milne are mentioned as having used a process of a graphical nature, although in neither case is there, I believe, a description of its precise character. It is quite possible that both these writers did not go beyond interpolation between fixed points." A reference to page 101 of Mr. Milne's work will dispel that notion. He describes most minutely the process he adopted, and he not only describes it verbally, but he gives a diagram of it. He performed a double process of graphical graduation; he took the population at certain intervals of life, and formed parallelograms, which are supposed to represent that population. These parallelograms he set beside each other, the bases representing the years of life and the areas the numbers of the population. Having done that, he then formed a rough series of steps. He then simply drew his curve, cutting off from one parallelogram a corner, and adding to it another corner of exactly equal area. In that way he graduated his table of population. He did the same for the deaths, and then divided the deaths by the population, and got the finally graduated ratios.

Mr. WALFORD—I have always had a fear about graduation. It may be applied to such an extent as to eliminate the vitality out of the table almost, or its individuality at all events, and that seems to me to suggest caution. I think, that if there are enough lives existing in an office, and sufficient deaths to make a fair average, a special table should be made up for that office; and in view of bringing out the distinctive features of the office experience, the proper method would be to apply Mr. Woolhouse's method, which preserves the actual features of the table in a more accurate way than any other. When you come to combine a number of these individual tables, or a large number of facts drawn from various contributing offices, with a view to getting a general result, then it seems to me you must take some such scientific principle as Mr. Makeham's, apply that, and then you will get a curve which will probably answer for all general actuarial purposes. In making these observations, one cannot escape from the consideration of the law of mortality; and I am one of those who believe that as to the general population, there does exist a law of mortality. But in speaking of individual life assurance associations, I doubt whether that can be said at all, for no life assurance association, and no combination of life assurance associations, represent all classes of the population, and probably never will; for even if you include the industrial offices, you do not represent all classes, because that class most affecting the results—namely, the lowest of all—even they do not go into industrial offices.

Mr. G. F. **Hardy**—It is quite clear that any graduation which will stand Professor McCay's severe tests must be a good one, but at the same time it seems to me we must look more or less to the object of the graduation before we can decide as to the best test of its success. Makeham's seems almost exclusively used at present for obtaining the values of joint-life annuities, not for obtaining the rate of mortality at isolated ages. What we want is that the values so obtained shall be as near as possible to the values we should have got if we had worked them out *in extenso* from the ungraduated table. In order to secure this, it seems to me the best method is to get the annuities on each of the single lives as near as possible to the annuities in the original table. The best test we can have therefore of such a graduation would appear to be the smallness of the errors in the monetary values. Mr. Sorley alludes to the serious difference between Makeham's hypothesis and the original facts, as regards the rate of mortality between the ages of 20 and 30. In the H^{MF} Table, for example, Makeham's formula gives the rate much too high for these ages. I think that may be fairly explained by the fact that the great majority of the experience at these ages consists of new entrants, where, owing to the effect of medical selection, the mortality would be below what we should consider the normal rate. I may add, that we have calculated the expected deaths from the "exposed to risk" given in the Institute Tables by means of our graduated H^{M} Table, and the total deaths differed by 27 only from the total deaths as shown by the original experience, and the numbers at each age run very near the original data.

Mr. R. P. **Hardy**—It has been for some time clear to me that there are two distinct schools of graduators—one seeking after the law of mortality as an actual fact, which will probably be discovered in the same day as the philosopher's stone; and the other one desiring simply to make the law run smoothly, so that we may not be stultifying ourselves by quoting results inconsistent with the preconceived opinions of the public—such as, for instance, that we shall not be charging less premiums at 30 years of age than at 21. Although I would never discourage any amount of independent study upon this or any other subject, yet it does seem to me that, as long as the results run moderately smoothly, it is very much a matter of which presents the greatest convenience at the time.

Mr. J. TERRY—As the expression "law of mortality" has been used often this evening, I may allude and refer you to the discussion upon the question whether there is such a law, which appeared in the *Journal of the Institute* some twenty years ago. I think there it was most conclusively shown that there is no such law of mortality beyond that law which exists in every species in assigning a limit to its structural endurance. There is no more a law amongst us than there is amongst ships. A ship will go on until it is worn out, subject, of course, to any accident befalling her. So it will be with human beings; and that law which assigns that limit of structural endurance, is the only physiological law of mortality that exists.

Mr. HARDY—Does that mean that all persons who are born on the same day will die on the same day?

Mr. TERRY—No, but there is no law.

Mr. SORLEY, in reply, said—Regarding the discovery of a law of life, people will of course differ as to what is, and what is not possible. We know that as yet no such law has been found; but we are also aware that, in other branches of science, problems whose solutions appeared most unlikely, have been solved. Still I adhere strongly to the view that no law of life can be discovered—that the fact of every change in the ever-varying conditions under which life is passed, having a greater or less effect on the rate of mortality, places vital phenomena practically beyond the limits of any mathematical analysis, and forbids the hope that the future will be an exact repetition of the past. Mr. King's remarks about Milne's graduation by a graphical method shall have my best attention. I recollect the diagram, but not the description to which Mr. King refers. Regarding the application of Mr. Makeham's formula to practical purposes, it is well to bear in mind that it may be used with two distinct objects—(1) To get a finally adjusted table from raw observations. (2) To obtain such a curve as will fairly represent a given table of mortality already in use. If, for instance, we wish to adjust the constants so as to reproduce the Carlisle Table, it will not do to defend the existence of considerable differences between the curve and the table, on the ground that the former is as likely to be correct as the latter; because, unless the two are substantially the same, inconsistent results must frequently be produced in intricate calculations made partly by Mr. Makeham's method and partly with the assistance of the ordinary Carlisle auxiliary tables. Referring now to the curves in Table D, we should not overlook that they are merely a comparison of methods for finding the constants. The ordinary curve there given is not submitted as a practical substitute for the one Messrs. King and Hardy have decided to adopt, but it seems to me to show very conclusively that if those gentlemen had been content to employ the ordinary method in their various trials, they would have obtained equally satisfactory results, with a much smaller expenditure of labour. Mr. King states that the first of the ordinary curves in their paper gives a less total error than mine; but this is entirely owing to the fact that, using the same ages as in Table D, they get the benefit of all the four points of coincidence with the actual l_x (owing to the ages selected for this coincidence being multiples of (5), while mine has only one of its four zero points represented. There can be little doubt that if the comparison were made for every age, their deviations would be considerably greater than mine. In contrasting the combined curve with my ordinary curve, Mr. King suggested the omission of the error in the former at age 15. This, however, would scarcely be fair, as of my four zero points that at age 15 is the only one included in Table D; while of the two points practically zero in the combined curve (that is, zero when compared with the Carlisle mortality graduated by Mr. Woolhouse's method) both appear in that table. The ordinary curve is more successful than the aggregate, even if the errors under age 30 be excluded; but I frankly admit that it is probable that the portion of the combined curve which Messrs. King and Hardy propose to use, practically runs closer to the unadjusted table than any of the others. I would just add, however, that if I had wished to get a curve to run from age 25, I should not have commenced with the data at age 15. (Mr. King—If you begin very young with the Carlisle Table you get a better curve than with the older ages.) In all these comparisons, I think it would have been better to have looked at the deviation from the deaths of the quinquennium, rather than from the numbers surviving. The former method would have shown more explicitly that the curve frequently makes the rate of mortality err in one direction for several quinquenniums at a time—a very serious defect, and one by no means neutralized by deviations in the opposite direction during subsequent quinquenniums, at least if the formula is to be used in connection with temporary or contingent assurances.

Note on the Calculation of Logarithms to a large number of places by means of Factor Tables. By J. W. L. Glaisher, M.A., F.R.S., *Fellow of Trinity College, Cambridge.*

[Extracted from a paper on Factor Tables in the Proceedings of the Cambridge Philosophical Society, vol. iii, pt. 4.]

THERE are several cases in which the factors of a number may be of use in ordinary mathematical processes; as for example, in the calculation of logarithms. This, although not the *raison d'être* of such a table, is so important an application as to deserve special notice, for by its means the number of numbers whose logarithms are known is greatly extended. For example, Abraham Sharp's table contains 61-decimal Briggian logarithms of primes to 1,100, so that the logarithms of all numbers whose greatest prime factor does not exceed this number, may be obtained by simple addition. Similarly, Wolfram's table gives 48-decimal hyperbolic logarithms of primes up to 10,009. Thus a factor table forms a very valuable complement to many-place logarithmic tables, the range of which must necessarily be comparatively limited; and even in the case of large prime numbers, or numbers exceeding the limits of the factor table, it affords a very convenient method of calculating logarithms.

I have always found that the factor tables of Burckhardt and Dase afforded the best practical method of obtaining an isolated logarithm to more places of decimals than can be obtained directly from the tables. The principle of the method is best exhibited by an example. I required the hyperbolic logarithms of the first eleven Bernoullian numbers to 24 places of decimals. Of these, the numerators and denominators are all composed of prime factors less than 10,009 (the limit of Wolfram's table*), except the numerator of the ninth Bernoullian number, which is the prime number 43,867. Now $50 \times 43{,}867 = 2{,}193{,}350$, and on looking in Burckhardt for a number near to this, which shall have no prime factor greater than 10,009, it appears that

$$2{,}193{,}349 = 23 \times 47 \times 2{,}029,$$

and thus $$43{,}867 = \tfrac{1}{50}(23 \times 47 \times 2{,}029 + 1),$$

and therefore

$$\log 43{,}867 = \log 23 + \log 47 + \log 2{,}029 - \log 50$$
$$+ \frac{1}{2{,}193{,}349} - \frac{1}{2}\,\frac{1}{(2{,}193{,}349)^2} + \frac{1}{3}\,\frac{1}{(2{,}193{,}349)^3} - \&\text{c.}$$

The first term of the series in the second line

$$= 0{\cdot}0000\ 0045\ 5923\ 7950\ 7319\ 6286;$$

dividing this by $2 \times 2{,}193{,}349$, we obtain

$$0{\cdot}0000\ 0000\ 0000\ 1039\ 3325\ 3457,$$

and the third term is

$$0{\cdot}0000\ 0000\ 0000\ 0000\ 0003\ 1590,$$

so that the series

$$= 0{\cdot}0000\ 0045\ 5923\ 6911\ 3997\ 4419;$$

---

* Wolfram's table gives hyperbolic logarithms of all numbers up to 2,200, and of primes (as well as of a great many composite numbers) up to 10,009, to 48-decimal places. It first appeared in Schulze's *Sammlung* (1778), and was reprinted in Vega's *Thesaurus* (1794).

whence, taking out the logarithms from Wolfram's tables,

$$\text{hyp.}\log 43{,}867 = 10{\cdot}6889\ 1760\ 7960\ 5681\ 0191\ 3661.$$

The principle of the method is to multiply the given prime (supposed to consist of 4, 5, or 6 figures) by such a factor that the product may be a number within the range of the factor tables, and such that when increased by 1 or 2, the prime factors may be all within the range of the logarithmic tables. The logarithm is then obtained by use of the formula

$$\log(x+d) = \log x + \frac{d}{x} - \tfrac{1}{2}\frac{d^2}{x^2} + \tfrac{1}{3}\frac{d^3}{x^3} - \&\text{c.},$$

in which, of course, the object is to render $d:x$ as small as possible.

If the number be incommensurable, or consist of more than seven figures, we can take the first seven figures of it (or multiply or divide the number by any factor, and take the first seven figures of the result), and proceed as before. An application to the hyperbolic logarithm of π is given by Burckhardt in the Introduction to his second million.*

I have often employed this method, and even with the gap from three millions to six millions I have never found any difficulty in rapidly obtaining the logarithms I required. Of course, there are methods which may be more expeditious, if the calculator is thoroughly conversant with them and accustomed to their use. But if it is only occasionally that a logarithm has to be calculated, the factor method possesses great advantages; for there is no rule to be remembered, and the reasoning is so elementary that there can be no doubt as to whether the principles have been correctly applied. Whenever a rule or method is but rarely used, great care is necessary in applying it, unless the reason for it is so self-evident that the work of itself shows that no error in principle has been committed.

In the calculation of a table that is to extend to n places, it is usual to calculate the values to $n+2$ places, and if the last two figures are 50, and it is desired that the last figure retained shall be always the nearest to the truth, it becomes necessary to extend the calculation in this case to $n+3$ or $n+4$ places, which probably exceeds the number of places for which the logarithmic tables are available. Recourse must then be had to some method of calculating logarithms, and it is in occasional cases of this kind, or such as that mentioned above, that I have found the factor method so convenient.

---

* See Mr. C. W. Merrifield's letter, vi. 298, *On Formulæ for using Tables of Logarithms*.—Ed. *J.I.A.*

For the purpose of comparison, we have calculated the common logarithm of 43,867 by means of Mr. Peter Gray's *Tables for the formation of Logarithms and Anti-Logarithms to 24 or any less number of places* (Laytons, 1876), and it may interest our readers if we give a brief outline of the process.

The number 43,867 is prepared by multiplying it by 3, which gives the product 131,601, and we have then to resolve 1·31601 into factors. Pursuing Mr. Gray's process, the factors are

1·316 . . . . . . (1)
1·000 007 . . . . . (2)
$1\cdot(000)^2598$ . . . . (3)
$1\cdot(000)^3780$ . . . . (4)
$1\cdot(000)^4002$ . . . . (5)
$1\cdot(000)^5602$ . . . . (6)
$1\cdot(000)^6412$ . . . . (7)
$1\cdot(000)^7340$ . . . . (8)

where $1\cdot(000)^2598$ is used as an abbreviation for 1·000,000,598, and similarly in the other cases.

Mr. Gray's Tables then give the logarithms of the factors, as follows; and addition gives us the logarithm required.

522 878 745 280 337 562 704 972 = Colog 3
119 255 889 277 936 685 553 913 = Log (1)
3 040 050 733 157 610 239 = „ (2)
259 708 022 525 453 597 = „ (3)
338 749 695 752 424 = „ (4)
868 588 964 = „ (5)
261 445 278 = „ (6)
178 929 = „ (7)
148 = „ (8)

4·642 137 934 655 780 757 288 464 = Log_{10} 43,867

Upon multiplying Mr. Glaisher's result by the modulus of the common system of logarithms, M = 434 294 481 903 251 827 651 128 919, we obtained a result which differed from Mr. Gray's merely by the last figure being 5 instead of 4. The general conclusion at which we have arrived on comparing the two processes, is that the labor required to obtain the common logarithm by means of Mr. Gray's tables, is rather less than the labor required by Mr. Glaisher's method to find the hyperbolic logarithm.

Mr. Glaisher writes to us that he has only used the factor method when either (1) he has wanted the common logarithm to only about 15 places, in which case the terms of the series can be calculated by logarithms, and the presence of the modulus as a factor is no inconvenience; or (2), as in the above example, when he has wanted hyperbolic logarithms to a great many places. He adds that the factor method for common logarithms is not a pleasant one, the multiplication by the modulus being very heavy. He also states that he has worked out the same logarithm by means of the relation, $90 \times 43{,}867 = 17^2 \times 19 \times 719 + 1$, which is got from the factor table of the Fourth Million. This leads to

$$\log 43{,}867 = 2\log 17 + \log 19 + \log 719 - \log 90$$
$$+ \frac{1}{3{,}948{,}029} - \tfrac{1}{2}\frac{1}{(3{,}948{,}029)^2} + \&c.$$

Ed. *J.I.A.*

Note on the preceding by Mr. Gray.

The Editor has courteously allowed me to see a revise of the foregoing, that is, of the extract from Mr. Glaisher's paper, and the editorial note by which it is followed; and he has further invited me to offer any remarks on the subject dealt with that occur to me.

The only point to which I desire to advert is one suggested by a remark of Mr. Glaisher's, to the effect that "the factor method for common logarithms is not a pleasant one, *the multiplication by the modulus being very heavy*". This is quite true, and especially so when the logarithm to be formed consists of many places. The labour will be lightened of course by the employment of a small table of the first *nine* multiples of M, which is easily formed; and a further saving of one half will be effected by the employment of a table of the first *hundred* multiples of this quantity. Such a table, extending to thirty-four places, has been formed, and is contained in Shortrede's *Logarithmic Tables*, p. 198.

A still further abridgment of the labour of conversion would be obtained by the employment of a table of the first *thousand* multiples of M, (if such a table were available) insomuch that when, if a table of the desiderated extent were used, the operation, even in the case of logarithms of *very many places,* would be far from deserving the imputation of being a "very heavy one".

I have said "*if* such a table were available". Well, it is, and it is not. To explain: I possess such a table, extending to thirty-three places.* But it is in MS., and its use at present consequently confined to myself. Whether it shall be hereafter put in print depends on circumstances. Meanwhile I give a few examples of its application.

My first example is the conversion of the hyp. log of 43,867, as formed by Mr. Glaisher to 24 places, into the com. log of the same number.

Log 43,867.

Hyp. log	10·688	917	607	960	568	101	913	661
	4·342	944	819	032	518	276	511	289
	298	794	603	549	437	257	423	977
		398	248	039	905	281	925	956
			263	616	750	515	273	859
				416	922	702	627	122
					246	679	265	721
						43	863	743
							396	511
								287
Com. log	4·642	137	934	655	780	757	288	465

As verification of a process involving so many figures will generally be considered desirable, if not indispensable, my second example consists of an easy method of verifying the above.

Hyp. log	10·688	917	607	960	568	101	913	661
×2	21·377	835	215	921	136	203	827	322
	9·120	184	119	968	288	380	673	707
	163	729	019	677	525	939	024	476
		362	635	892	389	215	276	089
			93	373	313	609	199	143
				399	985	217	832	895
					59	064	049	539
						88	161	780
							359	162
								140
÷2	9·284	275	869	311	561	514	576	931
Com. log	4·642	137	934	655	780	757	288	465

The foregoing needs no explanation. I now give a second verification of the same result by a different method, which consists in altering the points of separation of the given logarithm into periods.

---

* I formed this table upwards of twenty years ago, for a special purpose, which is explained on p. [39] of the work named in the foregoing editorial note. It answered its end most efficiently.

10·6	889	176	079	605	681	019	136	61
4·6	035	215	081	744	693	731	019	665
	386	087	794	411	990	874	781	854
		76	435	828	814	972	321	667
			34	309	264	070	356	894
				262	748	161	551	467
					295	754	542	176
						8	251	595
							59	064
								265
4·6	421	379	346	557	807	572	884	647

It is in this process virtually assumed that the last period of the given logarithm is 610, whereas it *may* be anything between 605 and 614. The last figure in the result has therefore no claim to be considered exact.

As a final example, to show the application of the table in its fullest extent, I give the conversion of hyp. log 3 into the com. log.

Log 3.

Hyp. log	1·09	861	228	866	810	969	139	524	523	692	253
	·47	338	098	527	454	449	213	973	052	161	910
		373	927	548	918	699	823	607	621	999	187
			99	019	141	873	941	416	704	457	394
				376	099	021	328	216	082	745	878
					351	778	530	341	633	980	397
						420	831	352	964	251	020
							60	366	932	984	552
								227	570	308	517
									227	136	014
										300	532
											110
Com. log	·47	712	125	471	966	248	729	502	790	325	511

P. GRAY.

London, 12 *Nov*. 1880.

On the Rates ov Marryge among Widos and Spinsters. *By* T. B. Sprague, M.A., F.I.A., F.R.S.E.

In my paper *On the Construction ov a Combined Marryge and Mortality Tabl* (*J.I.A.* xxi, 423), I shoed how the statistics suplyd by the Scottish Registrar-General may be employd to calculate the marryge rate among bachelors. I wil now use the same statistics to calculate the marryge rate among widos and spinsters. As I stated in the paper abov referd to, the Scottish marryge statistics apear to be furnisht with greater exactnes than the English, and the conclusions drawn from them as to the marryge rate, seem therefore much more trustworthy than the corresponding conclusions drawn from the English statistics. The investigation was partly suggested to me by reading the folloing passage in Mr. Meikle's report on the Ministers' Widos' Fund (Church ov Scotland) as at 22nd November 1875. In order to ascertain the liability ov the Fund in respect ov anuitys payabl to the widos ov contributors, it was necesary for him to investigate the rate ov remarryge among the widos, and he has treated the whole subject in his usual careful and exhaustiv manner. He says:—

"It has alreddy been stated that several ov the widos remarryd, and as the anuity then terminates, it is necesary to giv efect to the remarryge of widos in estimating the liabilitys ov the Fund. The number ov widos passing thro' each quinquennium ov life, and the number ov remarryges, wer as follos:—

Ages.	Number ov Widos under Observation.	Number ov Remarryges.	Corresponding Probability ov Remarryge.
20–24	9·083	0	·0
25–29	30·166	0	·0
30–34	96·000	5	·05208
35–39	178·916	7	·03912
40–44	260·833	4	·01534
45–49	373·583	2	·00535
50–59	1374·000	1	·00073

"The abov rate may be compared with the rate ov remarryge ov widos among the general population. The English Registrar-General, in his Annual Report, givs a note ov the number ov widos who marry each year, and in the report ov the Census (say in 1871) ther is givn a statement ov the number ov widos then living. The probability ov remarryge wil be fairly ascertaind, if the average number ov widos remarrying in the three years 1870, 1871, and 1872,—being the census year, and the years preceding and succeeding,—be compared with the number ov widos alive in the census year ov 1871.

"The folloing ar the facts:—

Ages.	Number ov Widos enumerated in 1871.	Average Number ov Widos Marryd 1870—1872.	Corresponding Probability ov Remarryge.
15–19	307	13	·04235
20–24	5,491	937	·17064
25–29	18,676	2,928	·15678
30–34	33,351	3,604	·10806
35–39	45,939	3,269	·07116
40–44	63,983	2,853	·04459
45–49	74,377	2,089	·02809
50–54	94,219	1,426	·01513
55–59	94,765	777	·00820
60–64	116,676	458	·00393
65–69	107,153	143	·00133
70–74	99,177	47	·00047
75–79	66,894	9	·00013
80 & upwards	58,165	1	·00002

"If the actual remarryges ov the widos ov the Fund be compared with this general probability ov remarryge, it wil be found that 67,526 miht hav been expected to marry, in place of 19, as follos:—

Ages.	Actual Number ov Marryges.	Estimated Number acording to		
		Own Experience ov Fund by Ajusted Probability.	Experience ov Population.	Experience ov Scottish Population.*
20–25	0	...	1·550	·842
25–29	0	...	4·729	2·866
30–34	5	4·974	10·374	6·294
35–39	7	6·433	12·732	7·474
40–44	4	3·990	11·630	5·304
45–49	2	1·814	10·494	4·386
50–59	1	·515	16·017	4·272
Totals	19	17·726	67·526	31·438

[* This colum is not containd in Mr. Meikle's tabl, but is added, as explaind belo, for the purpos ov comparison.]

Proceeding in the same manner as Mr. Meikle has don, we get the figurs containd in the folloing tabl, which has been compiled for me by Mr. John Notman, ov the Royal Insurance Company, Edinburgh:—

Ages.	Number ov Widos enumerated 1871.	Average Number ov Widos Marryd 1870-72.	Force ov Marryge.
15–19	37	1·	·027027
20–24	676	62·7	·092702
25–29	2,757	262·	·095031
30–34	5,440	356·7	·065564
35–39	7,676	320·7	·041775
40–44	10,917	222·	·020335
45–49	12,662	148·7	·011741
50–54	16,279	74·7	·004587
55–59	16,107	26·	·001614
60–64	20,687	15·7	·000757
65–69	17,283	3·	·000174
70–74	17,056	·7	·000039
75–79	10,549	·3	·000032
80 & upwards	10,115	...	...

It wil be observd that the marryge rate indicated by these figurs follos a very similar law to the rate obtaind from the English statistics, but is thro'out very much smaller. In order to sho the efect ov this, I hav added a colum to Mr. Meikle's tabl ov the estimated number ov remarryges, so as to sho what woud hav been the number ov remarryges among the ministers' widos if they had folloed the same law as obtains among the Scottish population.

The comparison shos us that the marryge rate among the Scottish widos is les than half the rate among the English widos, but the rate ov marryge among the Scottish ministers' widos is stil les. This result may be taken to justify the conclusion that the marryge rate among the widos ov the educated clases is greatly les than among the uneducated, who form the great bulk ov the population; and figurs as to the marryge rate deduced from statistics ov the general population shoud therefore be employd with very great caution in dealing with questions relating to the educated clases.

In my paper *On the Rate ov Remarryge among Widoers,* I shoed that at most ages the rate depends more upon the duration ov widoerhood than upon the age ov the widoer; but I am inclined to think that in the case ov widos the duration ov widohood wil be a very much les important element, and that after a woman has been a wido for (say) two, or at most three years, the probability ov remarryge woud practically depend entirely upon her age and not at al upon the length ov time she has been a wido.

The folloing tabl givs the corresponding figurs relating to spinsters. On comparing it with the previos one, we see that the marryge rate among widos is thro'out considerably hiher than among spinsters, the only real exception to this law being at the ages ov 20 to 24, when the spinsters marry at a somwhat hiher rate. This is perhaps an unexpected result, but upon consideration it seems to be one that is inherently probabl. It is wel knoen that the marryge rate among widoers is very much hiher than among bachelors, and the same causes which lead to this result no dout operate, tho to a much les degree, with the other sex.

Ages.	Number ov Spinsters enumerated 1871.	Average Number ov Spinsters Marryd 1870-72.	Force ov Marryge.
15–19	164,612	3,180	·019318
20–24	113,622	10,939	·096275
25–29	61,487	5,634	·091629
30–34	36,227	1,807	·049880
35–39	26,164	778·7	·029761
40–44	20,917	362·3	·017322
45–49	16,139	148	·009170
50–54	14,695	49·7	·003380
55–59	11,048	19	·001720
60–64	11,258	3·7	·000326
65–69	7,697	·3	·000043
70–74	6,616	...	...
75–79	3,556	·3	·000094
80 & upwards	2,995	...	...

On the Tradition Question in Probabilities. *By* ALEXANDER MACFARLANE, M.A., D.Sc., F.R.S.E.

In a book entitled *Principles of the Algebra of Logic,* which deals with the Sciences of Necessity and Probability as one whole, I have considered as an example the question in probabilities discussed in vol. xxi (p. 204) of this *Journal,* namely—"A says "that B says that a certain event took place: required the pro-"bability that the event did take place, p_1 and p_2 being A's and "B's respective probabilities of speaking the truth."

The conclusion at which I arrive is that the probability lies between the limits p_1p_2 and $p_1p_2+1-p_1$. A reports either correctly or incorrectly; if he reports correctly, B must have stated that the event in question took place; if he reports incorrectly, the event in question may or may not have taken place. Suppose A's statement correct, then B's was either correct or incorrect; the former supposition necessitates that the event took place; the latter supposition necessitates that it did not take place. But if p_1 is the probability of A making a correct statement, and p_2 of B making a correct statement; then the value of the former probability is p_1p_2, and of the latter $p_1(1-p_2)$. On the hypothesis that A reports incorrectly, the most favourable supposition to the happening of the event is that it always happens in that case, and the least favourable supposition is that it never happens in that case. Hence, a minor limit to the probability of the event having taken place, is p_1p_2; and a major limit is $p_1p_2+1-p_1$. Similarly, a minor limit to the probability of the event not having taken place, is $p_1(1-p_2)$; and a major limit, $p_1(1-p_2)+1-p_1$.

I have also considered the general problem—

> A_1 says that A_2 says that A_3 says .... that A_n says that a certain event took place. The probabilities of A_1, A_2, A_3, ... A_n, respectively speaking the truth, are p_1, p_2, p_3, ... p_n. Required the probability that the event took place.

It can be proved, very simply, by means of the *Algebra of Logic*, that the probability in this case lies between $p_1p_2\ldots p_{n-1}p_n$ and $p_1p_2\ldots p_{n-1}p_n+1-p_1p_2\ldots p_{n-1}$.

To give that proof would require a long discussion of principles, but the truth of the result may be seen from the following considerations. Suppose that we have three persons, A_1, A_2, A_3. Then, if A_1, A_2, and A_3 each speaks correctly, the event must have taken place. If A_1 and A_2 each speaks correctly, and A_3 incorrectly, the event cannot have taken place. If A_1 speaks correctly and A_2 incorrectly, the event may or may not have taken place. If A_1 speaks incorrectly, the event may or may not have taken place. Hence the necessary part of the probability of the event having taken place, is $p_1p_2p_3$; and the contingent part is $p_1(1-p_2)+1-p_1$, which is equal to $1-p_1p_2$. Similarly, when the tradition is transmitted through three agents, the necessary part of the probability is $p_1p_2p_3p_4$; and the contingent part is $p_1p_2(1-p_3)+p_1(1-p_2)+1-p_1$, which is equal to $1-p_1p_2p_3$.

It is evident that the contingent part will always reduce to

1 minus the necessary part without the last factor. It is equal to 1 minus the sum of the necessary and impossible parts.

Cor. 1.—Suppose that the original authority and each of the agents in the tradition always states correctly. Then $p_1, p_2, \ldots p_n$, each equal to 1, and the probability for the event having taken place is 1.

Cor. 2.—Suppose that each of the agents in the tradition always states correctly. Then the probability is equal to p_n.

Cor. 3.—Suppose that each of the persons states correctly and incorrectly with equal frequency. Then the minor limit of the probability is $(\frac{1}{2})^n$, and the major $(\frac{1}{2})^n + 1 - (\frac{1}{2})^{n-1}$, which is equal to $1 - (\frac{1}{2})^n$. When n is infinitely great, the minor limit is 0, and the major limit 1; thus the difference of the limits is 1; that is, there is complete uncertainty.

We must suppose that p_1, p_2, &c., are the probabilities of speaking, not with subjective veracity, but with objective correctness. Out of a number of statements which A has made, it has been found that p_1 is the ratio to the whole number of statements of those which are in accordance with fact—not merely in accordance with his belief when uttering them. If p_1 expressed A's veracity, it would be a much less effective datum in determining the probability of the event having taken place; for the veracity of A's statement would not necessitate that B said anything whatever. Mr. Woolhouse appears to take the subjective view of the matter.

It is not essential to the above investigation to suppose that the statement about the event admits of only one denial. The tradition may be—

A says that B says that he saw a man C at a place D at a time E.

Here the probability required is that of the statement about the event being in accordance with fact—not only the event taking place, but taking place as stated. Even the given problem requires that we should adopt this conception of truth and falsehood. For, though the event spoken of can be false only in one particular, the statement of A can be false in several particulars.

Note on the Probability that a Marriage entered into by a Man above Forty will be Fruitful. By THOMAS BOND SPRAGUE, M.A., F.R.S.E.

[From the Proceedings of the Royal Society of Edinburgh, vol. x, 1878-79.]

WHEN it is desired to disentail a landed estate, it is necessary for the heir in possession, after obtaining the consent of the first substitute heir, to pay to the second and third heirs the estimated value of their expectancy or interest in the estate. In the calculations that have to be made for the purpose of ascertaining this value, the actuary has often to take into account not only the probabilities of life, but the probabilities of marriage and of leaving issue. The heir in possession may be unmarried, in which case he may marry at some future time, and leave a child who would inherit the estate to the exclusion of the subsequent heirs; or the heir in possession may be married but have no children, and the probabilities then to be estimated are (1) that his present wife will have a child at some future time, and (2) that she will die before her husband, that he will then marry a second time, and have issue by his second marriage. Similar contingencies may occur with regard to the first substitute heir. The theory of the calculation of life contingencies is well understood; and much has been done in the way of accumulating marriage statistics and calculating the probability that a man of any age, bachelor or widower, will marry at some future time; but little, if anything, has been done in the direction of estimating whether the marriage so entered into will be fruitful or not. My object in the present note is to contribute towards the elucidation of this question.

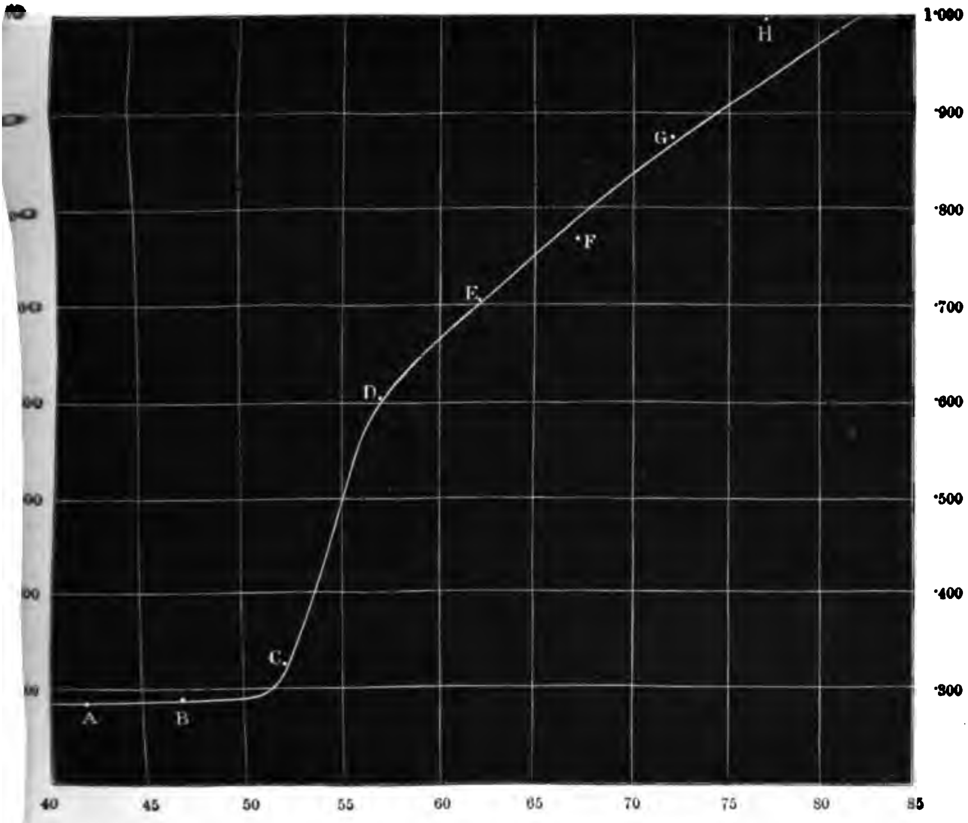
The statistics furnished in the Decennial Census Reports and in the Annual Reports of the Registrar-General, afford the means of calculating the marriage rate among the general population, but these are of no assistance to us in the present inquiry. First, as regards the rate of marriage, it would be clearly unsafe to assume that the rate deduced from statistics as to the general population, of which the working classes form a very large majority, would be applicable to the landed gentry. Secondly, the above-mentioned statistics give us no information whatever as to the fruitfulness of the marriages entered into. It is therefore necessary to seek statistics in some other quarter, and statistics very trustworthy and perfectly suitable for our purpose, are found among the records of the British Peerage as contained in the various Peerages published annually. The facts given in them are not so numerous as could be desired for special purposes, but it is of even more importance that our statistics should be accurate than that they should be very numerous. It therefore seems to me that the statistics obtained from a careful examination of the records of the British Peerage may safely be adopted as the basis of our calculations in the present subject. I have accordingly taken the volume of Lodge's Peerage for the year 1871, and gone carefully through it, noting all the cases of marriages of men, whether bachelors or widowers, entered into after the age of 40. The total number of such men, as to whom the necessary information was complete, was 339, of whom 132 were bachelors and 207 widowers. These may seem small numbers on which to base a general law; but, in default of larger numbers, I think we must do the best we can to see what conclusions may safely be drawn from them. The following table shows the ages at which the marriages were contracted, and how many of those contracted at each age were fruitful or unfruitful. Most of the marriages included in the observations were contracted many years ago, so that the information contained in the volume of Lodge was taken as conclusive as to whether the marriages were fruitful or not. For the comparatively few as to which there seemed a doubt, I referred to the volume of Burke's Peerage, published in 1878. In all cases, if a child was born, I have deemed the marriage fruitful, whether the child survived or not.

Age at which the Marriage is contracted.	BACHELORS.			WIDOWERS.			BACHELORS AND WIDOWERS.		
	No. of Marriages.	Of which were		No. of Marriages.	Of which were		No. of Marriages.	Of which were	
		Fruitful.	Unfruitful.		Fruitful.	Unfruitful.		Fruitful.	Unfruitful.
40	25	19	6	11	8	3	36	27	9
41	15	12	3	7	4	3	22	16	6
42	19	14	5	10	7	3	29	21	8
43	10	7	3	14	12	2	24	19	5
44	6	3	3	10	5	5	16	8	8
45	6	4	2	10	7	3	16	11	5
46	8	5	3	4	3	1	12	8	4
47	8	6	2	8	5	3	16	11	5
48	6	6	...	11	10	1	17	16	1
49	6	3	3	2	...	2	8	3	5
50	2	2	...	7	4	3	9	6	3
51	2	...	2	7	5	2	9	5	4
52	2	1	1	10	10	...	12	11	1
53	3	1	2	8	5	3	11	6	5
54	4	3	1	4	2	2	8	5	3
55	2	...	2	7	3	4	9	3	6
56	...	...	...	9	3	6	9	3	6
57	2	1	1	7	4	3	9	5	4
58	1	...	1	10	5	5	11	5	6
59	2	1	1	3	...	3	5	1	4
60	2	...	2	14	5	9	16	5	11
61	...	...	...	1	1	...	1	1	...
62	...	...	...	4	1	3	4	1	3
63	1	1	...	4	...	4	5	1	4
64	...	...	...	1	...	1	1	...	1
65	...	...	...	7	2	5	7	2	5
66	...	...	...	3	...	3	3	...	3
67	...	...	...	2	1	1	2	1	1
68	...	...	...	...	...	...	...	...	...
69	...	...	...	1	...	1	1	...	1
70	...	...	...	2	...	2	2	...	2
71	...	...	...	2	1	1	2	1	1
72	...	...	...	2	...	2	2	...	2
73	...	...	...	1	...	1	1	...	1
74	...	...	...	1	...	1	1	...	1
75	...	...	...	...	...	...	...	...	...
76	...	...	...	1	...	1	1	...	1
77	...	...	...	...	...	...	...	...	...
78	...	...	...	1	...	1	1	...	1
79	...	...	...	1	...	1	1	...	1
Total .	132	89	43	207	113	94	339	202	137

A glance at this table is sufficient to show that the probability of a marriage being fruitful is less as the age of the husband increases, and the same conclusion appears more plainly when we group the ages quinquennially, as is done in the following table:—

is remarkably regular; in fact so regular, that, notwithstanding the comparatively small numbers observed, we shall be justified in believing that we have here an indication of a law of nature, upon which we may safely base our calculations.

In order to make our conclusions practically serviceable, it is necessary to calculate from them the probability for every age from 40 onwards, or, in technical language, to graduate the probabilities. This I do by a graphic method. In the appended figure I take the abscissa to represent the age, and the ordinate the probability that a marriage entered into at that age will be unfruitful. Thus, for age 42, being the middle of the five ages 40–44, the probability is ·285, and this is represented by the point *A*. In the same way the points *B*, *C*, *D*, *E*, *F*, *G*, *H*, represent the probabilities for the ages 47, 52, 57, 62, 67, 72, 77. Then, joining these points by straight lines, I draw a curve which shall follow the general course of the points as faithfully as is consistent with the avoidance of irregularities in its form. Then, by estimating the ordinates of the



points where the curve cuts the vertical lines in the diagram, I obtain a first approximation to the adjusted probability for each age, and this is afterwards corrected by a process which it seems unnecessary to describe on the present occasion. The following are the values which I thus obtained:—

Probability that a Marriage entered into by a Man of any Age from 40 onwards will be Unfruitful.

Age.	Probability.	Age.	Probability.	Age.	Probability.	Age.	Probability.
40	·284	51	·295	62	·702	73	·882
41	·284	52	·315	63	·720	74	·896
42	·285	53	·365	64	·738	75	·910
43	·285	54	·430	65	·755	76	·924
44	·286	55	·500	66	·772	77	·937
45	·286	56	·562	67	·789	78	·950
46	·287	57	·600	68	·805	79	·963
47	·287	58	·626	69	·821	80	·976
48	·288	59	·646	70	·837	81	·988
49	·288	60	·665	71	·852	82	1·000
50	·290	61	·684	72	·867		

If we now multiply the number of marriages entered into at any age, by the probability in this table, the product will be the number of those marriages that may be expected to be unfruitful. Doing this for each age, and adding the products together in quinquennial groups, we get the following comparison of the expected and actual number of unfruitful marriages, the agreement between the results affording a very complete test of the goodness of the adjustment.

Ages.	No. of Unfruitful Marriages.	
	Actual.	Expected.
40–44	36	36·2
45–49	20	19·8
50–54	16	16·5
55–59	26	25·1
60–64	19	18·5
65–69	10	10·0
70–74	7	6·9
75–79	3	2·8
Total .	137	135·8

I will not attempt to discuss the various points on which the fruitfulness of a marriage depends. The most important of them is clearly the age of the wife; and consistently with this, we see that the probability of a marriage being unfruitful, increases with great rapidity between the ages of 50 and 60; obviously because a much larger proportion of the wives married by men of these ages are past child-bearing than is the case with those married by men under 50.

It is but very rarely that the information available to the public will enable us to form even a reasonable conjecture whether the absence of issue of a marriage is attributable to the husband or to the wife, but such cases do occasionally occur. For instance, when a man marries a widow of 30 who has already had several children, it may fairly be inferred that it is the fault of the husband if there are no children born of her second marriage. So, again, if there are no children born of a marriage, the wife dies, and the husband then marries a second time and has a family, it may fairly be inferred that the absence of children of the first marriage was not attributable to him. It is obvious that cases of this kind are so very few that we cannot attempt to base any general reasoning upon them, nor is it at all necessary that we should do so.

It only remains to mention that in obtaining the probabilities above set forth no account has been taken of the age of the wife at the date of marriage. Our conclusions therefrom will have no application to individual cases where the age of the wife is known, but are only applicable to the cases indicated at the outset, where the men we are considering are not contemplating marriage. In other words, we have calculated the probability that, if a man who is either now married or who is single and is not contemplating marriage, shall hereafter enter into a marriage at a certain age, the marriage will be unfruitful.

Scheme for the Encouragement of Life Insurance among the Employees of Life Offices.

SUGGESTIONS have been made at various times that the Directors of life insurance companies should encourage the practice of life insurance among the members of their own staff; and although we are aware that something of the kind has been done in a few instances, we do not know that this is at all general. We trust that, by publishing the following particulars of a scheme that has been lately adopted by an office with a numerous staff, and thus drawing attention to the general question, we shall be doing something to promote what must be acknowledged by all to be a very desirable object.

EMPLOYEES' INSURANCE SCHEME.

Extract from Minutes of the Board of Directors of the __________
________________ Life Assurance Society, *of* __________ 1880.

In order to encourage the practice of Life Insurance among the Employees of the Society, it is Resolved, That, subject to the conditions hereinafter stated, if any one of the salaried Employees has effected, or shall hereafter effect, a Policy or Policies on his life with the Society, on any one of the following Tables,—

Ordinary Whole Life, either with profits (not being half-credit) or without profits,
Limited payments, either with or without profits,
Endowment Assurance, with or without profits,
Non-Forfeitable,

the Society shall, upon his written application declaring his accession to this Scheme, pay, so long as he remains in their service, the premium in respect of one-half or a smaller portion of the sum assured under such Policy or Policies, but so that the premium paid by the Society for any Employee in the course of a year, shall not exceed five per-cent of his salary, nor in any event exceed £30, and that the portion of the sum assured in respect of which the Society pays the premium, shall not exceed £1000.

Any Policy, the premium on which is partly paid by the Society, shall, so long as the Employee remains in the service of the Society, be assigned to the Society on such trusts as the Directors shall prescribe, or shall be otherwise effectually settled to the satisfaction of the Directors for the benefit of the Employee's wife and family.

In the event of the Employee leaving the service of the Society, all sums paid by the Society as above in respect of any Policy shall, if the Directors in their absolute discretion think fit, be repayable by the Employee.

We print below the form of assignment to be executed by each employee who takes advantage of the Scheme, and it may be useful to some of our readers to know that the proper stamp duty has been adjudicated at 10*s*. The above Minute is printed on the back of the assignment, and the following note is appended:—

⁎ The Directors will be glad to be assisted in exercising their discretion under the form of assignment, by receiving from Employees intimation of their wishes as to the disposal of the policy moneys consistently with the trusts expressed in the form of assignment.*

I,

an Employee of the   LIFE ASSURANCE SOCIETY, Considering that the said Society have adopted a Scheme whereby they have undertaken on certain conditions to pay a portion of the premiums on Policies of Assurance effected with them by their Employees on their own lives, and that I have effected with the said Society the following Polic   on my life, namely No.   , dated   , for £   , the premium on which is £   , and No.   , dated   , for £   , the premium on which is £

the total sum assured under the said Polic   being £   , subject to premiums amounting to £   per annum, and am desirous of availing myself of the advantages of the said Scheme (a copy of which is endorsed hereon, and signed with reference hereto),— Therefore I, the said Employee, in consideration of the premises, do hereby assign unto the said Society the said Polic   of Assurance, and that for the purposes and under the conditions following:—

First, That in consideration of the said Society agreeing to pay yearly towards the premiums on the said Polic   , so long as I shall remain in their service, a sum of five per-cent on my salary for the time being, but not exceeding one-half of the said premiums, and in no event exceeding £30; such Polic   shall be held for the benefit of my wife and children, or wife, or children, as the Directors of the said Society shall in their absolute discretion think best; and accordingly (if I shall be in the employment of the said Society at the time of my death), the said Society shall pay to my wife and children, or apply for their benefit, or pay to any Trustee or Trustees for their behoof, to be approved of by the Directors, in such manner and at such times and in such proportions as the said Directors in their discretion may think best, the Policy moneys; the payment of the money by the Society being held to be an absolute and unconditional discharge to the Society, they being in no way responsible for the application thereof.

Second, That if I shall die, while in the employment of the said Society, without leaving either wife or children, the Policy moneys shall be paid to such person or persons as the Directors may in their absolute discretion think fit, payments under this power being held to be an absolute and unconditional discharge to the Society.

Third, So long as I shall remain in the service of the said Society, the Polic   shall be in no way at my disposal, nor subject in any way to my debts, control, or engagements.

Fourth, That whenever I shall leave the service of the said Society, the payment by the Society on account of the premiums shall forthwith cease and determine, and the trusts hereinbefore provided shall fall, and the Polic   shall be at my disposal, burdened with the repayment of the sums which the Society may have paid, if the Directors shall think proper to require such repayment, and I shall not be entitled to ask a reconveyance of the Polic   until these are paid.

In witness whereof, these presents are subscribed by me at   , on the   day of
in the year Eighteen hundred and

Witness,

On the Application of the Arithmometer to the Construction of Tables of the Values of Endowment Assurance Policies. By David Carment, *of the Australian Mutual Provident Society, Sydney, N.S.W.; Fellow of the Institute of Actuaries, and of the Faculty of Actuaries in Scotland.*

ENDOWMENT Assurances are now becoming so common, that the necessary tables for valuing such policies cannot fail to be of great use to offices which have a large proportion of these risks on their books. I have therefore ventured to attempt the construction of a series of Tables of the Values of Policies payable at 50, 55, 60, and 65, or at previous death—these being, if I mistake not, the ages most commonly chosen in effecting such assurances. The only similar tables I am at present acquainted with, are those contained in the Hon. Elizur Wright's *Valuation Tables*, published at Boston in 1871; but as they are calculated only at 4 per-cent interest, and on the basis of the old "Seventeen Offices'" Experience, which is now almost entirely superseded by the more recent H^M Table, I think it will be found that the tables now submitted will not be without value to the profession. Those which are appended to the present paper are calculated on a 3 per-cent basis. I hope, however, as time permits, to complete the construction of similar series at 3½ and 4 per-cent, and possibly also of the corresponding set of tables according to the $H^{M(5)}$ Table of mortality.

On reading Mr. Gray's paper on the use of the Arithmometer (*Journal*, vol. xvii, p. 249), it struck me that, by a slight adaptation, the process by which he deduces a table of the values of ordinary policies from the formula $1-\dfrac{1+a_{x+n}}{1+a_x}$ might be made available for endowment assurances as well. For the value of such a policy is represented by an expression exactly similar in form, namely, $1-\dfrac{1+{}_{|t-n-1}a_{x+n}}{1+{}_{|t-1}a_x}$, where $x+t$ is the age at which the assurance becomes payable if the life survives so long. Now, reasoning in the same manner as Mr. Gray does in the paper just referred to (pp. 255 and 256), we have

$$1-\frac{1+{}_{|t-n-1}a_{x+n}}{1+{}_{|t-1}a_x}=1-(1+{}_{|t-1}a_x)^{-1}(1+{}_{|t-n-1}a_{x+n})={}_nV_{x\overline{t}|}.$$

Hence $\qquad \Delta_n V_{x\overline{t}|}=-(1+{}_{|t-1}a_x)^{-1}\Delta_{|t-n-1}a_{x+n}\,;$

and $\qquad {}_{n+1}V_{x\overline{t}|}={}_nV_{x\overline{t}|}-(1+{}_{|t-1}a_x)^{-1}\Delta_{|t-n-1}a_{x+n}.$

Therefore all we have to do is to use the differences of the temporary annuities terminating at age $x+t$ in place of those of the whole-life annuities. Now these annuities being tabulated in Mr. Hardy's volume for the most commonly used values of $x+t$, we can easily take their differences, and are thus enabled to calculate values of endowment assurance policies quite as fast as ordinary policy-values. As, however, it requires four-place annuity-values to produce policy-values correct to the fifth decimal place (or, which is the same thing, to the *third* decimal place in the value of a policy for 100), I found it desirable, before proceeding further, to calculate the values of the required annuities to four places from the N and D columns given in the Institute Tables.

Then, the reciprocal of $1+{}_{|t-1}a_x$ being set on the machine, the successive terms of the series $\Delta_{|t-1}a_x$ are used as multipliers. The correctness of each column of values is easily proved, as the last result in each case ought to be $v-P_{x\overline{t}|}$, since the value of an endowment assurance policy one year before the age at which it becomes payable is evidently equal to the amount assured discounted for a year, less the annual premium which is just due. Another check on the correctness of the last value in each column is pointed out by Mr. Gray in his paper above referred to (p. 259), and consists in the fact that such value should also be equal to the difference between unity and the reciprocal which has been used in constructing the column; thus, at 3 per-cent,

$$\begin{aligned}{}_{29}V_{20.\overline{30}|}&=\cdot94566\\&=v-P_{20.\overline{30}|}=\cdot97087-\cdot02521\\&=1-\frac{1}{1+{}_{|29}a_{20}}=1-\cdot05434.\end{aligned}$$

In practice it was generally found that the last value in each column came out with perfect exactness, thus rendering any check values at intermediate points almost unnecessary, and establishing the correctness of the whole, with the exception of any errors committed in copying the results from the machine. In order finally to eliminate any errors arising from this source, each column was summed and the totals checked by means of a modification of the formula given for a similar purpose on page xii of Mr. Hardy's work. The formula there given is $\Sigma_n V_x = n - \frac{\Sigma(1+a_{x+1})}{1+a_x}$, which in the present case becomes transformed into

$$\Sigma_n V_{x\overline{t}|} = n - \frac{\Sigma(1+{}_{|t-2}a_{x+1})}{1+a_x}.$$

In order to facilitate the check calculations, as well as to find the necessary reciprocals and multipliers, I first of all constructed a table in the following form :—

$x+t=50.$

1	2	3	4	5	6
x	$N_{x-1}-N_{49}$	$1+{}_{\vert t-1}a_x$	$\Sigma(1+{}_{\vert t-2}a_{x+1})$	$(1+{}_{\vert t-1}a_x)^{-1}$	$-\Delta{}_{\vert t-1}a_x$
20	980501·4	18·4041	314·0238	5133572	·3637
1	927225·4	18·0404	295·9834	5543114	·3700
2	875827·4	17·6704	278·3130	5659182	·3816
3	826263·4	17·2888	261·0242	5784092	·3970
4	778471·4	16·8918	244·1324	5920032	·4139
5	732385·7	16·4779	227·6545	6068735	·4292
6	687939·2	16·0487	211·6058	6231034	·4443
7	645073·4	15·6044	196·0014	6408449	·4572
8	603734·4	15·1472	180·8542	6601880	·4704
9	553876·5	14·6768	166·1774	6813474	·4842
30	525457·0	14·1926	151·9848	7045925	·4984
1	488433·8	13·6942	138·2906	7302361	·5148
2	452766·5	13·1794	125·1112	7587599	·5322
3	418412·3	12·6472	112·4640	7906889	·5503
4	385329·0	12·0969	100·3671	8266580	·5691
5	353475·4	11·5278	88·8393	8674682	·5882
6	322812·6	10·9396	77·8997	9141102	·6077
7	293304·0	10·3319	67·5678	9678762	·6282
8	274915·9	9·7037	57·8641	1030535	·6503
9	237615·4	9·0534	48·8107	1104557	·6739
40	211369·4	8·3795	40·4312	1193389	·6995
1	186144·6	7·6800	32·7512	1302083	·7267
2	151907·0	6·9533	25·7979	1438166	·7549
3	138622·1	6·1984	19·5995	1613320	·7838
4	116258·0	5·4146	14·1849	1846858	·8144
5	94787·0	4·6002	9·5847	2173819	·8462
6	74182·2	3·7540	5·8307	2663825	·8802
7	51421·4	2·8738	2·9569	3479713	·9169
8	35484·5	1·9569	1·0000	5110123	·9569
9	17351·1	1·0000	...	1000000	...

Column (2) was formed by writing N_{49} on the lower edge of a slip of paper, and subtracting it successively from each of the values of N in the Institute volume from N_{19} onwards. It might also have been formed by successive addition of the D's, beginning at D_{49} and going backwards. Column (3) was formed on the machine by dividing each of the numbers in column (2) by the corresponding value of D taken from the Institute Table, and the results were checked as far as the third decimal place by comparison with those given in Hardy's book, p. 52. Column (4) contains the successive sums of the numbers in column (3), the number at any age in column (4) being equal to the sum of the numbers in column (3) at all subsequent ages. In column (5) are inserted the reciprocals of the annuity-values given in column (3), taken from Oakes's Tables; and in column (6) the differences of the annuity-values. In practice, this last column was written on a separate slip of paper, so that when calculating the policy-values it could be placed close alongside of each column in succession, as described at p. 260 of Mr. Gray's paper. As each column of values was completed it was summed up, and *before going on to the next* the formula for verifying the sums of the columns was brought into use in the following manner. Take, for example, the column of values under the heading of age 20, payable at death or 50. Here we will have $\Sigma_{29}V_{20.\overline{30}|}=29-(314\cdot0238)(\cdot05433572)$. Therefore, leaving on the face of the machine the same reciprocal which was used in constructing the column, setting 29 in the proper position on the slide, and putting the regulator at subtraction, we use 314·0238 as a multiplier, and get 1193·729 for the sum of the column, exactly agreeing with the result of the actual summation. As showing the degree of exactness attained by the use of this check, it may be mentioned that out of the 134 columns composing the present set of tables, in 39 the result of the verification formula agreed exactly with the sum of the column, in 62 the difference was 1 in the third decimal place, in 21 the difference was 2, in 10 it was 3, and in only 2 cases was it as much as 4. This may, I think, fairly be considered a crucial test of the accuracy of the whole work.

I found it generally desirable to use the reciprocals of the annuities to 7 significant figures, for on trying to work with only 6 I found that in certain cases the error accumulated to such an extent as occasionally to creep into the last place of the recorded

[*See page* 380.]

H^{M} 3 *per-cent—Death or* 50.

Duration.	20	21	22	23	24	25	26	Duration.
1	1·976	2·051	2·160	2·296	2·450	2·605	2·768	1
2	3·987	4·166	4·406	4·690	4·991	5·301	5·617	2
3	6·060	6·367	6·749	7·173	7·621	8·076	8·548	3
4	8·217	8·661	9·177	9·743	10·328	10·930	11·565	4
5	10·466	11·040	11·692	12·387	13·113	13·869	14·671	5
6	12·798	13·503	14·279	15·108	15·979	16·894	17·879	6
7	15·212	16·037	16·941	17·909	18·930	20·018	21·195	7
8	17·697	18·645	19·681	20·791	21·978	23·248	24·624	8
9	20·253	21·329	22·502	23·769	25·128	26·587	28·170	9
10	22·883	24·091	25·415	26·847	28·386	30·041	31·835	10
11	25·592	26·945	28·427	30·030	31·755	33·610	35·622	11
12	28·389	29·895	31·541	33·322	35·237	37·298	39·536	12
13	31·281	32·945	34·762	36·724	38·835	41·111	43·588	13
14	34·271	36·100	38·091	40·239	42·554	45·057	47·787	14
15	37·363	39·361	41·530	43·873	46·404	49·147	52·146	15
16	40·559	42·729	45·085	47·634	50·393	53·392	56·674	16
17	43·861	46·211	48·765	51·533	54·534	57·802	61·378	17
18	47·274	49·816	52·579	55·579	58·836	62·384	66·262	18
19	50·808	53·551	56·537	59·782	63·305	67·140	71·337	19
20	54·469	57·429	60·650	64·148	67·945	72·083	76·609	20
21	58·270	61·457	64·922	68·682	72·767	77·218	82·093	21
22	62·219	65·642	69·358	73·393	77·776	82·560	87·806	22
23	66·321	69·986	73·967	78·287	82·987	88·124	93·769	23
24	70·579	74·501	78·755	83·378	88·415	93·931	26	
25	75·004	79·191	83·737	88·682	94·080	25		
26	79·602	84·070	88·926	94·216	24		41	
27	84·385	89·153	94·341	23		42		
28	89·367	94·457	22		43		86·979	8
29	94·566	21		44		85·618	74·520	7
	20		45		83·867	71·857	62·581	6
				81·531	68·429	58·670	51·120	5
4	...	...	78·262	63·859	53·636	46·011	40·102	4
3	...	...	57·461	46·925	39·436	33·841	29·497	3
2	...	...	37·529	30·669	25·784	22·129	19·292	2
1	...	...	18·395	15·041	12·645	10·857	9·462	1
			45	44	43	42	41	

H^M 3 *per-cent—Death or* 50—(continued).

Duration.	27	28	29	30	31	32	33	Duration.
1	2·930	3·106	3·299	3·512	3·759	4·038	4·351	1
2	5·944	6·302	6·695	7·139	7·646	8·214	8·851	2
3	9·047	9·593	10·202	10·889	11·664	12·532	13·502	3
4	12·241	12·991	13·829	14·766	15·820	16·995	18·307	4
5	15·540	16·505	17·578	18·776	20·115	21·606	23·274	5
6	18·951	20·138	21·456	22·920	24·553	26·372	28·416	6
7	22·478	23·895	25·463	27·202	29·140	31·306	33·744	7
8	26·125	27·778	20·604	31·628	33·889	36·420	39·275	8
9	29·894	31·790	33·884	36·210	38·810	41·727	45·021	9
10	33·789	35·937	38·315	40·959	43·918	47·241	50·991	10
11	37·814	40·231	42·906	45·887	49·224	52·969	57·188	11
12	41·982	44·680	47·673	51·008	54·737	58·916	63·628	12
13	46·300	49·298	52·624	56·327	60·461	65·096	70·318	13
14	50·783	54·095	57·767	61·849	66·408	71·516	77·277	14
15	55·440	59·079	63·108	67·587	72·587	78·195	84·527	15
16	60·278	64·253	68·657	73·550	79·014	85·152	92·093	16
17	65·301	69·630	74·422	79·751	85·710	92·412	33	
18	70·520	75·217	80·419	86·212	92·698	32		
19	75·943	81·028	86·667	92·954	31			
20	81·583	87·081	93·187	30				
21	87·459	93·398	29					
22	93·592	28						
	27							

Duration.	40	39	38	37	36	35	34	Duration.
							91·733	15
						91·325	83·823	14
					90·859	83·025	76·243	13
				90·321	82·112	75·071	68·967	12
			89·695	81·060	73·730	67·435	61·972	11
		88·954	79·833	72·185	65·684	60·095	55·240	10
9	88·066	78·385	70·385	63·666	57·949	53·030	48·760	9
8	76·647	68·257	61·314	55·476	50·505	46·231	42·520	8
7	65·704	58·535	52·593	47·593	43·340	39·682	36·513	7
6	55·200	49·188	44·201	40·007	36·439	33·378	30·731	6
5	45·102	40·193	36·123	32·701	29·796	27·311	25·159	5
4	35·383	31·535	28·344	25·667	23·402	21·465	19·784	4
3	26·029	23·197	20·855	18·897	17·242	15·823	14·591	3
2	17·020	15·170	13·646	12·374	11·297	10·374	9·567	2
1	8·348	7·444	6·702	6·080	5·555	5·102	4·705	1
	40	39	38	37	36	35	34	

H^M 3 *per-cent—Death or* 55.

Duration.	20	21	22	23	24	25	26	27	Duration.
1	1·572	1·617	1·692	1·793	1·906	2·015	2·127	2·231	1
2	3·164	3·282	3·455	3·664	3·882	4·100	4·311	4·521	2
3	4·802	5·016	5·295	5·606	5·927	6·240	6·552	6·871	3
4	6·509	6·826	7·203	7·614	8·026	8·435	8·852	9·284	4
5	8·291	8·703	9·177	9·675	10·180	10·689	11·214	11·771	5
6	10·139	10·646	11·204	11·790	12·391	13·003	13·648	14·337	6
7	12·050	12·639	13·283	13·962	14·661	15·388	16·159	16·986	7
8	14·013	14·685	15·418	16·191	17·000	17·849	18·752	19·720	8
9	16·026	16·785	17·609	18·488	19·414	20·389	21·427	22·539	9
10	18·093	18·941	19·868	20·859	21·906	23·011	24·187	25·443	10
11	20·215	21·163	22·198	23·306	24·478	25·715	27·029	28·436	11
12	22·403	23·456	24·604	25·832	27·130	28·500	29·959	31·526	12
13	24·660	25·823	27·087	28·437	29·862	31·370	32·983	34·721	13
14	26·989	28·266	29·648	31·120	32·678	34·333	36·110	38·031	14
15	29·394	30·785	32·285	33·885	35·585	37·397	39·349	41·465	15
16	31·873	33·380	35·004	36·740	38·590	40·571	42·711	45·026	16
17	34·428	36·055	37·810	39·691	41·704	43·865	46·195	48·713	17
18	37·060	38·816	40·712	42·749	44·935	47·279	49·804	52·533	18
19	39·778	41·670	43·718	45·922	48·284	50·816	53·543	56·486	19
20	42·587	44·628	46·837	49·211	51·753	54·479	57·412	60·577	20
21	45·498	47·697	50·071	52·618	55·346	58·270	61·415	64·819	21
22	48·519	50·878	53·420	56·147	59·065	62·193	65·568	69·227	22
23	51·650	54·173	56·889	59·799	62·913	66·262	69·881	73·812	23
24	54·894	57·586	60·479	63·578	66·905	70·488	74·369	78·591	24
25	58·253	61·118	64·195	67·498	71·051	74·885	79·047	83·582	25
26	61·730	64·774	68·048	71·570	75·364	79·469	83·932	88·801	26
27	65·327	68·565	72·051	75·806	79·860	84·256	89·039	94·265	27
28	69·059	72·503	76·215	80·221	84·556	89·260	94·387	27	
29	72·935	76·600	80·556	84·832	89·464	94·500	26		
30	76·968	80·870	85·089	89·653	94·605	25			
					24				
31	81·171	85·330	89·829	94·702				44	
32	85·561	89·993	94·791	23			45		
33	90·150	94·876	22			46			
34	94·956	21			47				
	20			48					
								88·818	10
							87·918	78·163	9
						86·820	76·406	67·989	8
					85·453	74·263	65·413	58·257	7
			49	83·698	71·592	62·271	54·898	48·938	6
		50							
			81·364	68·166	58·356	50·801	44·829	39·998	5
4	...	78·101	63·607	53·334	45·696	39·817	35·169	31·404	4
3	...	57·235	46·650	39·146	33·572	29·280	25·884	23·131	3
2	...	37·309	30·431	25·561	21·941	19·151	16·946	15·155	2
1	...	18·251	14·900	12·528	10·762	9·401	8·328	7·447	1
		50	49	48	47	46	45	44	

H^M 3 *per-cent—Death or* 55—(continued).

Duration.	28	29	30	31	32	33	34	35	Duration.
1	2·342	2·461	2·591	2·742	2·908	3·092	3·293	3·512	1
2	4·745	4·988	5·261	5·570	5·911	6·284	6·689	7·129	2
3	7·213	7·593	8·017	8·490	9·009	9·575	10·188	10·858	3
4	9·757	10·281	10·861	11·504	12·205	12·965	13·793	14·707	4
5	12·382	13·055	13·797	14·612	15·496	16·459	17·516	18·686	5
6	15·091	15·918	16·824	17·813	18·889	20·066	21·364	22·809	6
7	17·887	18·871	19·942	21·113	22·391	23·796	25·351	27·087	7
8	20·771	21·913	23·156	24·519	26·012	27·660	29·489	31·522	8
9	23·741	25·048	26·474	28·041	29·763	31·669	33·777	36·116	9
10	26·803	28·284	29·905	31·689	33·656	35·825	38·219	40·873	10
11	29·963	31·630	33·459	35·475	37·691	40·130	42·820	45·797	11
12	33·231	35·097	37·147	39·400	41·871	44·589	47·582	50·893	12
13	36·616	38·694	40·970	43·465	46·200	49·203	52·510	56·178	13
14	40·130	42·423	44·929	47·675	50·681	53·979	57·621	61·668	14
15	43·771	46·285	49·031	52·033	55·317	58·932	62·930	67·379	15
16	47·543	50·285	53·275	56·542	60·126	64·076	68·453	73·333	16
17	51·449	54·426	57·668	61·219	65·121	69·429	74·211	79·550	17
18	55·493	58·710	62·224	66·077	70·318	75·008	80·223	86·050	18
19	59·677	63·154	66·956	71·132	75·735	80·835	86·509	92·856	19
20	64·016	67·770	71·880	76·400	81·392	86·926	93·092	35	
21	68·524	72·572	77·012	81·902	87·307	93·305	34		
22	73·214	77·578	82·371	87·655	93·500	33		36	
23	78·103	82·805	87·974	93·678	32		37		
24	83·208	88·270	93·842	31		38			
								92·596	18
25	88·545	93·993	30						
							92·308	85·542	17
26	94·134	29			39	91·986	84·979	78·806	16
	28			40					
			41		91·625	84·351	77·980	72·362	15
		42		91·215	83·644	77·059	71·285	66·192	14
	43		90·745	82·844	76·024	70·085	64·875	60·273	13
		90·202	81·928	74·850	68·735	63·406	58·725	54·583	12
11	89·568	80·867	73·507	67·204	61·754	56·999	52·814	49·106	11
10	79·628	71·952	65·453	59·883	55·058	50·840	47·123	43·825	10
9	70·136	63·426	57·740	52·859	48·622	44·912	41·636	38·721	9
8	61·057	55·260	50·341	46·107	42·426	39·195	36·334	33·790	8
7	52·363	47·427	43·229	39·608	36·452	33·671	31·211	29·030	7
6	44·022	39·898	36·382	33·341	30·678	28·334	26·265	24·434	6
5	36·005	32·649	29·781	27·286	25·100	23·181	21·490	20·000	5
4	28·288	25·661	23·402	21·434	19·715	18·207	16·883	15·727	4
3	20·846	18·907	17·238	15·785	14·515	13·407	12·444	11·602	3
2	13·656	12·382	11·288	10·332	9·499	8·782	8·159	7·613	2
1	6·708	6·082	5·543	5·070	4·666	4·318	4·014	3·749	1
	43	42	41	40	39	38	37	36	

H$^{\text{M}}$ 8 *per-cent—Death or* 60.

Duration.	20	21	22	23	24	25	26	27	28	Duration.
1	1·295	1·321	1·376	1·455	1·544	1·626	1·709	1·781	1·855	1
2	2·599	2·679	2·811	2·977	3·145	3·308	3·460	3·603	3·753	2
3	3·939	4·095	4·312	4·554	4·801	5·030	5·250	5·467	5·696	3
4	5·337	5·576	5·868	6·186	6·496	6·791	7·083	7·375	7·694	4
5	6·799	7·111	7·477	7·857	8·230	8·594	8·959	9·338	9·752	5
6	8·314	8·699	9·125	9·566	10·006	10·439	10·888	11·359	11·873	6
7	9·881	10·325	10·810	11·315	11·822	12·337	12·874	13·442	14·057	7
8	11·486	11·988	12·535	13·105	13·691	14·291	14·922	15·587	16·302	8
9	13·128	13·691	14·301	14·946	15·615	16·305	17·030	17·793	18·608	9
10	14·808	15·433	16·117	16·842	17·598	18·379	19·198	20·058	20·977	10
11	16·528	17·225	17·987	18·797	19·640	20·512	21·424	22·385	23·416	11
12	18·297	19·070	19·914	20·809	21·739	22·702	23·711	24·780	25·930	12
13	20·118	20·972	21·899	22·878	23·895	24·952	26·066	27·249	28·530	13
14	21·995	22·930	23·939	25·003	26·111	27·268	28·493	29·802	31·222	14
15	23·928	24·944	26·035	27·186	28·391	29·656	31·002	32·447	34·009	15
16	25·916	27·012	28·188	29·433	30·742	32·124	33·602	35·184	36·886	16
17	27·957	29·136	30·404	31·749	33·172	34·682	36·292	38·010	39·856	17
18	30·054	31·323	32·689	34·145	35·690	37·328	39·070	40·927	42·914	18
19	32·213	33·578	35·051	36·626	38·296	40·060	41·937	43·931	46·062	19
20	34·438	35·909	37·498	39·194	40·986	42·881	44·889	47·022	49·307	20
21	36·739	38·324	40·031	41·845	43·763	45·785	47·928	50·210	52·661	21
22	39·122	40·823	42·645	44·581	46·622	48·775	51·061	53·504	56·131	22
23	41·589	43·403	45·344	47·399	49·566	51·857	54·299	56·913	59·731	23
24	44·136	46·066	48·123	50·299	52·600	55·042	57·649	60·449	63·473	24
25	46·764	48·808	50·983	53·290	55·736	58·338	61·125	64·124	67·363	25
26	49·471	51·631	53·933	56·380	58·981	61·757	64·737	67·944	71·411	26
27	52·257	54·541	56·980	59·578	62·347	65·311	68·492	71·920	75·630	27
28	55·130	57·549	60·134	62·895	65·846	69·005	72·400	76·064	80·037	28
29	58·099	60·661	63·406	66·343	69·483	72·848	76·473	80·393	84·652	29
30	61·170	63·889	66·806	69·927	73·268	76·856	80·728	84·926	89·498	30
31	64·357	67·245	70·341	73·657	77·213	81·041	85·183	89·685	94·604	31
32	67·669	70·733	74·019	77·545	81·334	85·424	89·862	94·700	28	
33	71·112	74·362	77·854	81·606	85·649	90·027	94·790	27		
34	74·694	78·146	81·859	85·858	90·181	94·875	26			
35	78·429	82·099	86·053	90·323	94·954	25				
36	82·330	86·237	90·457	95·027	24					
37	86·415	90·583	95·096	23						
38	90·705	95·161	22							
39	95·223	21								
	20									

	55	54	53	52	51	50	49	48	47	
									89·995	12
								89·355	80·530	11
							88·600	79·284	71·546	10
						87·699	77·816	69·724	62·990	9
					86·599	76·061	67·579	60·621	54·819	8
				85·226	73·921	65·015	57·830	51·926	46·996	7
			83·465	71·250	61·887	54·494	48·520	43·603	39·492	6
		81·124	67·823	57·982	50·426	44·448	39·607	35·619	32·281	5
4	77·856	63·268	52·975	45·348	39·481	34·829	31·057	27·945	25·343	4
3	56·908	46·317	38·834	33·282	29·003	25·603	22·840	20·564	18·669	3
2	37·023	30·175	25·330	21·730	18·952	16·736	14·936	13·462	12·235	2
1	18·085	14·759	12·401	10·649	9·292	8·206	7·331	6·616	6·017	1
	55	54	53	52	51	50	49	48	47	

H^M 3 *per-cent—Death or* 60—(continued).

Duration.	29	30	31	32	33	34	35	36	37	Duration.
1	1·934	2·018	2·119	2·229	2·350	2·478	2·613	2·755	2·911	1
2	3·914	4·095	4·301	4·527	4·770	5·026	5·296	5·586	5·907	2
3	5·950	6·233	6·550	6·893	7·258	7·643	8·052	8·499	8·996	3
4	8·046	8·436	8·866	9·325	9·813	10·331	10·890	11·503	12·190	4
5	10·207	10·705	11·247	11·824	12·438	13·098	13·815	14·609	15·498	5
6	12·433	13·038	13·692	14·390	15·140	15·951	16·840	17·826	18·922	6
7	14·720	15·434	16·204	17·032	17·926	18·901	19·973	21·156	22·457	7
8	17·070	17·896	18·790	19·756	20·807	21·956	23·216	24·593	26·105	8
9	19·484	20·429	21·456	22·572	23·790	25·119	26·563	28·141	29·863	9
10	21·969	23·042	24·213	25·489	26·878	28·383	30·019	31·795	33·730	10
11	24·530	25·742	27·068	28·508	30·066	31·753	33·577	35·556	37·717	11
12	27·179	28·540	30·023	31·625	33·357	35·223	37·240	39·433	41·838	12
13	29·923	31·436	33·074	34·842	36·745	38·795	41·016	43·441	46·102	13
14	32·762	34·425	36·223	38·155	40·233	42·478	44·918	47·587	50·525	14
15	35·693	37·510	39·466	41·566	43·829	46·283	48·956	51·888	55·122	15
16	38·719	40·688	42·804	45·081	47·546	50·221	53·145	56·359	59·902	16
17	41·835	43·958	46·245	48·715	51·391	54·306	57·499	61·006	64·874	17
18	45·042	47·330	49·802	52·475	55·380	58·552	62·025	65·842	70·058	18
19	48·349	50·815	53·482	56·375	59·526	62·966	66·734	70·883	75·473	19
20	51·766	54·421	57·299	60·429	63·837	67·559	71·644	76·149	81·144	20
21	55·302	58·161	61·267	64·643	68·321	72·347	76·772	81·663	87·097	21
22	58·970	62·049	65·392	69·027	72·997	77·348	82·142	87·453	93·370	22
23	62·783	66·090	69·684	73·599	77·880	82·585	87·781	93·552	37	
24	66·746	70·296	74·158	78·373	82·994	88·084	93·721	36		
25	70·870	74·680	78·832	83·373	88·364	93·876	35			
26	75·169	79·259	83·725	88·623	94·020	34				
27	79·660	84·054	88·864	94·154	33					
28	81·362	89·089	94·278	32						
29	89·300	94·393	31							
30	94·502	30								
	29									

Duration.	46	45	44	43	42	41	40	39	38	Duration.
									93·171	21
								92·954	86·711	20
							92·714	86·288	80·578	19
						92·449	85·822	79·960	74·738	18
					92·154	85·306	79·280	73·934	69·161	17
				91·822	84·731	78·526	73·049	68·179	63·821	16
			91·450	84·086	77·685	72·069	67·099	62·669	58·699	15
		91·027	83·361	76·743	70·975	65·902	61·402	57·384	53·777	14
13	90·547	82·539	75·683	69·749	64·567	59·998	55·938	52·305	49·042	13
12	81·604	74·482	68·370	63·071	58·432	54·335	50·686	47·419	44·486	12
11	73·115	66·809	61·387	56·677	52·547	48·893	45·634	42·719	40·095	11
10	65·031	59·481	54·702	50·543	46·892	43·657	40·774	38·187	35·850	10
9	57·310	52·465	48·289	44·649	41·451	38·620	36·089	33·808	31·743	9
8	49·919	45·736	42·126	38·979	36·217	33·764	31·561	29·570	27·760	8
7	42·829	39·268	36·197	33·523	31·171	29·072	27·179	25·460	23·890	7
6	36·015	33·047	30·493	28·264	26·294	24·531	22·929	21·466	20·132	6
5	29·460	27·061	24·994	23·182	21·576	20·126	18·801	17·589	16·491	5
4	23·154	21·291	19·680	18·264	16·999	15·847	14·791	13·832	12·965	4
3	17·074	15·715	14·538	13·494	12·552	11·692	10·907	10·193	9·557	3
2	11·199	10·318	9·551	8·860	8·235	7·667	7·145	6·677	6·268	2
1	5·514	5·085	4·705	4·359	4·052	3·768	3·510	3·283	3·086	1
	46	45	44	43	42	41	40	39	38	

H^{M} 3 *per-cent—Death or* 65.

Duration.	20	21	22	23	24	25	26	27	28	29	30	Duration.
1	1·106	1·120	1·162	1·228	1·302	1·369	1·433	1·486	1·538	1·595	1·653	1
2	2·214	2·269	2·376	2·514	2·653	2·782	2·897	3·000	3·108	3·221	3·349	2
3	3·351	3·469	3·647	3·848	4·048	4·227	4·390	4·547	4·709	4·890	5·092	3
4	4·537	4·726	4·966	5·226	5·473	5·699	5·915	6·125	6·352	6·606	6·886	4
5	5·780	6·030	6·328	6·634	6·927	7·203	7·470	7·744	8·042	8·371	8·728	5
6	7·070	7·377	7·719	8·069	8·411	8·736	9·066	9·408	9·780	10·184	10·618	6
7	8·401	8·753	9·138	9·535	9·924	10·310	10·706	11·120	11·565	12·044	12·554	7
8	9·762	10·156	10·587	11·030	11·478	11·928	12·394	12·878	13·396	13·949	14·532	8
9	11·150	11·588	12·065	12·565	13·075	13·593	14·127	14·683	15·272	15·895	16·564	9
10	12·566	13·049	13·581	14·142	14·718	15·302	15·905	16·530	17·188	17·895	18·653	10
11	14·011	14·549	15·140	15·765	16·405	17·056	17·727	18·419	19·157	19·951	20·809	11
12	15·494	16·090	16·744	17·431	18·136	18·853	19·588	20·358	21·181	22·072	23·038	12
13	17·019	17·676	18·391	19·141	19·909	20·688	21·500	22·352	23·270	24·266	25·340	13
14	18·587	19·305	20·081	20·802	21·721	22·574	23·465	24·410	25·430	26·530	27·708	14
15	20·198	20·976	21·812	22·682	23·582	24·513	25·493	26·538	27·660	28·861	30·144	15
16	21·850	22·688	23·581	24·520	25·495	26·513	27·591	28·735	29·955	31·258	32·639	16
17	23·543	24·437	25·398	26·410	27·470	28·582	29·758	30·995	32·315	33·714	35·192	17
18	25·273	26·233	27·266	28·360	29·512	30·718	31·984	33·320	34·733	36·225	37·808	18
19	27·049	28·080	29·193	30·377	31·619	32·915	34·276	35·702	37·206	38·799	40·495	19
20	28·876	29·986	31·186	32·459	33·788	35·176	36·624	38·139	39·740	41·444	43·258	20
21	30·760	31·957	33·244	34·601	36·020	37·491	39·025	40·636	42·344	44·163	46·108	21
22	32·710	33·992	35·362	36·805	38·305	39·860	41·486	43·201	45·022	46·968	49·054	22
23	34·722	36·086	37·540	39·063	40·643	42·287	44·015	45·838	47·783	49·867	52·098	23
24	36·793	38·239	39·771	41·372	43·039	44·781	46·615	48·559	50·638	52·862	55·240	24
25	38·922	40·445	42·053	43·738	45·500	47·345	49·296	51·371	53·587	55·954	58·490	25
26	41·104	42·702	44·392	46·169	48·031	49·990	52·068	54·276	56·631	59·152	61·854	26
27	43·336	45·015	46·795	48·669	50·641	52·724	54·931	57·275	59·780	62·463	65·344	27
28	45·623	47·391	49·265	51·247	53·339	55·548	57·888	60·378	63·040	65·897	68·973	28
29	47·973	49·834	51·814	53·912	56·127	58·464	60·946	63·589	66·421	69·467	72·756	29
30	50·389	52·353	54·448	56·666	59·005	61·480	64·111	66·920	69·937	73·190	76·709	30
31	52·881	54·958	57·169	59·508	61·982	64·602	67·394	70·384	73·603	77·061	80·855	31
32	55·457	57·649	59·979	62·449	65·063	67·841	70·808	73·995	77·433	81·160	85·219	32
33	58·117	60·427	62·885	65·492	68·259	71·208	74·367	77·768	81·450	85·454	89·835	33
34	60·865	63·301	65·893	68·649	71·582	74·718	78·087	81·725	85·678	89·997	94·746	34
35	63·707	66·275	69·013	71·931	75·047	78·387	81·987	85·891	90·151	94·829	30	
36	66·648	69·360	72·258	75·354	78·668	82·234	86·093	90·297	94·909	29		
37	69·699	72·568	75·640	78·930	82·465	86·283	90·436	94·984	28			
38	72·872	75·913	79·175	82·680	86·462	90·567	95·056	27			51	
39	76·180	79·408	82·882	86·628	90·690	95·124	26			52		
40	79·636	83·073	86·783	90·804	95·187	25			53		90·250	13
					24					89·686	81·138	12
41	83·261	86·931	90·911	95·247				54	89·031	80·048	72·572	11
42	87·076	91·013	95·302	23			55					
43	91·113	95·354	22			56		88·261	78·780	70·986	64·475	10
44	95·406	21			57		87·341	77·290	69·142	62·420	56·783	9
	20			58		86·225	75·512	66·976	60·032	54·283	49·447	8
					84·838	73·353	64·390	57·227	51·379	46·523	42·427	7
			59	83·065	70·669	61·250	53·877	47·966	43·126	39·097	35·694	6
		60										
			80·713	67·239	57·348	49·810	43·891	39·133	35·228	31·975	29·218	5
4	..	77·439	62·690	52·360	44·755	38·943	34·367	30·681	27·653	25·125	22·976	4
3	..	56·356	45·744	38·295	32·794	28·579	25·253	22·575	20·368	18·522	16·945	3
2	..	36·535	29·727	24·935	21·386	18·661	16·512	14·777	13·345	12·142	11·114	2
1	..	17·798	14·511	12·193	10·469	9·149	8·104	7·262	6·560	5·974	5·467	1
		60	59	58	57	56	55	54	53	52	51	

H^M 3 *per-cent—Death or* 65—(continued).

Duration.	31	32	33	34	35	36	37	38	39	40	Duration.
1	1·725	1·804	1·890	1·978	2·071	2·166	2·262	2·377	2·504	2·650	1
2	3·497	3·660	3·831	4·008	4·192	4·379	4·586	4·822	5·087	5·390	2
3	5·321	5·566	5·823	6·087	6·359	6·652	6·975	7·344	7·759	8·220	3
4	7·195	7·521	7·862	8·212	8·585	8·990	9·440	9·952	10·518	11·131	4
5	9·116	9·524	9·947	10·394	10·874	11·401	11·989	12·645	13·356	14·125	5
6	11·085	11·571	12·088	12·638	13·236	13·896	14·621	15·416	16·276	17·193	6
7	13·096	13·673	14·289	14·952	15·679	16·470	17·330	18·266	19·266	20·331	7
8	15·162	15·835	16·560	17·347	18·200	19·120	20·115	21·186	22·326	23·546	8
9	17·287	18·065	18·909	19·819	20·795	21·845	22·969	24·172	25·461	26·850	9
10	19·478	20·372	21·334	22·362	23·464	24·637	25·888	27·233	28·681	30·247	10
11	21·745	22·753	23·829	24·978	26·198	27·493	28·879	30·377	31·993	33·750	11
12	24·085	25·203	26·396	27·658	28·994	30·419	31·952	33·611	35·409	37·372	12
13	26·493	27·724	29·025	30·399	31·860	33·426	35·112	36·945	38·940	41·113	13
14	28·970	30·306	31·715	33·208	34·804	36·517	38·371	40·392	42·588	44·976	14
15	31·508	32·947	34·471	36·094	37·832	39·706	41·741	43·958	46·354	48·972	15
16	34·103	35·653	37·302	39·062	40·955	43·002	45·221	47·629	50·250	53·107	16
17	36·763	38·433	40·214	42·123	44·183	46·407	48·814	51·432	54·281	57·898	17
18	39·495	41·292	43·217	45·287	47·517	49·923	52·531	55·368	58·464	61·858	18
19	42·305	44·241	46·321	48·555	50·960	53·559	56·378	59·452	62·813	66·509	19
20	45·203	47·289	49·528	51·930	54·521	57·323	60·369	63·697	67·347	71·369	20
21	48·198	50·438	52·839	55·421	58·207	61·227	64·518	68·124	72·086	76·465	21
22	51·293	53·689	56·263	59·033	62·030	65·287	68·845	72·749	77·054	81·829	22
23	54·488	57·052	59·808	62·782	66·006	69·519	73·366	77·599	82·284	87·504	23
24	57·793	60·533	63·485	66·678	70·151	73·943	78·106	82·705	87·817	93·541	24
25	61·213	64·144	67·308	70·741	74·482	78·580	83·096	88·107	93·702		
26	64·762	67·898	71·294	74·987	79·024	83·463	88·376	93·852			
27	68·451	71·812	75·460	79·439	83·805	88·628	93·991				
28	72·298	75·902	79·827	84·125	88·863	94·121					
29	76·318	80·191	84·425	89·083	94·243						
30	80·533	84·706	89·290	94·357							
31	84·970	89·483	94·464								
32	89·664	94·563									
33	94·657										

	50	49	48	47	46	45	44	43	42	41	
										93·365	23
									93·173	87·164	22
								92·962	86·792	81·334	21
							92·732	86·385	80·794	75·824	20
						92·478	85·939	80·202	75·124	70·589	19
					92·199	85·449	79·553	74·357	69·737	65·597	18
				91·892	84·910	78·840	73·517	68·805	64·600	60·820	17
			91·551	84·315	78·056	72·593	67·783	63·509	59·685	56·238	16
		91·170	83·656	77·192	71·578	66·659	62·314	58·442	54·970	51·831	15
14	90·740	82·918	76·233	70·458	65·424	61·000	57·081	53·582	50·436	47·583	14
13	82·086	75·159	69·216	64·062	59·555	55·584	52·062	48·908	46·065	43·479	13
12	73·950	67·826	62·551	57·962	53·938	50·390	47·234	44·402	41·841	39·511	12
11	66·259	60·860	56·194	52·124	48·552	45·394	42·581	40·048	37·758	35·668	11
10	58·953	54·216	50·111	46·526	43·371	40·578	38·084	35·840	33·804	31·947	10
9	51·986	47·858	44·277	41·141	38·377	35·925	33·738	31·763	29·976	28·348	9
8	45·318	41·760	38·665	35·950	33·552	31·427	29·528	27·817	26·273	24·859	8
7	38·924	35·895	33·256	30·935	28·887	27·071	25·453	24·000	22·682	21·466	7
6	32·773	30·242	28·030	26·086	24·369	22·853	21·510	20·299	19·191	18·163	6
5	26·845	24·780	22·977	21·390	19·995	18·773	17·688	16·700	15·792	14·939	5
4	21·116	19·499	18·084	16·844	15·764	14·817	13·971	13·196	12·475	11·788	4
3	15·579	14·385	13·347	12·446	11·662	10·971	10·352	9·777	9·233	8·712	3
2	10·215	9·433	8·764	8·182	7·673	7·226	6·821	6·435	6·068	5·722	2
1	5·023	4·644	4·321	4·036	3·790	3·572	3·370	3·172	2·990	2·815	1

results. It may here be mentioned, that when performing these calculations on an eight-figure machine care is sometimes required in taking down the results, owing to the limited carrying power of the instrument. For instance, ${}_{23}V_{21.\overline{29}|}$ appears on the machine as 69·986, and the next value, ${}_{24}V_{21.\overline{29}|}$, comes out as 64·501 in place of 74·501. In this and similar cases the wrong figure must be set right by hand before going further, and practice will soon enable one to detect when this is necessary. For a similar reason, care is requisite in calculating the sums of the columns from the verification formula. To take the example already referred to, in finding $\Sigma_{29}V_{20.\overline{30}|}$ the setting of the machine would naturally be as under—

0029000000000000
        05433572

but we find, on multiplying by 314·0238, that the result comes out as 1194·729 in place of 1193·729. The easiest way to get over this difficulty is to set the slide as follows—

0028990000000000,

and the result will then come out correctly.

It only remains to state that, without any assistance, I found it possible to construct a column of 44 values in 8 minutes, being at the rate of 5½ values a minute, not including the time required for setting the machine or verifying the sum of the column. The verification of the column just referred to was performed in 2 minutes. With the aid of an assistant to write down the results, the same column took only 6 minutes to construct.

When we consider that the process employed is one which is self-checking, and therefore obviates any necessity for the work being gone over a second time or performed in duplicate, I think I may safely challenge any one to produce equally accurate results with equal celerity by the use of any other method than that here described.

JOURNAL

OF THE

INSTITUTE OF ACTUARIES

AND

ASSURANCE MAGAZINE.

Opening Address by the President, A. H. BAILEY, Esq.

[Delivered 29 Nov. 1880.]

IT having been suggested that, in conformity with the practice of other societies, it should be one of the duties of the President of this Institute to open the Session with a somewhat more formal address than has been usual hitherto, it has occurred to me that your attention might for a short time be asked to a review of the past work of the Institute, to a consideration of what it can usefully undertake, and what objects it may hope to accomplish in the future.

In the spring of the year 1848, a preliminary meeting of gentlemen connected with life assurance institutions in London was held to consider a suggestion from one of the managers of the Scottish assurance companies, whether it would be desirable to form in this city a business organization on the model of the Managers' Association, which had been established in the city of Edinburgh about fifteen years previously, and which still exists. At this meeting a committee of ten, including some of the then senior actuaries, was formed, to which the consideration of the subject was referred. Their report, presented to a subsequent meeting convened by circular, was to the effect that it was not expedient to lay down rules for the formation of a society of any description, but that occasional meetings might be held with advantage if this should from experience be found to establish uniformity in dealing with points of practice. Of these, some examples were given.

This report gave rise to much discussion. The year 1848 was a year of revolutions in Europe. Monarchs were dethroned, governments were shaken; and the spirit then abroad seems to have penetrated into the discussions of the assurance representatives. At all events, a complete *bouleversement* ensued. It was resolved by a large majority that the report of the committee be not approved, and a resolution was carried—"That this meeting consider it desirable to establish a scientific and practical association amongst the actuaries, secretaries, and managers of the life assurance societies of Great Britain." Another committee of fifteen, chosen by ballot, was appointed to devise the best means of carrying this resolution into effect. Their labours resulted in the establishment of the Institute of Actuaries.

This short narrative of the origin of our Institute affords, I think, some evidence of the wisdom and judgment of its founders. It must be remembered that the life assurance companies were more numerous then than now; on an average, twenty new ones came every year into existence. Any one who pleased called himself an actuary; what were the real pretensions of some who so designated themselves may be seen in the returns to Parliament of the accounts registered under the Joint Stock Companies Act of 1844. These accounts are now well-nigh forgotten; but in justification of what has been said, I will take, almost at random, a specimen of a balance-sheet issued by an assurance company in 1848, and printed at the public expense for the information of Parliament and the country. This so-called "balance-sheet" is made up of two parts, one called "receipts and expenditure", the other "liabilities and assets". The first is in the main an abstract of the cash-book, and calls for no particular remark. The second is more interesting, as an example of the prevailing knowledge of the time.

Liabilities.				Assets.			
Amount assured.	£78,602	0	0	Amount advanced	£23,504	19	0
Annuities	146	9	6	Lease and Furniture	955	15	11
				Cash Balance	4,898	13	8
	£78,748	9	6		£29,359	8	7

Our founders believed that the remedy for this state of things was to be sought, not in standing aloof; but in associating together; that the Association should be, not a mere business organization, with its almost inevitable jealousies and heart-burnings, but one whose aim should be to raise the standard of the profession. Or, to borrow what has always seemed to me the happy language of our laws, that its purpose should be—" to elevate the attainments and promote the general efficiency of all who are engaged in occupations connected with the pursuits of an actuary; to extend and improve the data and methods of the science which has its origin in the application of the doctrine of probabilities to the affairs of life; and specially to investigate all monetary questions involving a consideration of the effects of interest and probability."

The Association was to be at once scientific and practical. Scientific, because on a scientific basis alone could its investigations be properly conducted; but yet its type was not to be found in the purely scientific societies, such as the "Royal", or "Astronomical". Practical, because it was to concern itself with the monetary affairs of actual life, and at the same time to carefully avoid interference with the conduct of any existing institutions. Its right position is, I think, with the professional societies, of which the "Royal Institute of British Architects", and the "Institution of Civil Engineers", are examples. Its founders were more ambitious. They wished to obtain legislative powers for regulating the profession, such as are possessed by the Incorporated Law Society; and, as several of you know, a bill for this purpose was drawn, which will be found in print in the first volume of the *Journal.* It can hardly be a matter of surprise that no encouragement was given to proceed with the bill, and that it should have been abandoned. A small profession like our own, one seemingly diminishing in numbers, can hardly hope to receive from the Legislature the consideration given to the great professions of law and medicine. But it has more than once been suggested, that it would be desirable to apply for a Royal Charter of Incorporation, such as has been granted to other similar societies. This is undoubtedly worth consideration, although, speaking for myself, the advantages which such a charter would confer are not very clear to my mind.

The idea of obtaining legislative powers having been abandoned, the Institute went on in its career. 131 members having been enrolled—94 as Fellows, and 37 as Associates, periodical meetings were held, at which subjects of interest were discussed.

Gradually, the regular practice was introduced at these meetings of reading papers, some of which were printed and circulated. Subsequently, the important step was taken of acquiring the *Assurance Magazine,* which had been started by private enterprise, and adopting it as the Journal of the Institute. Of this Journal it is hardly too much to say, that it has created a branch of literature which previously had no existence; and further, that, with scarcely an exception, every addition to our knowledge during this generation on the subject of life contingencies is to be found in this and other publications of the Institute, or in the works of its Members. In mentioning the other publications of the Institute, I refer especially to the materials it collected for determining the mortality among assured lives, and the tables that have been deduced therefrom. Having been one of your honorary secretaries at the time, I may perhaps be allowed to bear testimony to the difficulties that had to be overcome in obtaining the materials, and to express my conviction that without the influence of the Institute, they would not have been obtained at all. And it is therefore the more gratifying to observe how generally the results of our investigations are adopted, and our tables recognized as the standard.

The papers offered to the Council for reading are hardly as many as might be expected or desired. One reason sometimes assigned for this is, that the subject is limited, and is becoming exhausted. Surely this is not the case. Our investigations are not limited to the requirements of our ordinary work. Our aim should be to throw light upon subjects of public interest which are continually arising, on which our special studies can be brought to bear. For instance, the question of thrift is much discussed just now. A nobleman of high character, who has been a cabinet minister, and a clergyman whose zeal and sincerity are beyond doubt, contend that this virtue can be enforced by a compulsory process, that in several districts the poor rate may be materially diminished, if not dispensed with altogether, by taxing the wages of agricultural labourers in the first few years of their service to provide a fund for their support in old age. With a passing remark, that lectures on thrift might perhaps be more appropriately delivered to those who are higher in the social scale, surely this is very much a question of figures. The cost of a deferred life annuity, and the periodical payments required for such a provision, are actuarial problems of no great complexity, but are not well understood by the public. Who better than the members of this Institute can tell what the weight of such a tax would be? Others might then, with the aid of this information, be in a position to judge whether the tax could be borne.

Again, the Employers' Liability Act of last session has given greater prominence to the question of accident assurance. Surely this is a subject to which our attention might usefully be directed. And the finance of friendly societies generally, affecting as it does so many of our countrymen, has hardly received the attention it deserves. Having regard to the different amount of claims for sickness allowances arising from differences both of occupation and administration, can we be quite satisfied with the present system of contributions, the rates for which are often computed in farthings? Some of you might consider whether it would not be a wiser, as it certainly would be a simpler arrangement, to levy year by year a contribution sufficient to provide for the sickness claims of the year, and not attempt to accumulate a fund.

Again, the services of actuaries are wanted in compensation cases, both for companies and individuals. The principle upon which such compensation should be assessed is often a difficult question, to the solution of which the pages of our *Journal* at present afford little assistance.

The income tax, too, is a fertile subject for discussion. A paper on the principles on which this tax should be assessed on assurance companies might be both interesting and useful. The existing practice is sufficiently various.

And even in the ordinary work of an assurance company, how much room for improvement exists. Can anything be more unsatisfactory than the prevailing method (principle there is none) of charging premiums for what are called diseased lives? Is there any real knowledge of what is the *extent* of the extra risk in such common cases as gout, or a taint of consumption in the family? The experience of the companies has hitherto thrown very little light on the subject; more information may perhaps be forthcoming, but about this I am not very sanguine. Again, it seems to me that the present system of charging extra premiums for foreign residence cannot be maintained much longer. An attempt which has been made to cut the Gordian knot by dispensing with these premiums altogether, has deservedly failed. But, if proper inquiries were made into occupation and prospects in life when assurances are proposed, I believe it would be practicable to deal with exceptional cases by charging an extra annual premium, small in amount, but payable during the whole duration of the policy;

and that this would be a great practical improvement. Take, for instance, the case of military officers—a largely assuring class. At present these gentlemen are charged the ordinary premium, and then, if war breaks out and they are ordered abroad, an extra premium is imposed, fixed in utter ignorance of the real risk, but sometimes so high as to be unbearable. Why cannot an attempt, at least, be made to determine the general mortality of military officers, and to arrive at a uniform annual premium payable in war and peace, in tropical and temperate climates? And so with other occupations. I well know that there are difficulties in the way, but, in addressing the members of this Institute, I may surely adopt Lord Lyndhurst's definition—a difficulty means, something to be overcome. Once more, how absurd is the practice of quoting rates of premium in pence—an affectation of minute accuracy to which (as to other shams) no mercy should be shown. All our calculations are but approximations. A paper on approximations, showing the limits of error, and illustrated by numerical examples from cases occurring in practice, would be not unworthy of the powers both of the mathematician and the statist. Is it too much to hope that such a paper may be forthcoming from some one of our many competent members?

Nor do I think that discussions on points of common practice need always be avoided. I know that I am treading on dangerous ground in making a passing allusion to one matter of practice which is much in our minds, and which excites in some of us mingled feelings of apprehension and disgust. A controversy has long been carried on between mutual and proprietary assurance companies respecting the appropriation of the surplus. Of late years the vehemence of this controversy has somewhat abated; and its present state brings to mind an old fable. "A lion and a bear found the carcase of a fawn, and had a long fight for it. The contest was so hard and even that at last both of them, half blinded and half dead, lay panting on the ground without strength to touch the prize that was stretched between them. A fox coming by at the time, and seeing their helpless condition, stepped in between the combatants and carried off the booty." The lion and the bear are the two classes of assurance companies; you know very well the fox that carries off the booty.

This perhaps is hardly a subject for the Institute, but it is one for its individual members, because they can do something to remove the public ignorance which is at the root of the mischief. In popular publications, life assurance is sometimes spoken of as an investment, and contrasted with other modes of disposing of savings. But while a man who buys stock or shares knows exactly what he pays for commission, the life assurance investor has no such information. Or if he is aware that such a payment is made, he seldom realizes the fact that it is a deduction from his own savings, and *pro tanto* diminishes the benefits he is to receive. And it is remarkable that, while in mercantile transactions generally competition has tended so to reduce rates of commission, that it is a common complaint that men have to work harder for less money now than formerly, competition in life assurance has had the very opposite effect.

I gladly turn to the more agreeable subject of the relations of the Institute to its younger members. Its founders thought that, if the status of the profession were to be elevated, its younger members must be considered; that the Institute must be composed of students as well as practitioners, and must take something of an educational or collegiate character. But trades union notions constantly come to the surface, and this feature I have heard was very distasteful to some of the senior actuaries of the time; and even now I fear that the feeling is not wholly extinct. It would very ill become one, who, having been admitted as one of the early associates, and passed through all grades of office to the position in which by your kindness he has now been placed, not to express on this occasion his grateful sense of obligation to the more liberal views that prevailed. Examinations were instituted upon which to found the grant of certificates of competency. These examinations have now been held for a period of thirty years, and a considerable number of certificates granted. At a later period classes were formed for the instruction of students, of which several of the associates have availed themselves.

Owing, doubtless, to the prevailing fashion for examinations, which has grown up since the Institute was founded, this part of our work has attracted more attention than might have been expected. The conduct of the examinations has from time to time occupied the attention of the Council, and is not an easy task. My own notion is, that these examinations have been, and may be, of great service to those who hold appointments in assurance offices, but that they are of little use to outsiders, some of whom, I know, hope to obtain places thereby. The object of these examinations should be to encourage men to study, and to test their knowledge; but they should not be competitive examinations. The competition should begin when the certificate has been obtained; and, judging by the number of candidates who have lately presented themselves, and the possible number of appointments to be obtained, this competition is likely to be sufficiently severe.

Believing, as I do, that the British youth is now examined to distraction, that this mania has had an evil influence over places of education throughout the country, by tending to substitute for the acquisition of knowledge the power to pass an examination, it is not without a shudder that I read that last year not less than 17,942 wretched beings were examined by the Civil Service Commissioners alone. I hope that we shall keep clear of any such abuse as this. The idea in my mind, which may not be practically attainable, is, that the preliminary examination should be one in general knowledge, somewhat of the character of the London University Matriculation Examination, but with more regard to mathematics; and that it should be conducted by examiners outside. That the second year's examination should include the higher branches of mathematics; and that more subjects might be embraced in the syllabus for the final examination.

In urging for higher mathematical attainments, and especially for a knowledge of the differential and integral calculus, from the actuaries of the future, I do so partly because without such knowledge they cannot thoroughly master the principles of their profession, partly because of the value of mathematical studies as a training for the mind. But I do not forget that some very remarkable mathematicians, academic and non-academic, have been tried as actuaries, and failed. One reason for this I take to be that they did not sufficiently realize the responsibilities of their position. The work of an actuary is that of a financier rather than of a mathematician; he has to deal with the affairs of actual life, and must be able to hold his own in his intercourse with mankind. No doubt that he ought to be conversant with the doctrine of compound interest, but his more immediate work is actually to obtain compound interest for the money that passes through his hands, and to take especial care not to lose the principal. To use language which will be well understood here,—the actuary has to be a manager.

I am quite as desirous of insisting upon the converse of this proposition, namely, that the manager should be an actuary. It is by the combination of theoretical with practical knowledge that in this, as in other scientific professions, the really efficient man is made. Many are the mistakes that have arisen in practice from the want of sound knowledge, and from that contempt for general principles which, I think, is one of the defects of our national character; and such mistakes not unfrequently arise from attempting too much.

We know the practical manager. One thing upon which he prides himself is his skill in the selection of lives. He decides that such a sum may be taken upon one risk, such a smaller amount upon another. With his lips he may acknowledge that there is a general law of mortality; but the faith of his heart is his power to determine the length of life of the particular individual whose case is before him. To this end, he makes minute inquiries into personal health and family history, which formerly would have been resented as impertinent, and sometimes regarded as scarcely decent. And *cui bono?* Of the number of assurance societies that have passed out of existence, there is not one whose end has been brought about by an excessive mortality. And, on the other hand, are our manager's specially-selected lives better than those that were so carelessly (as he would say) admitted two generations ago? The evidence is the other way. No, the law is not to be baffled—

> "There's a divinity that shapes our ends,
> Rough-hew them how we will."

Before concluding, I have a few words to say upon a subject which has always presented more difficulties than some of you might suppose—I mean the obtaining suitable house accommodation for our purposes. The Institute had its first home in rooms above the office of the Family Endowment Society, in Chatham Place, Blackfriars. Events move in rapid succession. The Family Endowment Society has undergone the painful operation of having been wound up; the house in which its business was carried on has been pulled down; and, as if this were not enough, Chatham Place no longer exists, the name has dropped out of the London Directory. From time to time some of our friends have come forward with offers of similar accommodation, but it has been decided, I think wisely, that it is desirable that the Institute should not be connected in any way with any assurance society. In the year 1853 the Institute made an arrangement with the Statistical Society for the occupation, jointly with that Society, of the rooms in St. James's Square, which they held as tenants of the London Library. This arrangement continued until 1874, when, under a similar contract with the Statistical Society, we moved to our present quarters. It cannot be denied that the arrangement has not always worked quite smoothly; some friction has arisen from little misunderstandings, owing, no doubt, to mistakes on both sides; but I think that these have been removed. However, for different reasons, each society has been searching for other rooms, and I believe it is no secret that each has been unsuccessful; all attempts to obtain the required accommodation, in a suitable position, on reasonable terms, having failed. It seems to me that our aim should be to unite with two or three other societies, not to build a palace, but jointly to rent a house suited for our purposes; and I believe that with a little exertion, and the exercise of mutual forbearance, the affairs of all the societies might be efficiently and economically carried on under such an arrangement. This seems not to be feasible at present; and therefore an agreement has been made with the Council of King's College for a further temporary occupation of these rooms during the term of office of the present Principal, and for six months longer. Power has at the same time been obtained (which you will observe has been exercised,) to make the structural alterations, which you will probably admit have much improved the room in which we are assembled.

I think that we may fairly congratulate ourselves at this result. There are, no doubt, reasons why it would be desirable to be housed nearer to the places of our daily resort. But at all events our present situation is in the centre of London, close to a station of the District Railway, and near to other railway stations much used by those who live out of town. The rooms are quiet, light, and airy, and afford accommodation for our meetings, classes, examinations, and business purposes, which is all that can be reasonably desired, and on terms much more favourable than we are likely to meet with elsewhere.

And there are associations connected with this place which are not unsuited for the abode of the Institute of Actuaries. The museum of this College was the depository of Babbage's calculating machine. On the walls of this building were hung the four miles of copper wire for Wheatstone's experiments that ended in the electric telegraph. For nearly half a century almost every branch of human knowledge has been cultivated here; and of the many who, during that period, have taught and learned within these walls, not a few may be numbered with those—

> "Inventas aut qui vitam excoluere per artes,
> Quique sui memores alios fecere merendo."

And in connection with this subject, will you pardon a piece of egotism? For me, this building is associated with memories of the past which can never be effaced from the mind. Very slight is the effort with which I recall the day when, a child of eight years old, I arrived here in not a little trepidation, to become one of the early pupils in this then newly-founded Institution, to which I am indebted for all the education I have received. During the nine years of school and college life that here were passed, day-dreams of the future were indulged in from time to time. But it never for a moment entered into my boyish imagination to conceive that, after this long interval, I should find myself here again, in the position in which, through your kindness, I have been placed, telling you how highly I value the trust which you have reposed in me, and how deeply I feel my inability to discharge its duties aright.

On the motion of Mr. HUMPHREYS, seconded by Mr. H. A. SMITH, and supported by Mr. CUTCLIFFE, Mr. MANLY, and Mr. MESSENT, a cordial vote of thanks was given to the PRESIDENT for his address. In acknowledging this, he said that he had felt one difficulty in drawing up his address, and that was how to deal with the names of several gentlemen who have at different times been influentially connected with the Institute. Upon reflection he came to the conclusion that the best way to get over the difficulty was to omit altogether the mention of any names, but it was at some exercise of self-denial that he arrived at that conclusion. As regards the future of the Institute, he might mention that he had found that some gentlemen who have passed their examinations think they have done enough, but he hoped that in the future every actuary will pass his examinations as a matter of course, just as a man who wants to be a solicitor or a medical man has to pass examinations; and that he will then do something more for the profession, particularly in the way of compiling statistics as to the rate of mortality among special classes of persons.

On the Construction and Use ov a Series ov Select Mortality Tabls to be employd in combination with the Institute $H^{M(5)}$ *Tabl.* *By* T. B. SPRAGUE, M.A., *Manager ov the Scottish Equitable Life Assurance Society.*

[Read before the Institute, 31 January 1881.]

PART II.—*Construction* (continued).

IN Part I ov this paper I gave (vol. xxi, page 246) a tabl ov the finally ajusted probabilitys ov deth in the first five insurance years, for quinquennial ages at entry from 20 to 75. I gave also specimen mortality tabls for each ov those ages at entry, so ajusted as to join on, after the laps ov five years, to the Institute $H^{M(5)}$ Tabl; also values ov anuitys at 3 and 4 per-cent during the first five insurance years for lives entering at the same ages; and, lastly, commutation tabls calculated at 3 and 4 per-cent interest. I hav since found that ther is a misprint in the tabl ov the probabilitys ov deth, as $q_{[35]+3}$ shoud hav been printed ·010177 insted ov ·010077. This, however, is a mere misprint and dos not afect the other tabls. (It wil be notist that, insted ov riting q_{35+3}, as in the first part ov this paper, I now always enclose the age at entry in square brakets, thus $[x]$.)

In order to make the new tabls practically useful, it is necesary to hav results, not merely for quinquennial ages at entry, but for every age. The first step to be taken for this purpos is the calculation ov the probabilitys ov dying in the first five insurance years, for every age at entry. These calculations wer made by means ov the formula explaind in my paper in the number ov this *Jurnl* for last July, in which I gave (see page 280) the actual work ov interpolating between $q_{[30]+3}$ and $q_{[40]+3}$. But for the two quinquennial intervals at the beginning and at the end ov the series, namely, from age 20 to 30 and 65 to 75, I uzed constant fifth differences insted ov fourth differences as explaind in my paper. Whenever the calculated probability exceeded that acording to the $H^{M(5)}$ Tabl, the latter was substituted. The values thus obtaind wer carefully compared with each other, in order to ascertain whether they proceeded with suficient regularity to be safely uzed in the construction ov anuity values, and I found that the best means ov making this comparison was the grafic method ov representing the probabilitys as ordinats on cros-ruld paper, the age being the absissa. As the result ov this examination, it was found that the probabilitys wer on the whole very satisfactory; but in the folloing instances it was preferabl to substitute for the calculated probabilitys ov deth here set out, the larger ones shoen by the $H^{M(5)}$ Tabl:—

$q_{[27]+4}$	·009161	$q_{[60]+4}$	·041585	$q_{[63]+4}$	·050003	$q_{[65]+2}$	·050269	$q_{[72]+3}$	·099184
$q_{[28]+4}$	·009159	$q_{[61]+4}$	·044359	$q_{[64]+2}$	·046147	$q_{[71]+2}$	·081856	$q_{[73]+2}$	·096599
$q_{[31]+3}$	·009329	$q_{[62]+4}$	·047071	$q_{[64]+3}$	·050291	$q_{[71]+3}$	·091095	$q_{[73]+3}$	·107609
$q_{[31]+4}$	·009647	$q_{[63]+3}$	·046699	$q_{[64]+4}$	·053595	$q_{[72]+2}$	·088816	$q_{[74]+2}$	·105406

The efect ov the alterations thus made is to substitute the $H^{M(5)}$ probabilitys for the calculated ones in a good many cases, both when they wer greater and when they wer les; and this was don in order to carry out consistently the fundamental idea ov obtaining wel graduated tabls that woud smoothly join on to the $H^{M(5)}$ Tabl after a laps ov not more than 5 years. In two other cases sliht alterations wer made, namely, $q_{[32]+4}$ and $q_{[33]+4}$, the calculated values ov which wer ·009952 and ·010306, and for these wer substituted slihtly larger probabilitys, which ar, however, les than the $H^{M(5)}$ probabilitys. The probabilitys thus found ar containd in the folloing Tabl No. 1 (p. 394), in which ar inserted also the probabilitys for ages 15 to 19, estimated by the grafic method. In general, the $H^{M(5)}$ rate ov mortality is ataind after the laps ov five years; but in som instances, especially at the beginning and end ov the tabl, the efect ov selection wears of sooner. In al cases the $H^{M(5)}$ values ar printed in Clarendon type, so that it can be seen at a glance how soon after entry the $H^{M(5)}$ rate ov mortality is ataind. The Clarendon type is uzed with a similar meaning in the other tabls.

The arangement ov this tabl, it wil be observd, is different from that ov the tabl in Part I, page 246. I hav found by experience that the most convenient plan ov aranging tabls ov this kind is to place al the values relating to the same actual age in the same horizontal line, so that the values relating to the same age at entry ar redd diagonally downwards from left to right. Thus, $q_{[x]}$ denoting the probability that a life ov the age *x*, which has just been past as a select life, wil dy within a year, $q_{[x]+1}$ denotes the probability that such a life, if under observation after a year, wil dy in the second year; $q_{[x]+2}$ the probability that such a life, if stil under observation at the end ov two years, wil dy in the third year; and so on. The probability that a select life ov 35 wil dy in the first insurance year is ·004900, and the probabilitys ov deth in the folloing insurance years, as explaind abov, ar ·007585, ·009057, ·010177, ·010933, **·011316**, the last probability being equal to the probability givn by the $H^{M(5)}$ Tabl.

Upon the basis ov these probabilitys ar calculated the numbers living in Tabl No. 2, colums 2–6, and the numbers dying in Tabl No. 3; then the 4 per-cent commutation tabls, Nos. 4, 5, and 6; and the 4 per-cent anuity values, as shoen in Tabl No. 7; and lastly, the 4 per-cent singl and annual premiums for select lives, Tabl No. 8. It is to be notist that, since, as mentiond abov, I hav slihtly alterd som ov the probabilitys ov deth for entrants at 60 and 65, the tabular results for those ages givn in the first part ov this paper must be considerd as superseded by those now givn.

No. 1 : *being an extended Tabl ov the Values ov* q_x *for different Ages at Entry and different Periods since Entry.*

Age	YEARS ELAPST SINCE DATE OV INSURANCE.						Age
	0	1	2	3	4	5 or more	
x	$q_{[x]}$	$q_{[x-1]+1}$	$q_{[x-2]+2}$	$q_{[x-3]+3}$	$q_{[x-4]+4}$	q_x	x
15	·003000	...	...	...	...	·003253	15
16	·003700	·003774	...	...	...	·003774	16
17	·004300	·004402	·004402	...	...	·004402	17
18	·004700	·005450	·005450	·005450	...	·005450	18
19	·005000	·006600	·006720	·006720	·006720	·006720	19
20	·005000	·007800	·008327	·008327	·008327	·008327	20
21	·004869	·008550	·009657	·009657	·009657	·009657	21
22	·004750	·008246	·010281	·010281	·010281	·010281	22
23	·004647	·007972	·010020	·010709	·010709	·010709	23
24	·004563	·007734	·009749	·010611	·010825	·010825	24
25	·004500	·007533	·009485	·010396	·010506	·010506	25
26	·004459	·007371	·009242	·010064	·010064	·010064	26
27	·004439	·007249	·009029	·009842	·009943	·009943	27
28	·004440	·007166	·008853	·009585	·009704	·009704	28
29	·004461	·007120	·008719	·009377	·009458	·009458	29
30	·004500	·007108	·008628	·009203	·009203	·009203	30
31	·004554	·007128	·008581	·009160	·009172	·009172	31
32	·004624	·007174	·008575	·009159	·009257	·009257	32
33	·004708	·007248	·008603	·009225	·009225	·009225	33
34	·004801	·007346	·008674	·009431	·009431	·009431	34
35	·004900	·007460	·008781	·009507	·010002	·010002	35
36	·005005	·007585	·008911	·009733	·010150	·010347	36
37	·005116	·007723	·009057	·009964	·010421	·010701	37
38	·005235	·007872	·009222	·010177	·010645	·011065	38
39	·005362	·008036	·009406	·010376	·010933	·011189	39
40	·005500	·008216	·009612	·010560	·011176	·011316	40
41	·005649	·008415	·009841	·010752	·011317	·011317	41
42	·005811	·008635	·010098	·010994	·011551	·011576	42
43	·005988	·008878	·010385	·011313	·011816	·011844	43
44	·006183	·009149	·010706	·011708	·012211	·012252	44
45	·006400	·009452	·011067	·012186	·012730	·012943	45
46	·006641	·009792	·011474	·012734	·013385	·013659	46
47	·006908	·010174	·011936	·013328	·014140	·014402	47
48	·007205	·010604	·012459	·013955	·014932	·015315	48
49	·007534	·011085	·013053	·014628	·015724	·016267	49
50	·007900	·011625	·013724	·015358	·016535	·017116	50
51	·008305	·012229	·014477	·016157	·017374	·018005	51
52	·008754	·012904	·015323	·017048	·018264	·018786	52
53	·009250	·013659	·016274	·018047	·019246	·019911	53
54	·009797	·014500	·017343	·019161	·020351	·020941	54
55	·010400	·015435	·018545	·020390	·021579	·022187	55
56	·011062	·016474	·019892	·021773	·022926	·023506	56
57	·011789	·017626	·021400	·023363	·024436	·025075	57
58	·012584	·018901	·023082	·025194	·026172	·026577	58
59	·013453	·020311	·024944	·027280	·028168	·028360	59
60	·014400	·021869	·027028	·029658	·030447	·030638	60
61	·015433	·023587	·029393	·032261	·032916	·032916	61
62	·016554	·025478	·032078	·034975	·035583	·035583	62
63	·017770	·027557	·035099	·037742	·038500	·038500	63
64	·019083	·029836	·038492	·040596	·041710	·041710	64
65	·020500	·032330	·042206	·043537	·044614	·044614	65
66	·022025	·035055	·047836	·047836	·047836	·047836	66
67	·023662	·038027	·050957	·050957	·050957	·050957	67
68	·025418	·041264	·054449	·054449	·054449	·054449	68
69	·027295	·044782	·058118	·058118	·058118	·058118	69
70	·029300	·048601	·062836	·062836	·062836	·062836	70
71	·031437	·052740	·068559	·068559	·068559	·068559	71
72	·033710	·057220	·075551	·075551	·075551	·075551	72
73	·036126	·062062	·083480	·083480	·083480	·083480	73
74	·038687	·067289	·092231	·092231	·092231	·092231	74
75	·041400	·072924	·099494	·099494	·099494	·099494	75
76	...	·078991	·108146	·108146	·108146	·108146	76

The folloing remarks wil perhaps asist the reader to understand more clearly the exact construction ov the varios tabls :—

Tabl No. 2.—This tabl givs us separat mortality tabls for persons entering at each age from 15 to 75. Taking any age at entry, say 35, we see that, out ov 820,928 select lives ov 35, ther survive at the end ov the first year, 816,906; at the end ov the second, 810,710; at the end ov the third, 803,368; at the end ov the fourth, 795,194; and at the end ov five years, 786,500. The last number is 100 times the number living in the $H^{M(5)}$ Tabl, and the other numbers abov quoted ar deduced from it, in invers order, by means ov the probabilitys resulting from the values in Tabl No. 1. Thus $q_{[35]+4} = \cdot 010933$, $\therefore\ p_{[35]+4} = \cdot 989067$, and $l_{[35]+4} = 795{,}194 = 786{,}500 \div \cdot 989067$. This tabl posesses som remarkabl propertys, which wil be explaind hereafter.

Tabl No. 3 shos the number which dy in a year out ov the numbers containd in Tabl No. 2. Thus $d_{[35]+2} = 7{,}342 = q_{[35]+2} \times l_{[35]+2} = \cdot 009057 \times 810{,}710$.
Otherwise; $d_{[35]+2} = 7{,}342 = l_{[35]+2} - l_{[35]+3} = 810{,}710 - 803{,}368$.

Tabl No. 4 is obtaind by multiplying the figurs in Tabl No. 2 by the proper power ov $v (= 1 \div 1{\cdot}04)$. Thus $D_{[35]+2} = 189{,}947 = l_{[35]+2} v^{37} = 810{,}710 \times 1{\cdot}04^{-37}$.

Tabl No. 5 contains the values ov $\mathfrak{N}$ which ar deduced in the usual way from the values ov D in Tabl No. 4. Thus $\mathfrak{N}_{[35]+2} = 3{,}156{,}623 = \mathfrak{N}_{[35]+3} + D_{[35]+2} = 2{,}966{,}676 + 189{,}947$. It has been found convenient to adopt the initial form, $\mathfrak{N}$, rather than the usual one, N.

Tabl No. 6 contains the values ov M deduced in the usual way. Thus $M_{[35]+2} = 68{,}538{\cdot}4 = d_{[35]+2} v^{38} + d_{[35]+3} v^{39} + d_{[35]+4} v^{40} + \ldots\ldots = (l_{[35]+2} - l_{[35]+3}) v^{38} + (l_{[35]+3} - l_{[35]+4}) v^{39} + (l_{[35]+4} - l_{[35]+5}) v^{40} + \ldots\ldots$ It follos that $M_{[35]+2} = v\mathfrak{N}_{[35]+2} - \mathfrak{N}_{[35]+3} = 3{,}156{,}623 \div 1{\cdot}04 - 2{,}966{,}676 = 3{,}035{,}214{\cdot}4 - 2{,}966{,}676 = 68{,}538{\cdot}4$.

In Tabls Nos. 2, 3, 4, 5, and 6, the values in the last colums ar in every case 100 times the corresponding values acording to the $H^{M(5)}$ Tabl.

Taking $[x]$ as the age at entry and n the number ov years elapst, we hav the folloing general formulas :—

$$D_{[x]+n} = l_{[x]+n} v^{x+n}.$$

$$\mathfrak{N}_{[x]+n} = D_{[x]+n} + \mathfrak{N}_{[x]+n+1} = D_{[x]+n} + D_{[x]+n+1} + D_{[x]+n+2} + \&c.$$

$$\begin{aligned} M_{[x]+n} &= d_{[x]+n} v^{x+n+1} + d_{[x]+n+1} v^{x+n+2} + d_{[x]+n+2} v^{x+n+3} + \&c. \\ &= (l_{[x]+n} - l_{[x]+n+1}) v^{x+n+1} + (l_{[x]+n+1} - l_{[x]+n+2}) v^{x+n+2} + \&c. \\ &= v\mathfrak{N}_{[x]+n} - \mathfrak{N}_{[x]+n+1} = D_{[x]+n} - (1-v)\mathfrak{N}_{[x]+n}. \end{aligned}$$

Hence $(1+i)M_{[x]+n}=(1+i)D_{[x]+n}-iN_{[x]+n}$.

TABL NO. 7 givs the values ov anuitys at 4 per-cent interest, deduced from Tabls Nos. 4 and 5. Thus $a_{[35]}=N_{[35]}\div D_{[35]}-1$ = the value ov an anuity on a select life ov the age 35; $a_{[35]+1}=N_{[35]+1}\div D_{[35]+1}-1$ = the value ov an anuity on a life now 36 which was admitted a year ago as a select life. Consistently with a remark made abov, the values printed in Clarendon type ar the values ov anuitys obtaind from the $H^{M(5)}$ Tabl, that is to say, the values ov anuitys when the efect ov selection has entirely worn of.

TABL NO. 8 givs the annual and singl 4 per-cent premiums for select lives. These annual premiums, being loaded with a percentage and a constant adition, may be advantageosly employd in calculating offis premiums. For instance, if the offis premium is

$$1{\cdot}075\left(\varpi_{[x]}+\frac{{\cdot}01}{\mathbf{a}_{[x]}}+{\cdot}00125\right)=1{\cdot}086\varpi_{[x]}+{\cdot}00175,$$

we get the folloing, which seem very suitabl for adoption :—

Offis Premiums to asure £100 without Profits.

Age 20	.	£1 13 9	Age 50	.	£3 19 3
„ 30	.	2 0 9	„ 60	.	6 3 10
„ 40	.	2 14 10	„ 70	.	10 6 7

In order to facilitate the calculation ov anuitys and asurances upon two lives, I hav aded Tabl No. 9, the use ov which wil be best shoen by a numerical exampl. We lern from it that a 4 per-cent anuity on a select life ov 30 is equal in value to a similar anuity on an H^M life ov the age 30·83; or we may say that a select life ov 30 is equivalent to an H^M life ov 30·83; so also a select life ov 40 is equivalent to an H^M life ov 40·11; and we therefore conclude that a fair aproximation to the value ov an anuity for the joint existence ov two select lives ov 30 and 40, wil be got by finding the H^M anuity upon two lives ov 30·83 and 40·11. In symbols we may rite $a_{[30][40]}=a_{(30\cdot83)(40\cdot11)}(H^M)$.

The value ov the latter anuity will be found with suficient accuracy from the formula

$$a_{(30.83)(40.11)}=a_{30.40}-{\cdot}83(a_{30.40}-a_{31.40})-{\cdot}11(a_{30.40}-a_{30.41})$$
$$=13{\cdot}2324-{\cdot}83(13{\cdot}2324-13{\cdot}1660)-{\cdot}11(13{\cdot}2324-13{\cdot}0767)$$
$$=13{\cdot}2324-{\cdot}0551-{\cdot}0172=13{\cdot}1601.$$

A more exact formula may be obtaind as follos. Let x and y be integral ages, and h, k, fractions ov a year; then we hav aproximatly

$$a_{x+h.y}=a_{xy}-h(a_{xy}-a_{x+1.y})=(1-h)a_{xy}+ha_{x+1.y},$$

and now substituting $y+k$ for y,

$$\begin{aligned}a_{x+h.y+k}&=(1-h)a_{x.y+k}+ha_{x+1.y+k}\\&=(1-h)\{a_{xy}-k(a_{xy}-a_{x.y+1})\}+h\{a_{x+1.y}-k(a_{x+1.y}-a_{x+1.y+1})\}\\&=a_{xy}-h(a_{xy}-a_{x+1.y})-k(a_{xy}-a_{x.y+1})\\&\qquad+hk(a_{xy}-a_{x.y+1}-a_{x+1.y}+a_{x+1.y+1}).\end{aligned}$$

It wil be seen that we hav neglected the last term ov this formula, but its value is always smal. In the abov case, we hav $h={\cdot}83$, $k={\cdot}11$, and the value ov the term is

$$\begin{aligned}&{\cdot}83\times{\cdot}11(13{\cdot}2324-13{\cdot}1660-13{\cdot}0767+13{\cdot}0132)\\&={\cdot}0913(26{\cdot}2456-26{\cdot}2427)\\&={\cdot}0913\times{\cdot}0029={\cdot}0003.\end{aligned}$$

We infer that our result abov obtaind is not true beyond the third decimal place; but in fact the third decimal place is not corect, in consequence ov our aproximat ages, 30·83 and 40·11, being calculated only to the fraction ·01 ov a year.

As a further illustration ov the use that may be made ov Tabl 8, I wil calculate the singl and annual 4 per-cent premiums for a surviviership asurance on 30 agenst 60. Here both lives must be considerd as select, the one, 30, being selected by the insurance offis, and the other, 60, by the proposer; for, if the elder life is in a precarious state ov helth, it is not likely that the insurance wil be proposed. The ordinary formula for a surviviership asurance, wil require to be slihtly modified. We hav

$$\mathrm{A}^{1}_{xy}=\tfrac{1}{2}\left(\mathrm{A}_{xy}+\frac{a_{x-1.y}}{p_{x-1}}-\frac{a_{x.y-1}}{p_{y-1}}\right),$$

but if x and y ar both select lives, we can put no interpretation on a_{x-1}, p_{x-1}, &c. We see, however, that

$$\frac{a_{x-1.y}}{p_{x-1}}=\frac{vp_{x-1}p_y(1+a_{x.y+1})}{p_{x-1}}=vp_y(1+a_{x.y+1})=vp_y\mathbf{a}_{x.y+1},$$

and this presents no difficulty when x and y ar select lives. Transforming $a_{x.y-1}$ similarly, we get

$$\mathrm{A}^{1}_{[x][y]}=\tfrac{1}{2}\{\mathrm{A}_{[x][y]}+vp_{[y]}\mathbf{a}_{[x].[y]+1}-vp_{[x]}\mathbf{a}_{[x]+1.[y]}\},$$

and in this we hav to put $x=30$, $y=60$.

Then Tabl 8 informs us that

a life	[30]	is equivalent to an H$^{\mathrm{M}}$ life of	30·83
„	[30] + 1	„ „	32·16
„	[60]	„ „	59·40
„	[60] + 1	„ „	60·85

We find by the proces explaind abov that $a_{[30][60]}=a_{(30.83)(59.40)}=8{\cdot}948$, whence $\mathrm{A}_{[30][60]}={\cdot}61738$. Also $\mathbf{a}_{[30].[60]+1}=\mathbf{a}_{(30.83)(60.85)}=9{\cdot}547$, and $\mathbf{a}_{[30]+1.[60]}=\mathbf{a}_{(32.16)(59.40)}=9{\cdot}916$. Substituting, $\mathrm{A}_{\overset{1}{[30]}[60]}={\cdot}0867$, whence $\varpi_{\overset{1}{[30]}[60]}={\cdot}00871$.

No. 2: being an extended Tabl ov the Values ov l_x for different Ages at Entry and different Periods since Entry; shoing also the number ov Damaged Lives containd in them.

Age	SELECT LIVES.	MIXT LIVES.					DAMAGED LIVES.										Age
	Numbers Living.						Numbers Living.					Numbers Dying.					
	Years elapst since Date ov Insurance.																
	0	1	2	3	4	5 or more											
x	$l_{[x]}$	$l_{[x-1]+1}$	$l_{[x-2]+2}$	$l_{[x-3]+3}$	$l_{[x-4]+4}$	l_x	$l_x-l_{[x]}$ $=(nl)_x$	$l_x-l_{[x-1]+1}$ $=(nl)_{(x-1)+1}$	$l_x-l_{[x-2]+2}$ $=(nl)_{(x-2)+2}$	$l_x-l_{[x-3]+3}$ $=(nl)_{(x-3)+3}$	$l_x-l_{[x-4]+4}$ $=(nl)_{(x-4)+4}$	$d_x-d_{[x]}$ $=(nd)_x$	$d_x-d_{[x-1]+1}$ $=(nd)_{(x-1)+1}$	$d_x-d_{[x-2]+2}$ $=(nd)_{(x-2)+2}$	$d_x-d_{[x-3]+3}$ $=(nd)_{(x-3)+3}$	$d_x-d_{[x-4]+4}$ $=(nd)_{(x-4)+4}$	x
15	983452	...	...	...	...	983700	248	...	...	...	...	248	...	...	...	...	15
16	980428	980500	...	...	...	980500	72	...	...	...	...	72	...	...	...	...	16
17	976700	976800	976800	...	...	976800	100	...	...	...	...	100	...	...	...	...	17
18	971650	972500	972500	972500	...	972500	850	...	...	...	...	734	...	...	...	...	18
19	965015	967084	967200	967200	967200	967200	2185	116	...	...	...	1675	116	...	...	...	19
20	956418	960190	960700	960700	960700	960700	4282	510	...	...	...	3218	510	...	...	...	20
21	945306	951636	952700	952700	952700	952700	7394	1064	...	...	...	4598	1064	...	...	...	21
22	934548	940704	943500	943500	943500	943500	8952	2796	...	...	...	5262	1944	...	...	...	22
23	924270	930110	932948	933800	933800	933800	9530	3690	852	...	...	5705	2584	650	...	...	23
24	914390	919975	922694	923598	923800	923800	9410	3825	1106	202	...	5828	2885	1004	202	...	24
25	904880	910218	912860	913698	913800	913800	8020	3582	940	102	...	5528	2742	940	102	...	25
26	895828	900808	903360	904200	904200	904200	8372	3392	840	...	...	5106	2460	750	...	...	26
27	887214	891834	894168	895010	895100	895100	7886	3266	932	90	...	4962	2434	826	90	...	27
28	878918	883276	885368	886094	886200	886200	7282	2924	832	106	...	4696	2270	762	106	...	28
29	870746	875014	876946	877530	877600	877600	6854	2586	654	70	...	4416	2070	654	70	...	29
30	862820	866862	868784	869300	869300	869300	6480	2438	516	...	...	4116	1838	506	...	...	30
31	854970	858936	860700	861290	861300	861300	6330	2364	600	10	...	4006	1778	516	10	...	31
32	846500	851076	852814	853316	853400	853400	6900	2324	586	84	...	3986	1794	586	84	...	32
33	838098	842586	844970	845500	845500	845500	7402	2914	530	...	...	3856	1692	530	...	...	33
34	829492	834154	836478	837700	837700	837700	8208	3546	1222	...	...	3918	1772	644	...	...	34
35	820928	825510	828026	829222	829800	829800	8872	4290	1774	578	...	4278	2142	1030	416	...	35
36	812338	816906	819352	820756	821338	821500	9162	4594	2148	744	162	4435	2304	1198	512	162	36
37	803767	808273	810710	812050	812768	813000	9233	4727	2290	950	232	4589	2457	1358	610	232	37
38	795196	799656	802030	803368	803960	804300	9104	4644	2270	932	340	4738	2606	1504	726	340	38
39	786608	791034	793362	794634	795194	795400	8792	4366	2038	766	206	4682	2543	1438	654	206	39
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
40	777995	782390	784677	785900	786388	786500	8505	4110	1823	600	112	4622	2472	1357	600	112	40
41	769147	773717	775962	777134	777600	777600	8453	3883	1638	466	...	4455	2289	1163	444	...	41
42	760067	764802	767206	768325	768778	768800	8733	3998	1594	475	22	4183	2296	1152	453	22	42
43	750725	755650	758198	759458	759878	759900	9175	4250	1702	442	22	4505	2292	1126	409	22	43
44	740943	746230	748942	750324	750867	750900	9957	4670	1958	576	83	4620	2373	1183	414	33	44
45	730692	736363	739403	740925	741538	741700	11008	5337	2297	775	162	4923	2639	1417	570	162	45
46	720100	726015	729402	731220	731895	732100	12000	6085	2698	880	205	5220	2891	1631	688	205	46
47	709190	715320	718906	721033	721908	722100	12910	6780	3194	1067	192	5502	3122	1819	790	192	47
48	698088	704292	708042	710325	711423	711700	13612	7408	3658	1375	277	5870	3431	2078	988	277	48
49	686620	693058	696823	699220	700413	700800	14180	7742	3977	1580	387	6225	3719	2305	1172	387	49
50	674923	681445	685377	687728	688992	689400	14477	7955	4023	1672	408	6467	3879	2393	1238	408	50
51	662891	669590	673524	675970	677166	677600	14709	8010	4076	1630	434	6695	4011	2449	1278	434	51
52	650511	657386	661401	663773	665048	665400	14889	8014	3999	1627	352	6806	4016	2366	1181	352	52
53	637720	644817	648902	651267	652457	652900	15180	8083	3998	1633	443	7101	4192	2439	1247	443	53
54	624683	631821	636009	638341	639514	639900	15217	8079	3891	1559	386	7280	4239	2370	1169	386	54
55	611313	618563	622660	624979	626110	626500	15187	7937	3840	1521	390	7543	4353	2354	1155	390	55
56	597374	604956	609016	611114	612234	612600	15226	7644	3584	1486	366	7792	4434	2285	1095	366	56
57	582899	590766	594990	596901	597809	598200	15301	7434	3210	1299	391	8128	4587	2268	1057	391	57
58	567710	576027	580353	582258	582958	583200	15490	7173	2847	942	242	8356	4613	2104	830	242	58
59	551749	560566	565140	566957	567588	567700	15951	7134	2560	743	112	8677	4715	2003	633	112	59
60	534963	544326	549181	551043	551490	551600	16637	7274	2419	557	110	9196	4996	2037	557	110	60
61	517324	527259	532422	534338	534700	534700	17376	7441	2278	362	...	9617	5164	1951	362	...	61
62	499149	509341	514823	516773	517100	517100	17951	7759	2277	327	...	10137	5423	1885	327	...	62
63	480989	490886	496364	498308	498700	498700	17711	7814	2336	392	...	10652	5672	1779	392	...	63
64	462493	472441	477358	478943	479500	479500	17007	7059	2142	557	...	11175	5905	1625	557	...	64
65	442251	453668	458346	458983	459500	459500	17249	5832	1154	517	...	11434	5832	1154	517	...	65
66	421668	433185	439000	439000	439000	439000	17332	5815	...	...	...	11713	5815	...	...	...	66
67	400726	412381	418000	418000	418000	418000	17274	5619	...	...	...	11819	5619	...	...	...	67
68	379511	391245	396700	396700	396700	396700	17189	5455	...	...	...	11953	5455	...	...	...	68
69	357779	369864	375100	375100	375100	375100	17321	5236	...	...	...	12034	5236	...	...	...	69
70	335398	348013	353300	353300	353300	353300	17902	5287	...	...	...	12378	5287	...	...	...	70
71	312219	325571	331100	331100	331100	331100	18881	5529	...	...	...	12885	5529	..	...	...	71
72	288309	302404	308400	308400	308400	308400	20091	5996	...	...	...	13580	5996	...	...	...	72
73	263844	278589	285100	285100	285100	285100	21256	6511	...	...	...	14269	6511	...	...	...	73
74	239674	254313	261300	261300	261300	261300	21626	6987	...	...	...	14828	6987	...	...	...	74
75	215772	230402	237200	237200	237200	237200	21428	6798	...	...	...	14666	6798	...	...	...	75
76	...	206838	213600	213600	213600	213600	...	6762	...	...	...	...	6762	...	...	...	76

No. 3 : being an extended Tabl ov the Values ov d_x for different Ages at Entry and different Periods since Entry.

Age	YEARS ELAPST SINCE DATE OV INSURANCE.						Age
	0	1	2	3	4	5 or more	
x	$d_{[x]}$	$d_{[x-1]+1}$	$d_{[x-2]+2}$	$d_{[x-3]+3}$	$d_{[x-4]+4}$	d_x	x
15	2952	...	...	...	...	3200	15
16	3628	3700	..	...	...	3700	16
17	4200	4300	4300	...	...	4300	17
18	4566	5300	5300	5300	...	5300	18
19	4825	6384	6500	6500	6500	6500	19
20	4782	7490	8000	8000	8000	8000	20
21	4602	8136	9200	9200	9200	9200	21
22	4438	7756	9700	9700	9700	9700	22
23	4295	7416	9350	10000	10000	10000	23
24	4172	7115	8996	9798	10000	10000	24
25	4072	6858	8660	9498	9600	9600	25
26	3994	6640	8350	9100	9100	9100	26
27	3938	6466	8074	8810	8900	8900	27
28	3904	6330	7838	8494	8600	8600	28
29	3884	6230	7646	8230	8300	8300	29
30	3884	6162	7494	8000	8000	8000	30
31	3894	6122	7384	7890	7900	7900	31
32	3914	6106	7314	7816	7900	7900	32
33	3944	6108	7270	7800	7800	7800	33
34	3982	6128	7256	7900	7900	7900	34
35	4022	6158	7270	7884	8300	8300	35
36	4065	6196	7302	7988	8338	8500	36
37	4111	6243	7342	8090	8168	8700	37
38	4162	6294	7396	8174	8560	8900	38
39	4218	6357	7462	8246	8694	8900	39
40	4278	6428	7543	8300	8788	8900	40
41	4345	6511	7637	8356	8800	8800	41
42	4417	6604	7748	8447	8878	8900	42
43	4495	6708	7874	8591	8978	9000	43
44	4580	6827	8017	8786	9167	9200	44
45	4677	6961	8183	9030	9438	9600	45
46	4780	7109	8369	9312	9795	10000	46
47	4898	7278	8581	9610	10208	10400	47
48	5030	7469	8822	9912	10623	10900	48
49	5175	7681	9095	10228	11013	11400	49
50	5333	7921	9407	10562	11392	11800	50
51	5505	8189	9751	10922	11766	12200	51
52	5694	8484	10134	11316	12148	12500	52
53	5899	8808	10561	11753	12557	13000	53
54	6120	9161	11030	12231	13014	13400	54
55	6357	9547	11546	12745	13510	13900	55
56	6608	9966	12115	13305	14034	14400	56
57	6872	10413	12732	13943	14609	15000	57
58	7144	10887	13396	14670	15258	15500	58
59	7423	11385	14097	15467	15988	16100	59
60	7704	11904	14843	16343	16790	16900	60
61	7983	12436	15649	17238	17600	17600	61
62	8263	12977	16515	18073	18400	18400	62
63	8548	13528	17421	18808	19200	19200	63
64	8825	14095	18375	19443	20000	20000	64
65	9066	14668	19346	19983	20500	20500	65
66	9287	15185	21000	21000	21000	21000	66
67	9481	15681	21300	21300	21300	21300	67
68	9647	16145	21600	21600	21600	21600	68
69	9766	16564	21800	21800	21800	21800	69
70	9827	16913	22200	22200	22200	22200	70
71	9815	17171	22700	22700	22700	22700	71
72	9720	17304	23300	23300	23300	23300	72
73	9531	17289	23800	23800	23800	23800	73
74	9272	17113	24100	24100	24100	24100	74
75	8934	16802	23600	23600	23600	23600	75
76	...	16338	23100	23100	23100	23100	76

No. 4: being an extended Tabl ov the Values ov D_x for different Ages at Entry and different Periods since Entry.—4 per-cent.

Age	YEARS ELAPST SINCE DATE OV INSURANCE.						Age
	0	1	2	3	4	5 or more	
x	$D_{[x]}$	$D_{[x-1]+1}$	$D_{[x-2]+2}$	$D_{[x-3]+3}$	$D_{[x-4]+4}$	D_x	x
15	546075	...	...	...	...	546214	15
16	523458	523497	...	...	...	523497	16
17	501411	501463	501463	...	...	501463	17
18	479633	480053	480053	480053	...	480053	18
19	458037	459018	459074	459074	459074	459074	19
20	436496	438218	438451	438451	438451	438451	20
21	414832	417609	418077	418077	418077	418077	21
22	394336	396935	398115	398115	398115	398115	22
23	375000	377369	378521	378867	378867	378867	23
24	356722	358902	359963	360316	360394	360394	24
25	339435	341437	342428	342743	342782	342782	25
26	323115	324911	325832	326135	326135	326135	26
27	307700	309302	310111	310404	310436	310436	27
28	293098	294552	295250	295492	295528	295528	28
29	279204	280574	281194	281381	281404	281404	29
30	266022	267269	267862	268021	268021	268021	30
31	253464	254639	255162	255338	255341	255341	31
32	241301	242605	243101	243244	243268	243268	32
33	229717	230947	231601	231747	231747	231747	33
34	218613	219842	220455	220778	220778	220778	34
35	208035	209196	209834	210137	210284	210284	35
36	197940	199054	199650	199992	200134	200174	36
37	188320	189375	189947	190260	190429	190483	37
38	179145	180150	180685	180987	181120	181197	38
39	170395	171354	171858	172134	172255	172300	39
40	162047	162963	163410	163694	163796	163819	40
41	154043	154958	155408	155643	155736	155736	41
42	146369	147281	147744	147960	148048	148052	42
43	139010	139922	140394	140627	140705	140709	43
44	131922	132863	133346	133593	133689	133695	44
45	125093	126064	126584	126845	126950	126978	45
46	118538	119512	120069	120369	120480	120514	46
47	112252	113222	113790	114127	114266	114296	47
48	106245	107189	107760	108108	108275	108317	48
49	100480	101423	101974	102325	102499	102556	49
50	94969·8	95887·8	96440·8	96771·8	96949·8	97007·3	50
51	89689·2	90595·6	91128·0	91459·0	91620·8	91679·7	51
52	84629·2	85523·4	86045·8	86354·8	86520·4	86560·4	52
53	79774·0	80661·8	81172·8	81468·8	81617·8	81673·2	53
54	75137·7	75996·2	76500·2	76780·7	76921·7	76968·3	54
55	70701·5	71540·0	72013·8	72282·2	72413·0	72458·2	55
56	66132·1	67275·3	67726·9	67960·1	68085·0	68125·6	56
57	62329·3	63170·4	63622·3	63826·7	63923·6	63965·6	57
58	58370·3	59225·6	59670·3	59866·1	59938·1	59963·1	58
59	54547·3	55419·0	55871·4	56051·0	56113·4	56124·4	59
60	50853·5	51743·8	52205·4	52382·4	52425·0	52435·3	60
61	47285·4	48193·6	48665·6	48840·7	48873·8	48873·8	61
62	43869·3	44765·1	45247·0	45418·5	45447·2	45447·2	62
63	40647·5	41483·7	41946·8	42111·1	42144·3	42144·3	63
64	37581·2	38389·5	38789·0	38917·8	38963·2	38963·2	64
65	34554·3	35416·3	35811·7	35861·5	35902·0	35902·0	65
66	31678·9	32544·2	32981·0	32981·0	32981·0	32981·0	66
67	28947·7	29789·6	30195·5	30195·5	30195·5	30195·5	67
68	26360·6	27175·7	27554·7	27554·7	27554·7	27554·7	68
69	23895·4	24702·5	25052·3	25052·3	25052·3	25052·3	69
70	21539·0	22349·2	22688·7	22688·7	22688·7	22688·7	70
71	19279·3	20103·8	20445·2	20445·2	20445·2	20445·2	71
72	17118·2	17955·0	18311·1	18311·1	18311·1	18311·1	72
73	15063·1	15904·9	16276·6	16276·6	16276·6	16276·6	73
74	13156·9	13960·5	14344·1	14344·1	14344·1	14344·1	74
75	11389·2	12161·4	12520·3	12520·3	12520·3	12520·3	75
76	...	10497·8	10841·0	10841·0	10841·0	10841·0	76

No. 5 : being an extended Tabl ov the Values ov $\mathbf{N}_x$ *for different Ages at Entry and different Periods since Entry.—4 per-cent.*

Age	YEARS ELAPST SINCE DATE OV INSURANCE.						Age
	0	1	2	3	4	5 or more	
x	$\mathbf{N}_{[x]}$	$\mathbf{N}_{[x-1]+1}$	$\mathbf{N}_{[x-2]+2}$	$\mathbf{N}_{[x-3]+3}$	$\mathbf{N}_{[x-4]+4}$	$\mathbf{N}_x$	x
15	10847378	...	...	...	...	10847517	15
16	10301264	10301303	...	...	...	10301303	16
17	9777754	9777806	9777806	...	...	9777806	17
18	9275867	9276343	9276343	9276343	...	9276343	18
19	8795020	8796234	8796290	8796290	8796290	8796290	19
20	8334793	8336983	8337216	8337216	8337216	8337216	20
21	7893915	7898297	7898765	7898765	7898765	7898765	21
22	7474940	7479083	7480688	7480688	7480688	7480688	22
23	7076859	7080604	7082148	7082573	7082573	7082573	23
24	6698354	6701859	6703235	6703627	6703706	6703706	24
25	6338380	6341632	6342957	6343272	6343311	6343311	25
26	5996075	5998945	6000195	6000529	6000529	6000529	26
27	5670473	5672960	5674034	5674363	5674394	5674394	27
28	5360537	5362773	5363658	5363923	5363959	5363959	28
29	5065275	5067439	5068221	5068408	5068431	5068431	29
30	4784158	4786071	4786865	4787027	4787027	4787027	30
31	4516320	4518136	4518802	4519003	4519006	4519006	31
32	4260388	4262856	4263497	4263640	4263665	4263665	32
33	4016745	4019087	4020251	4020396	4020396	4020396	33
34	3784573	3787028	3788140	3788650	3788650	3788650	34
35	3563712	3565960	3567186	3567685	3567872	3567872	35
36	3353544	3355677	3356764	3357352	3357548	3357588	36
37	3153636	3155604	3156623	3157114	3157360	3157414	37
38	2963457	2965316	2966229	2966676	2966854	2966931	38
39	2782548	2784312	2785166	2785544	2785689	2785734	39
40	2610487	2612153	2612958	2613308	2613410	2613434	40
41	2446705	2448440	2449190	2449518	2449614	2449614	41
42	2290893	2292662	2293482	2293782	2293875	2293878	42
43	2142727	2144524	2145381	2145738	2145822	2145827	43
44	2001775	2003717	2004602	2004987	2005111	2005117	44
45	1867764	1869853	1870854	1871256	1871394	1871422	45
46	1740550	1742671	1743789	1744270	1744411	1744444	46
47	1619883	1622012	1623159	1623720	1623901	1623931	47
48	1505597	1507631	1508790	1509369	1509593	1509635	48
49	1397304	1399352	1400442	1401030	1401261	1401318	49
50	1294869	1296824	1297929	1298468	1298705	1298762	50
51	1197988	1199899	1200936	1201488	1201696	1201755	51
52	1106442	1108299	1109303	1109808	1110029	1110075	52
53	1019986	1021813	1022775	1023258	1023453	1023509	53
54	938524·1	940211·5	941150·7	941602·6	941788·8	941835·3	54
55	861808·9	863386·4	864215·3	864650·5	864821·9	864867·1	55
56	789543·8	791107·4	791846·4	792201·5	792368·3	792408·9	56
57	721603·7	723111·7	723832·1	724119·5	724241·4	724283·3	57
58	657756·5	659274·4	659941·3	660209·8	660292·8	660317·8	58
59	597849·1	599386·2	600048·8	600271·0	600343·7	600354·7	59
60	541734·8	543301·8	543967·2	544177·4	544220·0	544230·3	60
61	489281·4	490881·3	491558·0	491761·8	491795·0	491795·0	61
62	440467·9	441996·0	442687·7	442892·4	442921·1	442921·1	62
63	395313·1	396598·6	397230·9	397440·7	397473·9	397473·9	63
64	353491·9	354665·6	355114·9	355284·1	355329·6	355329·6	64
65	314581·8	315910·7	316276·1	316325·9	316366·3	316366·3	65
66	278756·3	280027·5	280464·4	280464·4	280464·4	280464·4	66
67	245856·6	247077·4	247483·3	247483·3	247483·3	247483·3	67
68	215744·0	216908·9	217287·8	217287·8	217287·8	217287·8	68
69	188236·8	189383·4	189733·2	189733·2	189733·2	189733·2	69
70	163189·8	164341·4	164680·9	164680·9	164680·9	164680·9	70
71	140470·2	141650·8	141992·2	141992·2	141992·2	141992·2	71
72	119982·4	121190·9	121547·0	121547·0	121547·0	121547·0	72
73	101638·8	102864·2	103235·9	103235·9	103235·9	103235·9	73
74	85413·22	86575·71	86959·28	86959·28	86959·28	86959·28	74
75	71140·96	72256·32	72615·21	72615·21	72615·21	72615·21	75
76	...	59751·76	60094·92	60094·92	60094·92	60094·92	76

No. 6: being an extended Tabl ov the Values ov M_x for different Ages at Entry and different Periods since Entry.—4 per-cent.

Age	YEARS ELAPST SINCE DATE OV INSURANCE.						Age
	0	1	2	3	4	5 or more	
x	$M_{[x]}$	$M_{[x-1]+1}$	$M_{[x-2]+2}$	$M_{[x-3]+3}$	$M_{[x-4]+4}$	M_x	x
15	128869	...	...	...	...	129002	15
16	127257	127293	...	...	...	127293	16
17	125345	125394	125394	...	...	125394	17
18	122870	123271	123271	123271	...	123271	18
19	119767	120703	120755	120755	120755	120755	19
20	115927	117565	117789	117789	117789	117789	20
21	111221	113829	114278	114278	114278	114278	21
22	106840	109278	110396	110396	110396	110396	22
23	102814	105039	106131	106461	106461	106461	23
24	99094·0	101138	102146	102484	102559	102559	24
25	95651·5	97529·0	98469·1	98771·4	98808·2	98808·2	25
26	92497·3	94182·8	95055·4	95345·6	95345·6	95345·6	26
27	89605·3	91112·1	91880·0	92159·5	92189·6	92189·6	27
28	86924·6	88292·1	88955·8	89187·6	89221·6	89221·6	28
29	84386·9	85672·8	86262·4	86442·5	86464·0	86464·0	29
30	82017·2	83189·6	83752·0	83905·0	83905·0	83905·0	30
31	79759·9	80865·8	81363·2	81530·4	81533·3	81533·3	31
32	77440·0	78649·9	79120·7	79258·3	79281·3	79281·3	32
33	75227·8	76367·2	76976·8	77116·0	77116·0	77116·0	33
34	73053·4	74188·3	74757·5	75060·3	75060·3	75060·3	34
35	70970·1	72044·8	72635·8	72918·8	73058·3	73058·3	35
36	68959·0	69990·1	70543·9	70863·8	70997·8	71035·9	36
37	67026·4	68006·6	68538·4	68833·1	68992·2	69044·3	37
38	65166·9	66100·2	66600·1	66884·3	67010·6	67084·4	38
39	63374·3	64265·3	64736·8	64997·9	65113·6	65156·4	39
40	61644·5	62495·8	62941·2	63182·5	63280·3	63302·7	40
41	59938·9	60787·7	61208·4	61430·5	61520·2	61520·2	41
42	58258·9	59102·2	59533·8	59737·7	59821·4	59825·5	42
43	56597·7	57440·9	57879·4	58099·1	58173·6	58177·5	43
44	54930·7	55797·4	56246·5	56477·5	56569·5	56575·1	44
45	53256·0	54146·6	54628·6	54874·0	54973·4	55000·1	45
46	51594·3	52486·1	53000·8	53281·6	53387·5	53419·8	46
47	49949·8	50837·7	51361·0	51676·1	51807·7	51837·0	47
48	48337·4	49203·9	49730·0	50055·0	50213·6	50254·1	48
49	46737·9	47601·8	48110·8	48439·0	48604·5	48659·0	49
50	45167·6	46009·7	46520·9	46831·0	46909·8	47054·9	50
51	43613·1	44146·0	44938·1	45248·1	45402·0	45458·4	51
52	42074·0	42896·9	43380·7	43669·6	43827·2	43871·2	52
53	40543·7	41361·7	41835·6	42113·0	42254·1	42307·5	53
54	39041·0	39834·2	40302·3	40565·3	40699·3	40743·8	54
55	37555·1	38333·2	38774·8	39026·6	39150·7	39194·1	55
56	36065·5	36848·2	37271·5	37490·8	37609·2	37648·3	56
57	34575·6	35358·9	35782·5	35976·0	36068·2	36108·5	57
58	33071·9	33869·1	34288·2	34473·5	34542·4	34566·2	58
59	31553·4	32365·7	32792·7	32963·8	33023·2	33033·9	59
60	30018·0	30847·8	31283·4	31452·6	31493·4	31503·4	60
61	28467·0	29318·8	29759·7	29926·8	29958·7	29958·7	61
62	26928·4	27765·4	28210·8	28384·3	28411·8	28411·8	62
63	25418·2	26230·1	26668·7	26825·1	26856·9	26856·9	63
64	23985·4	24748·6	25130·9	25253·1	25296·7	25296·7	64
65	22455·0	23295·9	23647·3	23695·2	23734·0	23734·0	65
66	20957·4	21773·8	22193·9	22193·9	22193·9	22193·9	66
67	19491·7	20286·6	20676·9	20676·9	20676·9	20676·9	67
68	18062·8	18833·1	19197·4	19197·4	19197·4	19197·4	68
69	16655·5	17418·5	17754·8	17754·8	17754·8	17754·8	69
70	15262·5	16028·4	16354·8	16354·8	16354·8	16354·8	70
71	13876·7	14655·7	14984·0	14984·0	14984·0	14984·0	71
72	12503·5	13293·9	13636·2	13636·2	13636·2	13636·2	72
73	11153·9	11948·6	12306·0	12306·0	12306·0	12306·0	73
74	9871·78	10630·7	10999·5	10999·5	10999·5	10999·5	74
75	8653·05	9382·37	9727·39	9727·39	9727·39	9727·39	75
76	...	8199·62	8529·61	8520·61	8529·61	8529·61	76

No. 7 : being an extended Tabl ov the Values ov a_x for different Ages at Entry and different Periods since Entry.—4 per-cent.

Age	Years elapst since Date ov Insurance.						Age
	0	1	2	3	4	5 or more	
x	$a_{[x]}$	$a_{[x-1]+1}$	$a_{[x-2]+2}$	$a_{[x-3]+3}$	$a_{[x-4]+4}$	a_x	x
15	18·8643	...	...	...	...	18·8594	15
16	18·6793	18·6779	...	...	...	18·6779	16
17	18·5005	18·4986	18·4986	...	...	18·4986	17
18	18·3395	18·3236	18·3236	18·3236	...	18·3236	18
19	18·2016	18·1631	18·1609	18·1609	18·1609	18·1609	19
20	18·0948	18·0248	18·0152	18·0152	18·0152	18·0152	20
21	18·0292	17·9132	17·8931	17·8931	17·8931	17·8931	21
22	17·9558	17·8421	17·7903	17·7903	17·7903	17·7903	22
23	17·8716	17·7631	17·7100	17·6941	17·6941	17·6941	23
24	17·7776	17·6733	17·6221	17·6049	17·6010	17·6010	24
25	17·6734	17·5734	17·5235	17·5074	17·5054	17·5054	25
26	17·5571	17·4634	17·4151	17·3989	17·3989	17·3989	26
27	17·4286	17·3412	17·2967	17·2806	17·2789	17·2789	27
28	17·2893	17·2065	17·1665	17·1525	17·1505	17·1505	28
29	17·1418	17·0610	17·0239	17·0126	17·0112	17·0112	29
30	16·9841	16·9073	16·8707	16·8606	16·8606	16·8606	30
31	16·8184	16·7433	16·7095	16·6982	16·6980	16·6980	31
32	16·6559	16·5712	16·5381	16·5282	16·5266	16·5266	32
33	16·4857	16·4026	16·3586	16·3483	16·3483	16·3483	33
34	16·3118	16·2261	16·1832	16·1605	16·1605	16·1605	34
35	16·1305	16·0461	16·0000	15·9779	15·9669	15·9669	35
36	15·9422	15·8583	15·8133	15·7874	15·7765	15·7734	36
37	15·7463	15·6633	15·6186	15·5937	15·5803	15·5758	37
38	15·5422	15·4603	15·4165	15·3917	15·3807	15·3741	38
39	15·3300	15·2489	15·2062	15·1824	15·1719	15·1680	39
40	15·1095	15·0292	14·9873	14·9646	14·9553	14·9532	40
41	14·8833	14·8007	14·7598	14·7381	14·7293	14·7293	41
42	14·6514	14·5665	14·5234	14·5027	14·4942	14·4938	42
43	14·4142	14·3265	14·2811	14·2584	14·2505	14·2501	43
44	14·1739	14·0811	14·0330	14·0082	13·9984	13·9977	44
45	13·9311	13·8325	13·7795	13·7523	13·7411	13·7382	45
46	13·6835	13·5816	13·5231	13·4911	13·4788	13·4750	46
47	13·4308	13·3259	13·2645	13·2273	13·2117	13·2081	47
48	13·1709	13·0651	13·0014	12·9617	12·9422	12·9372	48
49	12·9063	12·7972	12·7334	12·6920	12·6710	12·6639	49
50	12·6346	12·5244	12·4583	12·4178	12·3956	12·3883	50
51	12·3571	12·2446	12·1786	12·1369	12·1160	12·1082	51
52	12·0740	11·9590	11·8920	11·8518	11·8296	11·8234	52
53	11·7860	11·6678	11·6000	11·5601	11·5396	11·5317	53
54	11·4908	11·3718	11·3026	11·2635	11·2435	11·2367	54
55	11·1894	11·0686	11·0007	10·9621	10·9429	10·9361	55
56	10·8850	10·7593	10·6918	10·6569	10·6379	10·6316	56
57	10·5773	10·4470	10·3770	10·3451	10·3298	10·3230	57
58	10·2687	10·1316	10·0598	10·0281	10·0163	10·0121	58
59	9·9602	9·8155	9·7398	9·7094	9·6988	9·6969	59
60	9·6528	9·4999	9·4198	9·3885	9·3809	9·3791	60
61	9·3474	9·1856	9·1008	9·0687	9·0625	9·0625	61
62	9·0404	8·8736	8·7838	8·7514	8·7458	8·7458	62
63	8·7254	8·5603	8·4698	8·4379	8·4313	8·4313	63
64	8·4061	8·2386	8·1550	8·1290	8·1196	8·1196	64
65	8·1040	7·9124	7·8316	7·8207	7·8119	7·8119	65
66	7·7994	7·6045	7·5038	7·5038	7·5038	7·5038	66
67	7·4931	7·2941	7·1960	7·1960	7·1960	7·1960	67
68	7·1843	6·9817	6·8857	6·8857	6·8857	6·8857	68
69	6·8775	6·6666	6·5735	6·5735	6·5735	6·5735	69
70	6·5765	6·3533	6·2583	6·2583	6·2583	6·2583	70
71	6·2861	6·0460	5·9450	5·9450	5·9450	5·9450	71
72	6·0091	5·7497	5·6379	5·6379	5·6379	5·6379	72
73	5·7476	5·4675	5·3426	5·3426	5·3426	5·3426	73
74	5·4919	5·2015	5·0624	5·0624	5·0624	5·0624	74
75	5·2463	4·9414	4·7998	4·7998	4·7998	4·7998	75
76	...	4·6918	4·5433	4·5433	4·5433	4·5433	76

No. 8 : *Tabl ov Annual and Singl Premiums for Select Lives— 4 per-cent.*

x	$\varpi_{[x]}$	$A_{[x]}$	x	$\varpi_{[x]}$	$A_{[x]}$
(1)	(2)	(3)	(1)	(2)	(3)
15	·011880	·235992	46	·029642	·435255
16	·012354	·243108	47	·030835	·444974
17	·012819	·249984	48	·032105	·454963
18	·013246	·256175	49	·033449	·465144
19	·013618	·261479	50	·034882	·475600
20	·013909	·265586	51	·036405	·486270
21	·014089	·268111	52	·038026	·497158
22	·014293	·270936	53	·039749	·508233
23	·014528	·274171	54	·041598	·519593
24	·014794	·277791	55	·043577	·531178
25	·015091	·281797	56	·045679	·542893
26	·015426	·286268	57	·047915	·554725
27	·015802	·291211	58	·050280	·566588
28	·016216	·296572	59	·052778	·578460
29	·016660	·302241	60	·055411	·590284
30	·017144	·308309	61	·058181	·602026
31	·017660	·314680	62	·061136	·613831
32	·018177	·320928	63	·064362	·625947
33	·018729	·327481	64	·067853	·638230
34	·019303	·334168	65	·071380	·649847
35	·019915	·341146	66	·075182	·661557
36	·020563	·348382	67	·079281	·673341
37	·021254	·355919	68	·083723	·685218
38	·021990	·363765	69	·088482	·697018
39	·022776	·371926	70	·093526	·708598
40	·023614	·380410	71	·098787	·719772
41	·024498	·389105	72	·104211	·730423
42	·025431	·398026	73	·109740	·740482
43	·026414	·407149	74	·115577	·750312
44	·027441	·416389	75	·121633	·759762
45	·028513	·425733			

No. 9.—Tabl shoing the Age ov the H^{M} *Life which is ov the same Value as a Select Life that was insured at a givn Age and has been insured for a certain number ov Years.*

Age x	Years elapst since Date ov Insurance.						Age x
	0	1	2	3	4	5 or more	
15	18·43	...	...	...	...	18·46	15
16	19·74	19·75	...	...	...	19·75	16
17	21·10	21·11	21·11	...	...	21·11	17
18	22·33	22·45	22·45	22·45	...	22·45	18
19	23·35	23·63	23·64	23·64	23·64	23·64	19
20	24·10	24·57	24·64	24·64	24·64	24·64	20
21	24·54	25·30	25·43	25·43	25·43	25·43	21
22	25·03	25·76	26·08	26·08	26·08	26·08	22
23	25·57	26·25	26·58	26·67	26·67	26·67	23
24	26·16	26·80	27·11	27·21	27·24	27·24	24
25	26·80	27·40	27·70	27·80	27·81	27·81	25
26	27·50	28·06	28·35	28·44	28·44	28·44	26
27	28·27	28·78	29·04	29·13	29·14	29·14	27
28	29·08	29·56	29·79	29·88	29·89	29·89	28
29	29·94	30·40	30·61	30·67	30·68	30·68	29
30	30·88	31·26	31·46	31·52	31·52	31·52	30
31	31·75	32·16	32·34	32·40	32·41	32·41	31
32	32·63	33·08	33·26	33·31	33·31	33·31	32
33	33·53	33·96	34·18	34·23	34·23	34·23	33
34	34·42	34·85	35·07	35·18	35·18	35·18	34
35	35·33	35·74	35·97	36·08	36·13	36·13	35
36	36·25	36·65	36·87	37·00	37·05	37·06	36
37	37·19	37·58	37·79	37·91	37·97	37·99	37
38	38·15	38·53	38·73	38·84	38·89	38·92	38
39	39·12	39·49	39·68	39·79	39·83	39·85	39
40	40·11	40·46	40·64	40·74	40·78	40·79	40
41	41·09	41·43	41·60	41·69	41·73	41·73	41
42	42·05	42·39	42·57	42·65	42·69	42·69	42
43	43·01	43·35	43·53	43·62	43·65	43·65	43
44	43·95	44·31	44·49	44·59	44·63	44·63	44
45	44·88	45·26	45·46	45·56	45·60	45·61	45
46	45·82	46·20	46·42	46·54	46·58	46·60	46
47	46·76	47·15	47·37	47·51	47·57	47·58	47
48	47·72	48·10	48·33	48·48	48·55	48·57	48
49	48·68	49·07	49·30	49·45	49·52	49·55	49
50	49·65	50·04	50·27	50·41	50·49	50·51	50
51	50·62	51·01	51·24	51·38	51·45	51·48	51
52	51·59	51·99	52·21	52·34	52·42	52·44	52
53	52·56	52·96	53·18	53·31	53·38	53·41	53
54	53·54	53·93	54·16	54·28	54·35	54·37	54
55	54·52	54·92	55·13	55·26	55·32	55·34	55
56	55·51	55·91	56·13	56·24	56·30	56·32	56
57	56·49	56·90	57·13	57·23	57·28	57·30	57
58	57·47	57·90	58·12	58·22	58·26	58·27	58
59	58·44	58·89	59·12	59·22	59·25	59·26	59
60	59·40	59·87	60·12	60·22	60·24	60·25	60
61	60·35	60·85	61·12	61·22	61·24	61·24	61
62	61·31	61·83	62·11	62·21	62·23	62·23	62
63	62·29	62·81	63·10	63·20	63·22	63·22	63
64	63·30	63·83	64·09	64·18	64·21	64·21	64
65	64·26	64·87	65·12	65·16	65·19	65·19	65
66	65·23	65·85	66·17	66·17	66·17	66·17	66
67	66·20	66·84	67·15	67·15	67·15	67·15	67
68	67·19	67·83	68·13	68·13	68·13	68·13	68
69	68·16	68·82	69·12	69·12	69·12	69·12	69
70	69·11	69·81	70·11	70·11	70·11	70·11	70
71	70·02	70·79	71·11	71·11	71·11	71·11	71
72	70·90	71·75	72·12	72·12	72·12	72·12	72
73	71·75	72·69	73·12	73·12	73·12	73·12	73
74	72·61	73·63	74·13	74·13	74·13	74·13	74
75	73·47	74·59	75·13	75·13	75·13	75·13	75
76	...	75·54	76·12	76·12	76·12	76·12	76

Exampl.—A $4^{\circ}/_{\circ}$ anuity on the life ov [35]+2 is equal to a $4^{\circ}/_{\circ}$ H^{M} anuity on a life ov 37·79.

PART III.—*Use.*

By means ov these tabls we may anser with more or les accuracy a variety ov questions to which hitherto no numerical ansers coud be givn: in particular, we ar abl to investigate in a complete manner the efect upon life insurance finance ov the gradual wearing out ov the benefit ov selection. As is wel knoen, a body ov insured lives who ar at the outset al select, wil, after the laps ov som years, contain a number ov damaged lives, the majority, however, being stil good lives. For som purposes, as for instance, when we ar estimating the liabilitys ov a life insurance company, we may, without inconvenience or error, proceed as if al lives ov the same age and standing wer equally deteriorated; but in other cases, as for instance, when we hav to fix a rule for the calculation ov surender values, it is important to bear in mind the real character ov the deterioration which has taken place, namely, that, while the majority ov the lives stil remain in good helth, a certain smal proportion ov them hav becom more or les diseasd. It is no new sugestion that the lives which aply to surender their policys, wil on the average be in a better state ov helth than those which continue their policys; that those who surender shoud therefore be treated as select lives, and the values ov their policys calculated acordingly. It is obvios that while, by the use ov such a tabl as the $H^{M(5)}$, we may calculate *average* policy-values, that shal be aplicabl to the policys taken as a whole, yet the liability in respect ov a policy on a life that has becom damaged, must be considerably greater than this average, and therefore the liability under a policy on a life which is stil select must be les than the average; but I am not aware that any atemt has ever been made hitherto to calculate the different values ov policys on select and damaged lives, or the respectiv numbers ov helthy and damaged lives containd in a body ov mixt lives that hav been asured for any length ov time. It is my object to do this in the present paper.

In the first place, it wil be convenient to consider the net premiums givn by my tabls. It wil be found that up to the age ov 43 my net premiums ar greater than the H^M premiums, and abov that age they ar les.

This is a point ov so much importance that it seems desirabl to sho by comparison with other tabls that my premiums ar not exceptional in the abov respect, but that the H^M premiums ar really too lo at the yunger ages and too hih at the older ages.

In the folloing tabl ar givn the 4 per-cent net premiums, (1) acording to the H^M tabl; (2) as calculated by Mr. Spens* from Mr. Higham's data (*J.I.A.*, x. 79); (3) acording to Mr. Spens's Scottish Amicable Experience* (*J.I.A.*, x. 79); (4) acording to Mr. King (*J.I.A.*, xx 247); (5) as givn in my paper *On the premiums for the insurance ov recently selected lives* (*J.I.A.*, xx 95); and lastly, as determind from my new tabls. It wil be seen that the premiums in the last five colums ov the tabl, al agree in being greater than the H^M premiums at the yunger ages, that is to say, up to the age ov 30 inclusiv without any exception, and with the singl exception ov Mr. King's values, up to the age ov 40 inclusiv; and for the hiher ages, 50 onwards, they al agree in being les than the H^M. The differences between them ar generally much les than the difference between the average ov them and the H^M premiums; and their general agreement, calculated as they ar by widely different methods and from different data, may be taken as establishing beyond al question that the H^M premiums ar too lo at the yunger ages and too hih at the older ages. I may be permitted to ad that the agreement between my oen two sets ov premiums is much closer than I shoud hav anticipated, bearing in mind how very different ar the processes by which they wer calculated. At the oldest ages, 65–75, where the values ov the two sets ov premiums differ most, I believ that the "1876" values ar the more likely to be corect.

Tabl ov 4 per-cent net Premiums for the Insurance ov 100.

Age at Entry.	H^M.	Higham.	Spens.	King.	Sprague.	
					"1876".	Select.
15	1·052	...	...	...	...	1·188
20	1·245	...	1·37	1·411	1·330	1·391
25	1·428	1·58	1·55	1·532	1·517	1·509
30	1·669	1·79	1·78	1·718	1·715	1·714
35	1·969	2·05	2·05	1·964	1·976	1·992
40	2·352	2·40	2·39	2·325	2·365	2·361
45	2·865	2·88	2·86	2·896	2·914	2·851
50	3·542	3·51	3·49	3·370	3·433	3·488
55	4·458	4·28	4·30	4·270	4·350	4·358
60	5·715	5·39	5·34	5·231	5·634	5·541
65	7·427	...	...	6·270	6·637	7·138
70	9·866	...	...	...	8·930	9·353
75	13·299	...	...	...	10·666	12·163

---

* Mr. Spens has givn only the anuitys; but it is so simpl a proces to deduce the premiums from them, that I think I may properly speak ov the premiums as being calculated by Mr. Spens.

We hav next to consider the average policy-values resulting from my tabls. The formula for the net-premium value ov a policy taken out n years ago at the age x, wil be

$$A_{[x]+n} - \mathbf{a}_{[x]+n} \cdot \varpi_{[x]} = \mathbf{a}_{[x]+n}(\varpi_{[x]+n} - \varpi_{[x]}) = 1 - \frac{\mathbf{a}_{[x]+n}}{\mathbf{a}_{[x]}};$$

and when n is greater than 4, these formulas becom

$$A_{x+n} - \mathbf{a}_{x+n} \cdot \varpi_{[x]} = \mathbf{a}_{x+n}(\varpi_{x+n} - \varpi_{[x]}) = 1 - \frac{\mathbf{a}_{x+n}}{\mathbf{a}_{[x]}},$$

where the functions A_{x+n}, $\mathbf{a}_{x+n}$, ϖ_{x+n}, ar al taken from the $H^{M(5)}$ tabl. If in the second set ov formulas we substitute insted ov $\varpi_{[x]}$ and $\mathbf{a}_{[x]}$ the premium and anuity-due ov the H^{M} tabl, we shal get the values ov policys as found from the combined H^{M} and $H^{M(5)}$ tabls; and this shos us that the policy-values as found from my tabls, wil be greater or les than those found from the combined H^{M} and $H^{M(5)}$ tabls, acording as $\varpi_{[x]}$ is les or greater than the net premium deduced from the H^{M} tabl; that is, up to age at entry 43 the policy-values givn by my tabls ar les than those givn by the combined H^{M} and $H^{M(5)}$, and abov that age they ar greater.

Colum (4) ov the folloing tabls shos the 4 per-cent policy-values calculated by these formulas, for ages at entry 20, 25, ... 75, and for values ov n, 1, 2, 3, ... 10, 15, 20, 30; and for the purpos ov comparison colum (2) ov the same tabls shos the 4 per-cent values calculated by the H^{M} tabl; and colum (3) the values calculated by the combined H^{M} and $H^{M(5)}$ tabls. It wil be notist that ther ar no values opposit the first four ages in colum (3). In practis the blank woud be suplyd by taking the values from colum (2), since in valuing policys under five years' standing the H^{M} tabl would be uzed alone. An examination ov the tabl shos that for such policys, whatever the age at entry, my policy-values greatly exceed the H^{M} values. Combining this result with the one abov, it seems probabl that the adoption ov my tabls woud hav the efect ov considerably increasing the estimated liability ov a company under its policys. I hav, however, made no calculations as to this; and being at the present time too fully occupyd to do so, I shal be pleasd if som other member ov the *Institute* wil investigate the point.

Values ov Policys for 100, *on Mixt and on Helthy Lives, calculated at* 4 *per-cènt Interest acording to Different Formulas.*

n Years elapst	Mixt Lives.			Helthy Lives.	Mixt Lives.			Helthy Lives.	*n* Years elapst
	100_nV_x			$100_n(hV)_x$	$100_{n-1}V_{x+1}$		$100V_{([x+1],[x]+n)}$	$100_{n-1}(hV)_{x+1}$	
	H^M	Combined H^{MS} and		Select.	H^M	Combined H^{MS} and		Select.	
		H^M	Select.			H^M	Select.		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Age at Entry, 20.									
1	·665	...	·951	·343	·000	...	·610	·000	1
2	1·323	...	1·595	·728	·662	...	1·255	·386	2
3	1·999	...	2·099	1·169	1·343	...	1·761	·828	3
4	2·715	...	2·586	1·661	2·064	...	2·259	1·322	4
5	3·477	5·795	3·087	2·207	2·831	5·164	2·753	1·870	5
6	4·276	6·337	3·644	2·816	3·635	5·710	3·312	2·481	6
7	5·107	6·948	4·273	3·489	4·471	6·325	3·943	3·156	7
8	5·957	7·602	4·945	4·218	5·327	6·983	4·618	3·888	8
9	6·821	8·311	5·675	4·991	6·197	7·697	5·350	4·663	9
10	7·702	9·078	6·463	5·817	7·083	8·469	6·141	5·492	10
15	12·457	13·627	11·144	10·287	11·870	13·049	10·838	9·978	15
20	17·864	18·788	16·453	15·634	17·313	18·244	16·165	15·343	20
30	31·093	31·845	29·885	28·595	30·631	31·388	29·643	28·349	30
Age at Entry, 25.									
1	·828	...	1·124	·623	·000	...	·505	·000	1
2	1·688	...	2·017	1·311	·868	...	1·403	·693	2
3	2·568	...	2·789	2·057	1·755	...	2·180	1·443	3
4	3·464	...	3·546	2·847	2·659	...	2·942	2·238	4
5	4·376	5·802	4·353	3·691	3·579	5·016	3·753	3·088	5
6	5·305	6·660	5·223	4·579	4·515	5·881	4·630	3·981	6
7	6·260	7·564	6·141	5·449	5·478	6·792	5·553	4·856	7
8	7·244	8·504	7·096	6·360	6·470	7·740	6·514	5·773	8
9	8·259	9·494	8·102	7·292	7·493	8·739	7·526	6·711	9
10	9·303	10·515	9·139	8·262	8·546	9·769	8·569	7·688	10
15	14·905	15·862	14·567	13·730	14·194	15·160	14·032	13·189	15
20	21·414	22·270	21·074	20·041	20·758	21·621	20·579	19·540	20
30	36·487	37·048	36·080	34·723	35·957	36·523	35·679	34·314	30
Age at Entry, 30.									
1	·971	...	1·339	·921	·000	...	·421	·000	1
2	1·970	...	2·480	1·825	1·009	...	1·573	·912	2
3	2·999	...	3·535	2·771	2·047	...	2·638	1·867	3
4	4·060	...	4·579	3·738	3·119	...	3·692	2·843	4
5	5·152	6·420	5·656	4·746	4·222	5·502	4·779	3·861	5
6	6·272	7·487	6·732	5·793	5·352	6·580	5·865	4·917	6
7	7·416	8·577	7·831	6·883	6·507	7·680	6·974	6·017	7
8	8·583	9·690	8·952	8·018	7·687	8·804	8·106	7·162	8
9	9·780	10·826	10·098	9·197	8·895	9·952	9·262	8·353	9
10	11·010	12·011	11·293	10·423	10·137	11·148	10·468	9·590	10
15	17·817	18·712	18·048	16·976	17·011	17·915	17·287	16·204	15
20	25·343	26·158	25·555	24·185	24·611	25·433	24·862	23·480	20
30	42·314	42·755	42·287	40·765	41·748	42·193	41·751	40·215	30

Values ov Policys for 100, *&c.*—(continued).

Years elapst n	MIXT LIVES.			HELTHY LIVES.	MIXT LIVES.			HELTHY LIVES.	Years elapst n
	$100\,{}_nV_x$			$100\,{}_n(hV)_x$	$100\,{}_{n-1}V_{x+1}$		$100V_{([x+1],[x]+n)}$	$100\,{}_{n-1}(hV)_{x+1}$	
	H^M	Combined H^{MS} and		Select.	H^M	Combined H^{MS} and		Select.	
		H^M	Select.			H^M	Select.		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Age at Entry, 35.									
1	1·180	...	1·589	1·099	·000	...	·495	·000	1
2	2·386	...	2·988	2·243	1·220	...	1·910	1·156	2
3	3·618	...	4·313	3·434	2·466	...	3·249	2·361	3
4	4·879	...	5·596	4·673	3·743	...	4·547	3·613	4
5	6·176	7·232	6·872	5·960	5·055	6·123	5·837	4·915	5
6	7·518	8·534	8·180	7·280	6·413	7·441	7·159	6·250	6
7	8·911	9·903	9·554	8·634	7·823	8·827	8·549	7·619	7
8	10·353	11·320	10·977	10·019	9·282	10·261	9·987	9·019	8
9	11·833	12·788	12·450	11·422	10·780	11·746	11·477	10·437	9
10	13·353	14·297	13·965	12·839	12·318	13·273	13·009	11·870	10
15	21·288	22·147	21·845	20·407	20·347	21·217	20·977	19·523	15
20	29·972	30·591	30·322	28·844	29·135	29·762	29·548	28·053	20
30	48·419	48·758	48·560	46·855	47·803	48·146	47·988	46·264	30
Age at Entry, 40.									
1	1·430	...	1·917	1·404	·000	...	·520	·000	1
2	2·915	...	3·638	2·844	1·507	...	2·266	1·460	2
3	4·451	...	5·283	4·316	3·065	...	3·934	2·953	3
4	6·029	...	6·897	5·808	4·666	...	5·571	4·466	4
5	7·649	8·655	8·512	7·315	6·310	7·330	7·209	5·995	5
6	9·295	10·287	10·146	8·852	7·979	8·985	8·866	7·554	6
7	10·960	11·941	11·803	10·421	9·669	10·663	10·547	9·145	7
8	12·648	13·620	13·485	12·034	11·381	12·367	12·253	10·781	8
9	14·363	15·314	15·181	13·677	13·121	14·085	13·973	12·447	9
10	16·106	17·022	16·892	15·363	14·889	15·818	15·708	14·158	10
15	25·362	26·022	25·906	24·334	24·279	24·949	24·851	23·257	15
20	35·177	35·672	35·572	33·873	34·237	34·739	34·654	32·931	20
30	54·799	55·014	54·944	52·969	54·144	54·362	54·302	52·299	30
Age at Entry, 45.									
1	1·782	...	2·341	1·658	·000	...	·694	·000	1
2	3·585	...	4·464	3·351	1·836	...	2·853	1·721	2
3	5·413	...	6·492	5·091	3·697	...	4·916	3·491	3
4	7·270	...	8·440	6·864	5·587	...	6·896	5·293	4
5	9·157	10·149	10·333	8·683	7·509	8·519	8·821	7·144	5
6	11·085	12·028	12·209	10·542	9·472	10·433	10·728	9·033	6
7	13·059	13·940	14·116	12·438	11·481	12·379	12·668	10·961	7
8	15·070	15·897	16·070	14·367	13·529	14·372	14·655	12·923	8
9	17·111	17·877	18·046	16·344	15·607	16·387	16·664	14·933	9
10	19·180	19·895	20·059	18·362	17·714	18·441	18·711	16·986	10
15	29·808	30·344	30·487	28·654	28·534	29·080	29·314	27·451	15
20	40·470	40·862	40·983	39·027	39·390	39·789	39·988	37·999	20
30	60·856	61·076	61·156	58·166	60·146	60·370	60·501	57·460	30

Values ov Policys for 100, *&c.*—(continued).

Years elapst n	MIXT LIVES.			HELTHY LIVES.	MIXT LIVES.			HELTHY LIVES.	Years elapst n
	100_nV_x			$100_n(hV)_x$	$100_{n-1}V_{x+1}$		$100V_{([x+1].[x]+n)}$	$100_{n-1}(hV)_{x+1}$	
	H^M	Combined H^{M5} and		Select.	H^M	Combined H^{M5} and		Select.	
		H^M	Select.			H^M	Select.		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
				Age at Entry, 50.					
1	2·122	...	2·860	2·035	·000	...	·842	·000	1
2	4·294	...	5·446	4·112	2·220	...	3·482	2·119	2
3	6·509	...	7·881	6·224	4·482	...	5·967	4·275	3
4	8·755	...	10·203	8·389	6·777	...	8·337	6·486	4
5	11·033	11·820	12·457	10·599	9·104	9·908	10·638	8·742	5
6	13·334	14·069	14·691	12·832	11·455	12·206	12·918	11·021	6
7	15·657	16·349	16·954	15·089	13·828	14·536	15·228	13·325	7
8	17·999	18·646	19·234	17·352	16·222	16·882	17·556	15·635	8
9	20·362	20·974	21·546	19·615	18·636	19·261	19·916	17·945	9
10	22·732	23·322	23·877	21·869	21·057	21·660	22·295	20·246	10
15	34·469	34·900	35·371	33·229	33·048	33·489	34·028	31·841	15
20	46·121	46·378	46·765	44·432	44·954	45·215	45·660	43·277	20
30	65·985	66·110	66·355	...	65·247	65·375	65·656	...	30
				Age at Entry, 55.					
1	2·587	...	3·528	2·497	·000	...	1·058	·000	1
2	5·197	...	6·665	5·022	2·680	...	4·274	2·589	2
3	7·831	...	9·527	7·553	5·383	...	7·210	5·185	3
4	10·486	...	12·229	10·084	8·109	...	9·981	7·781	4
5	13·150	13·813	14·851	12·606	10·844	11·525	12·670	10·368	5
6	15·813	16·442	17·449	15·111	13·578	14·224	15·334	12·937	6
7	18·469	19·072	20·047	17·630	16·305	16·923	17·999	15·520	7
8	21·113	21·684	22·627	20·214	19·018	19·604	20·645	18·171	8
9	23·733	24·272	25·184	22·834	21·708	22·261	23·268	20·857	9
10	26·342	26·827	27·708	25·312	24·386	24·884	25·857	23·399	10
15	39·440	39·728	40·454	37·843	37·832	38·128	38·929	36·251	15
20	51·567	51·839	52·419	48·756	50·281	50·560	51·201	47·444	20
30	69·626	69·820	70·184	...	68·820	69·019	69·420	...	30
				Age at Entry, 60.					
1	3·066	...	4·386	2·867	·000	...	1·564	·000	1
2	6·125	...	8·157	5·749	3·155	...	5·447	2·967	2
3	9·168	...	11·405	8·706	6·295	...	8·790	6·011	3
4	12·186	...	14·392	11·703	9·408	...	11·866	9·097	4
5	15·190	15·748	17·281	14·539	12·507	13·083	14·839	12·017	5
6	18·188	18·694	20·173	17·398	15·600	16·122	17·817	14·960	6
7	21·188	21·637	23·062	20·274	18·695	19·158	20·792	17·920	7
8	24·198	24·604	25·975	23·172	21·800	22·219	23·790	20·905	8
9	27·236	27·589	28·906	26·052	24·934	25·298	26·808	23·870	9
10	30·271	30·602	31·865	28·878	28·065	28·407	29·854	26·779	10
15	44·234	44·547	45·556	41·365	42·470	42·793	43·949	39·634	15
20	55·978	56·139	56·937	...	54·585	54·752	55·666	...	20
30	74·148	74·538	75·001	...	73·330	73·732	74·263	...	30

Values ov Policys for 100, *&c.*—(continued).

Years elapst n	Mixt Lives.			Helthy Lives.	Mixt Lives.			Helthy Lives.	Years elapst n
	100_nV_x			$100_n(hV)_x$	$100_{n-1}V_{x+1}$	$100V_{[x+1],[x]+n}$		$100_{n-1}(hV)_{x+1}$	
	H^M	Combined H^{M5} and		Select.	H^M	Combined H^{M5} and		Select.	
		H^M	Select.			H^M	Select.		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
				Age at Entry, 65.					
1	3·535	...	5·486	3·346	·000	...	2 215	·000	1
2	7·073	...	9·974	6·710	3·667	...	6·857	3·481	2
3	10·622	...	13·382	10·102	7·346	...	10·384	6·990	3
4	14·204	...	16·811	13·472	11·059	...	13·932	10·477	4
5	17·782	18·173	20·273	16·778	14·769	15·174	17·514	13·897	5
6	21·325	21·705	23·715	19·968	18·442	18·836	21·074	17·198	6
7	24·784	25·167	27·088	23·011	22·027	22·425	24·564	20·346	7
8	28·113	28·496	30·332	25·883	25·478	25·876	27·920	23·317	8
9	31·266	31·655	33·409	28·692	28·747	29·150	31·104	26·223	9
10	34·246	34·616	36·294	31·389	31·836	32·219	34·089	29·014	10
15	48·093	48·284	49·611	...	46·191	46·388	47·867	...	15
20	58·764	59·027	60·079	...	57·252	57·526	58·697	...	20
30	84·097	84·784	85·175	...	83·515	84·226	84·661	...	30
				Age at Entry, 70.					
1	4·310	...	7·002	3·833	·000	...	3·295	·000	1
2	8·516	...	12·388	7·489	4·396	...	8·896	3·802	2
3	12·565	...	16·286	10·941	8·628	...	12·949	7·391	3
4	16·401	...	19·984	14·315	12·636	...	16·795	10·900	4
5	20·025	20·474	23·450	17·557	16·423	16·893	20·399	14·271	5
6	23·568	23·991	26·836	...	20·126	20·568	23·920	...	6
7	27·005	27·355	30·073	...	23·718	24·083	27·286	...	7
8	30·359	30·706	33·299	...	27·223	27·585	30·640	...	8
9	33·660	33·965	36·436	...	30·672	30·991	33·903	...	9
10	36·867	37·099	39·452	...	34·024	34·266	37·039	...	10
15	49·845	50·166	52·031	...	47·586	47·922	50·119	...	15
20	62·925	63·484	64·851	...	61·255	61·840	63·450	...	20
				Age at Entry, 75.					
1	4·430	...	8·877	...	·000	...	...	...	1
2	8·729	...	15·182	...	4·498	...	...	...	2
3	12·922	...	19·094	...	8·886	...	...	...	3
4	17·049	...	22·900	...	13·204	...	...	...	4
5	21·059	21·349	26·558	...	17·400	17·703	...	...	5
6	24·864	25·117	30·077	...	21·381	21·646	...	...	6
7	28·368	28·625	33·353	...	25·048	25·317	...	...	7
8	31·615	31·861	36·373	...	28·445	28·702	...	...	8
9	34·547	34·864	39·178	...	31·513	31·845	...	...	9
10	37·287	37·688	41·815	...	34·380	34 800	...	...	10
15	53·642	54·341	57·365	...	51·493	52·225	...	...	15
20	75·815	76·859	78·392	...	74·694	75·787	...	...	20

I hav stated abov how the values in colums (2), (3), and (4), ar got, and a ful explanation ov the values in the other colums wil be givn presently; but it may be useful to giv here a short description ov the varios formulas by which the different values ar calculated.

As alreddy explaind, the values in colums (2), (3), (4), ar al got by the wel-knoen formulas

$${}_{n}V_{x}=A_{x+n}-\mathbf{a}_{x+n}\cdot\varpi_{x}=\mathbf{a}_{x+n}(\varpi_{x+n}-\varpi_{x})=1-\mathbf{a}_{x+n}\div\mathbf{a}_{x}.$$

In colum (2) the values ov al the quantitys ar taken from the H^{M} Tabl.

In (3), n being > 4, the values ov A_{x+n}, $\mathbf{a}_{x+n}$, ϖ_{x+n}, ar taken from the $H^{M(5)}$ Tabl, but those of $\mathbf{a}_{x}$ and ϖ_{x} from the H^{M} Tabl.

In (4) the formulas becom

$${}_{n}V_{[x]}=A_{[x]+n}-\mathbf{a}_{[x]+n}\cdot\varpi_{[x]}=\mathbf{a}_{[x]+n}(\varpi_{[x]+n}-\varpi_{[x]})=1-\mathbf{a}_{[x]+n}\div\mathbf{a}_{[x]};$$

and the values ov $A_{[x]+n}$, $\mathbf{a}_{[x]+n}$, $\varpi_{[x]+n}$, ar taken from my Select Tabls when $n < 5$ and from the $H^{M(5)}$ Tabl when $n =$ or > 5; the values ov $\mathbf{a}_{[x]}$ and $\varpi_{[x]}$ being always taken from the Select Tabls.

In (5) the life is suposed to be stil select (or helthy) at the date ov valuation, and the formulas ar

$${}_{n}(hV)_{[x]}=A_{[x+n]}-\mathbf{a}_{[x+n]}\cdot\varpi_{[x]}=\mathbf{a}_{[x+n]}(\varpi_{[x+n]}-\varpi_{[x]})=1-\mathbf{a}_{[x+n]}\div\mathbf{a}_{[x]},$$

the values ov al the quantitys being taken from my Select Tabls. Since a life is always suposed to be select at entry, we may omit the brakets round the age at entry x, here and abov, and rite without any risk ov confusion, ${}_{n}V_{x}$ and ${}_{n}(hV)_{x}$.

Passing now to the second set ov values—those in colums (6) to (9), we supose that the first year's premium has been absorbd by the initial expenses and the first year's risk. The values in (6) and (7) ar got by the formulas

$$A_{x+n}-\mathbf{a}_{x+n}\cdot\varpi_{x+1}=\mathbf{a}_{x+n}(\varpi_{x+n}-\varpi_{x+1})=1-\mathbf{a}_{x+n}\div\mathbf{a}_{x+1}.$$

In (6) the values ov al the quantitys ar taken from the H^{M} Tabl.

In (7), n being > 4, the values ov A_{x+n}, $\mathbf{a}_{x+n}$, ϖ_{x+n}, ar taken from the $H^{M(5)}$ Tabl, and those ov $\mathbf{a}_{x+1}$, ϖ_{x+1}, from the H^{M} Tabl.

In (8) the policy-values ar calculated by the new formula,

$$A_{[x]+n}-\mathbf{a}_{[x]+n}\cdot\varpi_{[x+1]}=\mathbf{a}_{[x]+n}(\varpi_{[x]+n}-\varpi_{[x+1]})=1-\mathbf{a}_{[x]+n}\div\mathbf{a}_{[x+1]}.$$

It is desirabl to hav a symbol to denote this value, and as the ordinary symbol for a policy-value, ${}_{n}V_{x}$, dos not admit ov being suitably modifyd, I propose $V([x+1], [x]+n)$; consistently with which we miht hav the ordinary policy-value denoted by $V(x, x+n)$.

In (9), as in (5), the life is suposed to be still helthy, and the policy-value is calculated by the formula

$$
{}_{n-1}(hV)_{[x+1]}=A_{[x+n]}-a_{[x+n]}\cdot\varpi_{[x+1]}=a_{[x+n]}(\varpi_{[x+n]}-\varpi_{[x+1]})
$$
$$
=1-a_{[x+n]}:a_{[x+1]},
$$

in which the values ov al the quantitys ar got from the Select Tabls. This value miht, if desired, be denoted by the symbol $V([x+1],\ [x+n])$. It wil be notist that in the case ov a helthy life, the value obtaind in this way is the same as that calculated by the formula ov colum (5) for a policy on a life still helthy, which was taken out at the age $x+1$ and had been in force $n-1$ years; but a similar relation dos not exist between the policy-values for mixt lives.

I submit that by means ov my new tabls we can estimate the liability ov a life offis under its policys more corectly than has ever been possibl hitherto. When the *Institute* tabls wer publisht, it seemd very probabl that we shoud get aproximatly true results by uzing the H^{M} tabl for policys ov les than five years' duration, and the combined H^{M} and $H^{M(5)}$ tabls for policys ov greater duration. On the asumtion that after the laps ov five years the efect ov selection has worn of, it was clear that for policys ov five years or more duration, the $H^{M(5)}$ anuitys and reversions wer the proper ones to uze; but what was the proper net premium? In practis the H^{M} premium was adopted, but this was only becaus no more accurate premiums wer availabl. It coud not be maintaind that the H^{M} premiums wer the corect ones; in fact, it was impossibl to say with certainty whether the corect net premiums woud be greater or les than these. As regards policys ov les than five years' standing, the uncertainty was stil greater, as it coud not even be knoen whether the H^{M} reversions and anuitys wer greater or les than the corect values when proper alowance was made for the efect ov selection. The only legitimat conclusion that coud be drawn was that, as the mortality in the erly years ov insurance is liht, probably the reserv that shoud be made for policys quite recently efected, o't to be greater than that givn by the H^{M} tabl. Al these douts and difficultys ar remoovd by the use ov my Select Tabls. We ar abl to calculate from them for the first time the proper reserv to make for the liability under policys on recently selected lives ov any age, when alowance is made for the gradual wearing out ov the benefit ov selection, and we altogether get rid ov the awkward break in the series ov policy-values between the fourth and fifth years, which is causd by the use ov the combined H^{M} and $H^{M(5)}$ tabls. My tabls also, as I shal sho further on, enabl us to make proper alowance for the initial expenses ov the new bisnes.

On these grounds I ask my professional brethren to giv a frendly reception to my tabls, and to consider for themselvs, first, whether these tabls ar, as I hold, so great an improovment on al previosly existing tabls, that they deserv to be employd in al life insurance calculations; and then, how they may best be practically aplyd in making life offis valuations. Ther wil be no difficulty in uzing them for this purpos when the values ov policys ar determind individually by means ov a tabl; and I cannot help thinking that this method ov valuation is at present unduly discredited. Ther is one very great benefit that ataches to it, the value ov which is strikingly exemplified by recent events; and that is, that it most efectually prevents the introduction ov any negativ policy-values into a valuation. If a tabl ov policy-values wer calculated on the basis ov my Select Mortality Tabls, its extent woud be precisely the same as that ov a tabl calculated on any other basis, and the use ov it woud be atended with precisely the same amount ov labor. If, however, the policys ar valued, as is now more usual, in classes, separat classifications woud hav to be made ov policys ov les and more than five years' duration; and policys ov les than five years' duration woud hav to be aranged both acording to the age at entry and the endurance, unles it shoud be found suficiently corect to groop al policys under five years' standing simply acording to the age ataind, and then to value the policys at any age, say y, by means ov the anuitys and asurances for the age $[y-2]+2$. I thro out this sugestion in the hope that som member ov the *Institute* wil take it up, examin it practically, and publish the result ov his examination. So many interesting lines ov enquiry ar sugested by a study ov my new tabls that I cannot myself undertake to follo them al up. It seems probabl, indeed, that a complete examination ov them woud aford employment for many investigaters during a long period; and having laid unreservedly before the actuarial profession my methods ov investigation and the results at which I hav thus far arivd, I trust other volunteers wil now com forward and asist in the exploration ov the new region into which I may at al events claim to hav opend a practicabl road.

I now pas on to consider what is the value ov a policy on a life which is stil select. Taking the general formula for the net value ov a policy, $A_{x+n}-\mathbf{a}_{x+n}.\varpi_x$, it is obvios that the fact ov the life being stil select is no reason for altering the value ov ϖ_x;

but the values ov the asurance and the anuity must be those for select lives. The formulas for the value ov a policy on a select life, which I wil denote by ${}_n(hV)_x$, becom

$${}_n(hV)_x = A_{[x+n]} - \mathbf{a}_{[x+n]}\varpi_{[x]} = \mathbf{a}_{[x+n]}(\varpi_{[x+n]} - \varpi_{[x]}) = 1 - \mathbf{a}_{[x+n]} \div \mathbf{a}_{[x]}\,;$$

and since $\mathbf{a}_{[x+n]} >$ both $\mathbf{a}_{[x]+n}$ and $\mathbf{a}_{x+n}$, we see that the policy-values givn by these formulas wil always be les, as miht hav been anticipated, than those givn by the others. The values resulting from this formula ar givn in colum 5 ov the tabl; and these values, it wil be observd, ar the sums that an offis shoud hav in hand, acording to the net-premium method ov valuation, in order to provide for its liability under a policy upon a life that is stil select. It is obvios that these values wil hav a most important bearing on the calculation ov surender values; but since, in calculating them, no alowance is made for the initial expenses, I do not consider them so suitabl for practical adoption as those which I shal presently explain.

It woud be interesting to compare the policy-values resulting from my tabls with those that result from Mr. Higham's Class Mortality Tabls publisht in 1851, (*Asurance Magazine*, i, 199); but to do this fully woud require more time and labor than I coud devote to the purpos, since Mr. Higham's anuity values ar calculated at 3 and 3½ per-cent, while mine ar calculated at 4 per-cent, and he has givn tabls ov anuity values only for the quinquennial ages at entry. I hav, however, made the folloing comparison for age at entry 35, and durations ov 5, 10, and 20 years.

Years elapst n	${}_nV_{35}$		Percentage ov (2) to (3).	${}_nV_{35}$		Percentage ov (5) to (6).
	Sprague 4%	H^M^ 4%		Higham 3½%	H^M^ 3½%	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
5	·06872	·06176	111·27	·07144	·06668	107·14
10	·13965	·13353	104·58	·14729	·14283	103·12
20	·30322	·29972	101·17	·32061	·31521	101·71

Years elapst n	${}_n(hV)_{35}$	${}_nV_{35}$	Percentage ov (2) to (3).	${}_n(hV)_{35}$	${}_nV_{35}$	Percentage ov (5) to (6).
	Sprague 4%	H^M^ 4%		Higham 3½%	H^M^ 3½%	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
5	·05960	·06176	96·503	·05870	·06668	88·032
10	·12839	·13353	96·150	·13178	·14283	92·264
20	·28844	·29972	96·237	·29458	·31521	93·454

In al the preceding formulas I hav asumed, in acordance with the fundamental principl ov the net-premium method ov valuation, that the net premium valued is that for the original age at entry, namely, $\varpi_{[x]}$. I am ov opinion, however, as I hav on varios ocasions explaind, that, having regard to the hiher rate ov expenditure which necesarily ataches to the first year ov a policy, this is not the corect method ov procedure. If, as is offen the case, the first year's premium is wholly absorbd by the current risk and expenses, I hav sugested that, in the formula for a policy value, ${}_nV_x = \mathbf{a}_{x+n}(\varpi_{x+n} - P)$, P shoud be put $= \varpi_{x+1}$, so as to value the net premium for an age one year greater than the age at entry, thus making no reserv for policys ov only one year's standing. This proces cannot be adopted without som modification when we make use ov the new tabls. We then hav

$$
\begin{aligned}
{}_1V_x &= A_{[x]+1} - \mathbf{a}_{[x]+1} . P = \mathbf{a}_{[x]+1}(\varpi_{[x]+1} - P) \\
{}_1(hV)_x &= A_{[x+1]} - \mathbf{a}_{[x+1]} . P = \mathbf{a}_{[x+1]}(\varpi_{[x+1]} - P).
\end{aligned}
$$

These equations sho us that, if we put $P = \varpi_{[x]+1}$, we make ${}_1V_x = 0$; but, since $\varpi_{[x]+1} > \varpi_{[x+1]}$, the same substitution givs a negativ value for ${}_1(hV)_x$, which is ov cours inadmissibl. We may, however, put $P = \varpi_{[x+1]}$; and we then get

$$
\begin{aligned}
{}_1V_x &= \mathbf{a}_{[x]+1}(\varpi_{[x]+1} - \varpi_{[x+1]}) = 1 - \frac{\mathbf{a}_{[x]+1}}{\mathbf{a}_{[x+1]}}; \\
{}_1(hV)_x &= \mathbf{a}_{[x+1]}(\varpi_{[x+1]} - \varpi_{[x+1]}) = 0.
\end{aligned}
$$

We thus make no reserv for lives which ar stil select, but, looking at the lives as a whole, we do make a reserv for policys which hav been one year in force; and, as we shal hereafter see, this reserv is exactly enuf to meet the liability under the policys on the lives that hav becom damaged in the year.

These results enabl us to calculate the value ov the option that a policyholder can exercise at the end ov the first insurance year. The company, in granting a whole-life policy, enters into a contract which the policyholder is at liberty to continue or discontinue, as he may think proper; and the premiums receivd by the company in the first year must therefore be suficient to provide, not only for the hevy initial expenses and for the risk ov deth during the first year, but also for the risk ov the life becoming damaged in the first year, since in that event it is morally certain that the policyholder wil continue the policy in force. Let us uze the fraze "first year's risk" to include both ov the risks just mentiond, and investigate what is the minimum net premium, Q, which must be receivd from each ov $l_{[x]}$ select lives ov the age x, in order that this risk may be exactly met. The number ov deths which wil ocur among the $l_{[x]}$ lives is $l_{[x]}-l_{[x]+1}$, and the number ov survivers at the end ov a year is $l_{[x]+1}$, for each ov whom, as we hav seen, must be made a reserv equal to $1-\mathbf{a}_{[x]+1}\div\mathbf{a}_{[x+1]}$. The total sum, therefore, which the company must hav in hand at the end ov the year, to provide for the claims then due and for the policy-values, is

$$l_{[x]}-l_{[x]+1}+l_{[x]+1}\cdot\left(1-\frac{\mathbf{a}_{[x]+1}}{\mathbf{a}_{[x+1]}}\right)=l_{[x]}-l_{[x]+1}\cdot\frac{\mathbf{a}_{[x]+1}}{\mathbf{a}_{[x+1]}};$$

and this must be equal to the acumulated amount ov the $l_{[x]}$ premiums, $=(1+i)l_{[x]}Q$. Hence

$$Q=v\left(1-\frac{l_{[x]+1}}{l_{[x]}}\cdot\frac{\mathbf{a}_{[x]+1}}{\mathbf{a}_{[x+1]}}\right)=v\left(1-p_{[x]}\cdot\frac{1+a_{[x]+1}}{\mathbf{a}_{[x+1]}}\right)=v-\frac{a_{[x]}}{\mathbf{a}_{[x+1]}}.$$

The folloing tabl shos the values ov this quantity for quinquennial ages at entry, and the values here shoen seem deserving of careful study, as indicating what is the real risk run by a life offis during the first year's currency ov a whole-life policy. It follos that the offis can aford to spend only the exces ov the premium receivd over the amount here shoen; and that, if it spends more than this, it runs a risk ov its new bisnes resulting in a los, by the dropping ov the policys on the helthy lives at the end ov the first year.

Age at Entry.	First Year's Risk.	Age at Entry.	First Year's Risk.
15	·002945	45	·012783
20	·010640	50	·015632
25	·009161	55	·020064
30	·008360	60	·028662
35	·009450	65	·040568
40	·010258	70	·058932

We may now say that, on the asumption that the first year's premiums ar absorbd by the first year's risk and the initial expenses, the proper net premium to value is $\varpi_{[x+1]}$; and then, if the expenses in future years ar aproximatly constant, we must value the same net premium for policys ov al durations, and we hav the formulas

$${}_{n}V_{x}=A_{[x]+n}-\mathbf{a}_{[x]+n}\cdot\varpi_{[x+1]}=\mathbf{a}_{[x]+n}(\varpi_{[x]+n}-\varpi_{[x+1]})=1-\frac{\mathbf{a}_{[x]+n}}{\mathbf{a}_{[x+1]}};$$

$${}_{n}(hV)_{x}=A_{[x+n]}-\mathbf{a}_{[x+n]}\cdot\varpi_{[x+1]}=\mathbf{a}_{[x+n]}(\varpi_{[x+n]}-\varpi_{[x+1]})=1-\frac{\mathbf{a}_{[x+n]}}{\mathbf{a}_{[x+1]}}.$$

The values in colums (8) and (9) ov the tabl ov policy-values ar calculated by these formulas; and these ar the values which, in my opinion, may most advantageosly be uzed in life offis calculations.

The values in colum (8) ar suitabl for calculating policy-values in the quinquennial or other periodical investigations, and a tabl ov them miht be formd and uzed in the same way as the wel-knoen tabls ov policy-values which ar in use in a good many offises. The values in colum (9) ar useful for calculating surender values, but must not be employd without som further ajustment. They represent the sum that the company shoud hav in hand to meet its liability under a policy, on the suppositions that the first year's premium is wholly absorbd by the first year's risk and expenses, and that the life is stil in good helth. It is therefore clear that an offis cannot aford to giv for the surrender ov the policy a larger sum than shoen by these values; but it may very properly giv a smaller value. I wil not, however, stay on the present ocasion to consider on what principl this smaller value shoud be calculated.

In finding the value ov a policy on a life that is stil select, we hav taken the values ov $A_{[x]}$ and $\mathbf{a}_{[x]}$, our object being to determin with the greatest atainabl accuracy the real value ov the sum asured and the premium; but when we bear in mind that the rate ov mortality among insured lives depends upon the rate ov laps, in such a way that, the greater the rate ov laps, the greater wil be the rate ov mortality, we see that, in order to determin the true value ov the policy, we o't to kno the values ov asurances and anuitys upon select lives among which ther ar no withdrawals. At present, however, I believ ther ar no tabls in existens that wil giv us the values ov these quantitys. The fact is that, altho we hav ampl materials for determining the exact rate ov mortality among insured lives, and ar abl to determin with great accuracy the rate ov mortality during the first insurance year for entrants ov any age, and to trace the gradual increas in the rate ov mortality among them causd by the combined operation ov advancing age and the withdrawals ov helthy lives, yet we hav no means at present ov determining how the rate ov mortality woud increas if ther wer no such withdrawals. Al we can say is, that ther is good reason for believing that the rate ov mortality woud be les than is now found to prevail among insured lives, and consequently the values ov asurances woud be les, and the values ov anuitys greater, than those givn by our tabls. It follos also that the net premiums for the insurance ov such lives woud be les than those abov calculated; but whether the values ov policys woud be greater or les is not obvios.

Before the subject ov selection can be thoroly understood, it wil be esential to obtain information as to the rate ov mortality among select lives, non-ov which withdraw from observation. Three sources ocur to me whence the necesary information miht be obtaind. One ov these I hav mentiond in former papers, namely, the experience among the Government anuitants. These anuitants when nominated ar probably, on the average, a very select body ov lives; and no withdrawals take place among them, since the Government wil never repurchase an anuity which it has granted. The second source from which information miht be gaind is the mortality experience ov the Peerage Familys. We may fairly asume that men who marry ar, taken as a whole, a body ov select lives, altho ther wil, no dout, be ocasional exceptions. We miht therefore investigate the rate ov mortality among the marryd men ov the Peerage Familys (suposing each to com under observation at the date of marryge), grooping together al those who marry at the same age, and tracing the rate ov mortality in each ov these groops. I hav myself extracted a very considerabl number ov facts from the Peerage records, which I may, perhaps, som day utilize in this way. The third source from which information miht be obtaind, is from the publisht experience ov individual life offises, among which I woud particularly indicate the experience ov the *New York Mutual*, and the very extensiv experience ov the *Gotha Life Offis* during 50 years recently publisht by the Manager, Dr. Emminghaus. If we asume that al the persons who withdraw from observation by the laps or surrender ov their policys, ar select lives, and ar subject, after withdrawal, to the rate ov mortality which we find to prevail among recently selected lives, we miht by a long and somwhat complicated series ov calculations determin aproximatly what woud be the number ov persons under observation in any year, and the number ov deths among them, suposing that no withdrawals took place; and comparison ov these woud giv us the rate ov mortality we require.

Thus far we hav seen that, after the laps ov any number ov years, the survivers ov a body ov lives originally select, form a mixt body, consisting ov comparativly few damaged lives, with a majority ov lives stil helthy; and that the value ov a policy on a life stil select, is les than the average value calculated on the asumtion that al the lives ov the same age, and insured for the same length ov time, ar in the same state ov helth. We hav also seen how we can calculate both the average values ov policys and the values ov policys on lives that ar stil select. It remains to complete this part ov the subject by shoing how to calculate the number ov damaged lives among a body ov persons that hav been asured for any time, and the values ov their policys; and we shal then see that the sum ov the values ov the policys on the select lives and ov the values ov the policys on the damaged lives, is together equal to the sum ov the average policy-values for the total number ov lives.

In the first place it wil be convenient to consider lives that hav been five years or more insured, and hav therefore, acording to our fundamental supposition, ataind the ultimat rate ov mortality for their age. Turning to Tabl No. 2, let us in the first instance consider the 862,820 select lives ov 30. We see that out of these ther survive at the end ov 5 years 829,800, who ar then ov the age 35. Som ov these wil be stil select lives and some wil be damaged, and the rates ov mortality prevailing among these two classes wil, ov cours, be widely different; but the 829,800 wil as a whole be subject to the mortality ov the $H^{M(5)}$ tabl. (It is true that acording to the tabl the $H^{M(5)}$ mortality is reacht at the age ov 33, but this dos not in any way afect our argument.) After the laps ov 5 more years, or at the age ov 40, the number ov survivers wil be 786,500. Now acording to the construction ov the tabl, this is also the number ov survivers out ov 820,928 select lives ov 35, that is to say, the 820,928 select lives and the 829,800 mixt lives ov 35, wil giv us after 5 years the same number ov survivers, who wil thenceforward be subject to the $H^{M(5)}$ mortality. It is therefore clear that the difference, 8,872, between these numbers must be the number ov damaged lives included among the latter; and consistently with our supposition that the efect ov selection wears of in 5 years at most, these damaged lives must al be ded before the end ov 5 years, or non ov them wil atain the age ov 40. It wil be useful to consider this point a litl more closely. When we say that the benefit ov selection wears of in five years, we mean that after 5 years a body ov lives originally select wil contain the proportion ov damaged lives that wil giv us the $H^{M(5)}$ rate ov mortality; and that after 5 years the the survivers ov a body ov select lives wil be subject to the same rate ov mortality as the survivers ov a body ov mixt lives ov the same age; and it is clear that this can only be the case if the damaged lives containd among these mixt lives al dy within the 5 years. If we now trace from year to year the number ov survivers ov the 820,928 select lives and 829,800 mixt lives, as set out for greater distinctnes in the folloing tabl, it is easy to see that the differences, as shoen in colum (5), must be the survivers from year to year out ov the 8,872 damaged lives, until, when we com to the end ov the fifth year, ther is no difference, or al the 8,872 ar ded.

Age x $=35+t$	Years elapst t	Survivers ov			Number ov Damaged Lives dying in a Year
		Select Lives $l_{[35]+t}$	Mixt Lives $l_x=l_{35+t}$	Damaged Lives $(ul)_{(35)+t}$	$(ud)_{(35)+t}$
(1)	(2)	(3)	(4)	(5)=(4)—(3)	(6)
35	0	820,928	829,800	8,872	4,278
36	1	816,906	821,500	4,594	2,304
37	2	810,710	813,000	2,290	1,358
38	3	803,368	804,300	932	726
39	4	795,194	795,400	206	206
40	5	786,500	786,500	0	0

The differences ov the numbers in the fifth colum wil giv us the number ov damaged lives dying in each year, as shoen in the last colum. These figurs may also be got by means ov the numbers dying, as containd in Tabl No. 3. Thus we hav

Age x $=35+t$	Years elapst t	Numbers Dying in the folloing Year.		
		Select $d_{[35]+t}$	Mixt $d_x=d_{35+t}$	Damaged $(ud)_{(35)+t}$
(1)	(2)	(3)	(4)	(5)=(4)—(3)
35	0	4,022	8,300	$4{,}278=(ud)_{35}$
36	1	6,196	8,500	$2{,}304=(ud)_{(35)+1}$
37	2	7,342	8,700	$1{,}358=(ud)_{(35)+2}$
38	3	8,174	8,900	$726=(ud)_{(35)+3}$
39	4	8,694	8,900	$206=(ud)_{(35)+4}$
40	5	8,900	8,900	$0=(ud)_{(35)+5}$

We thus see that, if we denote by $(ul)_x$ the number ov damaged (or *un*helthy) lives among the l_x mixt lives, we shal hav $(ul)_x=l_x-l_{[x]}$; if, furthermore, $(ul)_{(x)+t}$ denotes the number ov these damaged lives that ar alive after the laps ov t years, $(ul)_{(x)+t}=l_{x+t}-l_{[x]+t}$; and if $(ud)_{(x)+t}$ denotes the number ov these lives that dy in the $(t+1)$th year, $(ud)_{(x)+t}=d_{x+t}-d_{[x]+t}$. Acording to the suppositions we hav made, when $t=$ or >5, $(ul)_{(x)+t}=0$, $(ud)_{(x)+t}=0$. Colum (8) ov Tabl No. 2 contains the values ov $(ul)_x$, or the number ov damaged lives calculated in this way; colums (9) to (12) sho the numbers ov these surviving after 1, 2, 3, 4 years; and colums (13) to (17) sho how many ov them dy in successiv years up to the fifth. In each case the numbers ar, as in the previos tabls, to be redd diagonally downwards from left to riht.

This tabl, it is to be observd, puts our ajusted tabl to a very severe test, which I think it stands on the whole very satisfactorily. It is true that the numbers in colums (11), (12), (16), (17), sho considerabl iregularitys; but these arise from the unsatisfactory graduation ov the $H^{M(5)}$ tabl, and they wil afect the money values very slihtly.

In what precedes we hav shoen that, if l_x represents the survivers ov a number ov lives now ov the age x, who hav been insured for 5 years or more, then the number ov those lives that ar stil select (or helthy) is $l_{[x]}$, and the number ov damaged lives among them, $(ul)_x$, is the difference, $l_x - l_{[x]}$. We hav next to determin the number ov helthy and damaged lives among a body ov lives ov any age that hav been insured for les than 5 years. In order to fix the ideas, let us first supose the lives to be now ov the age 35, and to hav been insured for 1 year. Then Tabl No. 2 shos us that 825,510 such lives wil giv after the laps ov 5 years 786,500 mixt lives; but, from the construction ov the tabl, this is also the number ov survivers out ov 820,928 select lives ov the age 35. Tracing by means ov the figurs in Tabl No. 2, the survivers out ov the two bodys, as shoen in the folloing tabl, we get the numbers ov these damaged lives who survive after the laps ov 1, 2, 3, 4, 5 years.

Age x	Years elapst t	Respectiv number ov Survivers after n Years.			Number ov the Damaged Lives who Dy in a Year $(ud)_{([34]+1)+t}$
		$l_{[35]+t}$	$l_{[34]+t+1}$	$(ul)_{([34]+1)+t}$	
(1)	(2)	(3)	(4)	(5)=(4)−(3)	(6)
35	0	820,928	825,510	4,582	2,136
36	1	816,906	819,352	2,446	1,106
37	2	810,710	812,050	1,340	748
38	3	803,368	803,960	592	386
39	4	795,194	795,400	206	206
40	5	786,500	786,500	0	0

The numbers in the last colum may also be got from the numbers dying givn in Tabl No. 3, as follos:—

Age x	Years elapst t	$d_{[35]+t}$	$d_{[34]+t+1}$	Numbers Dying in the folloing Year.
(1)	(2)	(3)	(4)	(5)=(4)−(3)
35	0	4,022	6,158	$2{,}136 = (ud)_{([34]+1)}$
36	1	6,196	7,302	$1{,}106 = (ud)_{([34]+1)+1}$
37	2	7,342	8,090	$748 = (ud)_{([34]+1)+2}$
38	3	8,174	8,560	$386 = (ud)_{([34]+1)+3}$
39	4	8,694	8,900	$206 = (ud)_{([34]+1)+4}$
40	5	8,900	8,900	0

We now see that for any age x, $l_{[x]}$ is the number ov helthy lives among $l_{[x-1]+1}$ lives who hav been insured for 1 year; and the same reasoning wil sho us that it is also the number ov helthy lives among $l_{[x-2]+2}$ lives who hav been insured for 2 years, among $l_{[x-3]+3}$ lives who hav been insured for 3 years, and among $l_{[x-4]+4}$ lives who hav been insured for 4 years; and we saw previosly that it is also the number among l_x lives who hav been insured for 5 years or more. It wil be convenient to hav a distinctiv notation to denote the varios numbers ov damaged lives. If we put $(ul)_{[x]+n}$ to denote the number ov damaged lives containd in $l_{[x]+n}$ mixt lives who hav been insured for n years, we hav $(ul)_{[x]+n}=l_{[x]+n}-l_{[x+n]}$; and putting $(ul)_{([x]+n)+t}$ for the number ov the abov-mentiond damaged lives that stil survive after t years, al being ov the age $x+n+t$; and $(ud)_{([x]+n)+t}$ for the number ov them who dy in the $(t+1)$th year, that is to say, between the ages $x+n+t$ and $x+n+t+1$, we hav

$$(ul)_{([x]+n)+t}=l_{[x]+n+t}-l_{[x+n]+t};$$

and $$(ud)_{([x]+n)+t}=d_{[x]+n+t}-d_{[x+n]+t}=l_{[x]+n+t}-l_{[x]+n+t+1}$$
$$-l_{[x+n]+t}+l_{[x+n]+t+1}=l_{[r]+n+t}-l_{[x+n]+t}-(l_{[x]+n+t+1}-l_{[x+n]+t+1})$$
$$=(ul)_{([x]+n)+t}-(ul)_{([x]+n)+t+1}.$$

Taking a general vew now ov our extended tabl ov l_x, we see that, since the radix ov the $H^{M(5)}$ tabl is 10,000 persons alive at the age 10, the number at any age x in our colum l_x, is the number ov survivers out ov 1,000,000 select lives ov 10, and is also the number ov survivers out ov $l_{[y]}$ select lives ov the age y, where $y=$ or $<x-5$; for instance, 551,600 mixt lives ov the age ov 60 ar the survivers out ov 611,313 select lives ov 55, or out ov 624,683 select lives ov 54, or out ov 674,923 select lives ov 50, or out ov 862,820 select lives ov 30, and so on. Again, we see that $l_{[x]}$ is the number ov lives stil select among al the numbers ov the same age x in the different colums ov our tabl; and the number ov damaged lives among the number ov mixt lives in any ov those colums, is found by subtracting from it the corresponding number $l_{[x]}$. We further see that $l_{[x]}$ is the number ov helthy lives among the survivors of $l_{[x-1]}$ select lives who hav been insured for one year, among $l_{[x-2]}$ select lives who hav been insured for two years, and in general among $l_{[x-n]}$ select lives who hav been insured for n years. If we supose that the ultimat rate ov mortality is not ataind after 5 years, but after a longer term, for instance 10 years, we may construct a tabl with similar propertys, which ar suficiently obvios without explanation.

It wil now be useful to sho by an exampl the gradual deterioration that takes place among a body ov lives originally select. Taking 820,928 select lives ov 35, we hav seen that the number ov damaged lives at the end ov a year is $l_{[35]+1}-l_{[36]}=4{,}568$. The number ov select lives among the survivers being $l_{[36]}$, the number ov those that becom damaged in the next insurance year is obviosly $l_{[36]+1}-l_{[37]}=4{,}506$, the number that becom damaged in the folloing insurance year is $l_{[37]+1}-l_{[38]}=4{,}460$, and so on; and we hav seen abov how to ascertain the number ov survivers out ov these varios numbers ov damaged lives, after the laps ov any number ov years. The folloing tabl shos (in colum (2)) the number ov the 820,928 lives becoming damaged in each of the first 5 insurance years, and (in colums (3) to (6)) the respectiv numbers ov them surviving, until they ar al ded; and the last colum shos the total numbers ov these survivers at the end ov each ov the 10 years from the date ov entry. It wil be notist that the number ov damaged lives existing at the age 40, and the numbers ov them surviving in successiv years, agree with the figurs givn in colums (8) to (12) ov Tabl No. 2.

Age x	$(ul)_{[x-1]+1}$	$(ul)_{([x-2]+1)+1}$	$(ul)_{([x-3]+1)+2}$	$(ul)_{([x-4]+1)+3}$	$(ul)_{([x-5]+1)+4}$	
(1)	(2)	(3)	(4)	(5)	(6)	(7)=(2)+(3)+(4)+(5)+(6)
35	...	...	...	...	...	...
36	4,568	...	...	...	...	$4{,}568=(ul)_{[35]+1}$
37	4,506	2,437	...	...	...	$6{,}943=(ul)_{[35]+2}$
38	4,460	2,374	1,338	...	...	$8{,}172=(ul)_{[35]+3}$
39	4,426	2,328	1,272	560	...	$8{,}586=(ul)_{[35]+4}$
40	4,395	2,287	1,223	488	112	$8{,}505=(ul)_{40}$
41	...	2,245	1,172	466	0	$3{,}883=(ul)_{(40)+1}$
42	...	...	1,119	453	22	$1{,}594=(ul)_{(40)+2}$
43	...	...	...	420	22	$442=(ul)_{(40)+3}$
44	...	...	...	...	33	$33=(ul)_{(40)+4}$
45	...	...	...	...	...	...

Having by the proces explaind abov determind the number ov damaged lives, $(ul)_{[x]+n}$, among a body ov mixt lives ov any age and standing, $l_{[x]+n}$, and the number ov these damaged lives who dy in each year until they ar al ded, it is obvios that we can calculate without difficulty the rate ov mortality among these damaged lives in each year. The folloing tabls sho these rates ov mortality (1) among the damaged lives ov ages 15, 20, 25, &c., containd in bodys ov lives ov those ages who hav been insured for five years or upwards, and (2) among the damaged lives ov the same ages containd in bodys ov lives who hav been insured for only one year.

Values ov $(uq)_{(x)+t}$, *for Quinquennial Ages.*

x	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	x
15	1·000	...	...	...	...	15
20	·752	1·000	...	...	...	20
25	·620	·725	·886	1·000	...	25
30	·635	·752	1·000	...	...	30
35	·482	·502	·593	·779	1·000	35
40	·543	·589	·723	·925	1·000	40
45	·447	·475	·570	·719	1·000	45
50	·447	·501	·592	·764	1·000	50
55	·497	·580	·707	·881	1·000	55
60	·553	·694	·828	1·000	...	60
65	·663	1·000	...	...	...	65
70	·691	1·000	...	...	...	70
75	·684	1·000	...	...	...	75

Values ov $(uq)_{([x]+1)+t}$, *for Quinquennial Ages.*

$[x]+1$	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$[x]+1$
15	1·000	...	...	...	...	15
20	·718	1·000	...	...	...	20
25	·522	·670	·874	1·000	...	25
30	·564	·715	1·000	...	...	30
35	·466	·452	·558	·652	1·000	35
40	·489	·502	·625	·921	1·000	40
45	·403	·372	·484	·648	1·000	45
50	·397	·397	·498	·676	1·000	50
55	·440	·529	·634	·840	1·000	55
60	·449	·622	·799	1·000	...	60
65	·491	1·000	...	...	...	65
70	·562	1·000	...	...	...	70
75	·538	1·000	...	...	...	75

The iregularitys which exist among the results here tabulated proov that our results admit ov being amended. As I hav previosly intimated, this is to a great extent due to the iregularitys in the $H^{M(5)}$ Tabl; and I believ I shoud hav obtaind much better results if, insted ov adhering strictly to that tabl, I had taken the original observations and myself graduated them afresh. It was, however, my desire to adopt without alteration a tabl which apears to be very hihly apreciated by the profession.

We ar now in a position to calculate the increas in the rate ov mortality that is produced by the withdrawal ov any number ov helthy lives. Taking, for instans, the 829,800 mixt lives ov the age ov 35, we hav seen that the number ov damaged lives among them is 8,872, and the number ov helthy lives, 820,928. The mixt lives, taken as a whole, ar subject to the $H^{M(5)}$ rate ov mortality; but, as we hav seen, the damaged and the select lives considerd separatly ar subject to entirely different rates ov mortality. Supose now that one-tenth ov the helthy lives (82,093) withdraw, leaving 738,835 under observation; then the deaths among the remaining helthy lives wil be reduced proportionatly, namely, from 4022 to 3620, while the deths among the damaged lives wil remain as before, namely, 4278. Adding these together, we get a total ov 7898 deths out ov 747,707 lives, so that the rate ov mortality is ·010563, which is about 5 per-cent greater than the $H^{M(5)}$ rate ov mortality, ·010002. In the same way we may calculate the rate ov mortality in each ov the five years until the 8,872 damaged lives ar al ded. The rates ov mortality thus resulting ar containd in colum (3) ov the subjoind tabl. If we supose a larger number ov the helthy lives to withdraw, it is clear that the rate ov mortality among the lives remaining under observation wil be increast; and the greater the proportion ov helthy lives that withdraw, the hiher wil be the rate ov mortality. In the folloing tabl I hav givn the rates ov mortality that woud result, acording to our tabls, from the withdrawal, not only ov 10 per-cent, but also ov 20, 30, 40, and 50 per-cent, ov the helthy lives; and it wil be seen that the withdrawal ov one-haf the helthy lives has the efect ov increasing the rate ov mortality in the folloing year almost 50 per-cent.

Tabl shoing the Rate ov Mortality among a body ov Lives, age 35, that hav been Insured for 5 or more years, on the Asumption that a varying Proportion ov the Helthy Lives withdraw.

Age.	PROPORTION OV HELTHY LIVES WITHDRAWING.						Age.
	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
35	·010002	·010563	·011261	·012157	·013346	·014999	35
36	·010347	·010651	·011031	·011519	·012168	·013073	36
37	·010701	·010883	·011111	·011404	·011793	·012340	37
38	·011065	·011165	·011289	·011448	·011661	·011958	38
39	·011189	·011218	·011254	·011300	·011363	·011451	39
40	·011316	·011316	·011316	·011316	·011316	·011316	40

These ar calculated by the formula: Probability ov dying in the $(n+1)$th year, if $m:10$ ov the helthy lives withdraw at the age 35,

$=\dfrac{d_{35+n}-\frac{1}{10}md_{[35]+n}}{l_{35+n}-\frac{1}{10}ml_{[35]+n}}$. When $n=$ or >5, $d_{[35]+n}=d_{35+n}$, $l_{[35]+n}=l_{35+n}$; and the formula becoms $\dfrac{d_{35+n}(1-\frac{1}{10}m)}{l_{35+n}(1-\frac{1}{10}m)}=\dfrac{d_{35+n}}{l_{35+n}}=q_{35+n}$.

We ar now in a position to sho how to calculate the values ov anuitys and asurances on the damaged lives ov our tabl. Confining our atention, in the first instance, to the $(ul)_x$ damaged lives containd in l_x lives that hav been insured for five or more years, and asuming that they ar al in the same state ov helth, we shal hav the value ov an asurance upon any one ov them equal to

$$\{(ud)_x v+(ud)_{(x)+1}v^2+(ud)_{(x)+2}v^3+(ud)_{(x)+3}v^4+(ud)_{(x)+4}v^5\}\div(ul)_x$$
$$=\{(ud)_x v^{x+1}+(ud)_{(x)+1}v^{x+2}+(ud)_{(x)+2}v^{x+3}+(ud)_{(x)+3}v^{x+4}+(ud)_{(x)+4}v^{x+5}\}\div(ul)_x v^x$$
$$=(u\mathrm{M})_x\div(u\mathrm{D})_x,\text{ supose.}$$

This shos us how the columnar method may be extended to the values ov benefits dependent upon the damaged lives. Bearing in mind that $(ul)_x=l_x-l_{[x]}$, we see that $(u\mathrm{D})_x=\mathrm{D}_x-\mathrm{D}_{[x]}$. Also since $(ud)_x=d_x-d_{[x]}$, and $(ud)_{(x)+t}=d_{x+t}-d_{[x]+t}$, we hav

$$\begin{aligned}(u\mathrm{M})_x&=d_x v^{x+1}+d_{x+1}v^{x+2}+d_{x+2}v^{x+3}+d_{x+3}v^{x+4}+d_{x+4}v^{x+5}\\&\quad-d_{[x]}v^{x+1}-d_{[x]+1}v^{x+2}-d_{[x]+2}v^{x+3}-d_{[x]+3}v^{x+4}-d_{[x]+4}v^{x+5}\\&=\mathrm{M}_x-\mathrm{M}_{[x]},\end{aligned}$$

so that we finally get the equation $(u\mathrm{A})_x=\dfrac{(u\mathrm{M})_x}{(u\mathrm{D})_x}=\dfrac{\mathrm{M}_x-\mathrm{M}_{[x]}}{\mathrm{D}_x-\mathrm{D}_{[x]}}$.

Similarly the value ov an anuity-due upon a damaged life is

$$\{(ul)_x+(ul)_{(x)+1}v+(ul)_{(x)+2}v^2+(ul)_{(x)+3}v^3+(ul)_{(x)+4}v^4\}\div(ul)_x$$
$$=\{(ul)_x v^x+(ul)_{(x)+1}v^{x+1}+(ul)_{(x)+2}v^{x+2}+(ul)_{(x)+3}v^{x+3}+(ul)_{(x)+4}v^{x+4}\}\div(ul)_x v^x;$$

and proceeding as before, it is easy to see that we hav

$$(u\mathbf{a})_x=\frac{(u\mathrm{N})_x}{(u\mathrm{D})_x}=\frac{\mathrm{N}_x-\mathrm{N}_{[x]}}{\mathrm{D}_x-\mathrm{D}_{[x]}}.$$

Since $(u\mathbf{a})_x$ and $(u\mathrm{A})_x$ ar the ordinary whole-life anuity-due and asurance, on lives which ar subject to a certain rate ov mortality, the usual wel-knoen relation wil exist between them, $\mathrm{A}=1-(1-v)\mathbf{a}$. This may be reddily proovd. We hav $\mathrm{M}_x=\mathrm{D}_x-(1-v)\mathrm{N}_x$, and $\mathrm{M}_{[x]}=\mathrm{D}_{[x]}-(1-v)\mathrm{N}_{[x]}$. Hence by subtraction, $(u\mathrm{M})_x=(u\mathrm{D})_x-(1-v)(u\mathrm{N})_x$; and dividing by $(u\mathrm{D})_x$, we hav $(u\mathrm{A})_x=1-(1-v)(u\mathbf{a})_x$. When these relations ar uzed for the purpos ov cheking the accuracy ov the values ov the quantitys, it is convenient to put them into the form $(1+i)\mathrm{A}+i\mathbf{a}=1+i$, and $(1+i)\mathrm{M}+i\mathrm{N}=(1+i)\mathrm{D}$.

Colums (2) (3) and (4) ov the folloing tabl giv the values ov $(u\mathrm{D})_x$, $(u\mathrm{N})_x$, and $(u\mathrm{M})_x$, and colums (5) and (6) the values ov $(u\mathbf{a})_x$ and $(u\mathrm{A})_x$ for al ages from 15 to 75. If D_x, N_x, M_x, and

TABL 11.

x (1)	$(uD)_x$ (2)	$(uN)_x$ (3)	$(uM)_x$ (4)	$(ua)_x$ (5)	$(uA)_x$ (6)	x (7)
15	137·71	137·71	132·39	1·0000	·96154	15
16	38·441	38·441	36·963	1·0000	·96154	16
17	51·337	51·337	49·363	1·0000	·96154	17
18	419·58	474·64	401·33	1·1312	·95649	18
19	1037·09	1269·85	988·25	1·2244	·95291	19
20	1954·25	2421·17	1861·13	1·2389	·95235	20
21	3244·74	4849·01	3058·24	1·4944	·94252	21
22	3777·34	5744·21	3556·41	1·5207	·94151	22
23	3866·57	5711·40	3646·90	1·4771	·94319	23
24	3671·04	5348·90	3465·32	1·4571	·94396	24
25	3346·04	4928·08	3156·50	1·4728	·94335	25
26	3019·69	4452·29	2848·45	1·4744	·94329	26
27	2735·00	3919·79	2584·24	1·4332	·94488	27
28	2428·38	3419·65	2296·86	1·4082	·94584	28
29	2197·75	3151·25	2076·54	1·4339	·94485	29
30	1997·91	2865·78	1887·68	1·4344	·94483	30
31	1876·59	2684·34	1773·35	1·4304	·94498	31
32	1966·90	3273·62	1840·99	1·6644	·93599	32
33	2028·85	3648·61	1888·51	1·7984	·93083	33
34	2163·24	4072·97	2006·58	1·8828	·92758	34
35	2248·30	4158·85	2088·35	1·8498	·92886	35
36	2232·49	4040·67	2077·08	1·8099	·93039	36
37	2163·26	3775·93	2018·03	1·7455	·93287	37
38	2051·00	3474·04	1917·38	1·6938	·93485	38
39	1904·53	3184·20	1782·06	1·6719	·93570	39
40	1771·50	2943·86	1658·27	1·6618	·93608	40
41	1692·95	2908·31	1581·09	1·7179	·93393	41
42	1681·76	2983·76	1567·00	1·7742	·93176	42
43	1698·92	3098·89	1579·73	1·8240	·92984	43
44	1772·81	3341·67	1644·28	1·8850	·92750	44
45	1884·55	3657·68	1743·87	1·9409	·92535	45
46	1975·37	3893·88	1825·60	1·9712	·92418	46
47	2043·43	4046·88	1887·78	1·9804	·92383	47
48	2071·68	4037·07	1916·40	1·9487	·92505	48
49	2075·12	4013·05	1920·77	1·9339	·92562	49
50	2037·10	3891·82	1887·41	1·9105	·92652	50
51	1990·14	3765·48	1845·31	1·8921	·92723	51
52	1937·01	3632·77	1797·29	1·8755	·92787	52
53	1898·91	3521·85	1763·46	1·8547	·92866	53
54	1830·33	3310·64	1703·00	1·8088	·93043	54
55	1756·46	3057·70	1638·86	1·7408	·93305	55
56	1693·24	2864·79	1583·06	1·6919	·93493	56
57	1636·14	2679·68	1533·07	1·6378	·93701	57
58	1592·64	2560·97	1494·14	1·6080	·93815	58
59	1576·96	2505·39	1480·60	1·5887	·93889	59
60	1581·52	2494·91	1485·56	1·5775	·93933	60
61	1588·24	2512·84	1491·59	1·5822	·93915	61
62	1577·69	2452·49	1483·36	1·5545	·94021	62
63	1496·73	2160·49	1413·63	1·4435	·94448	63
64	1381·96	1837·63	1311·28	1·3297	·94886	64
65	1347·71	1784·58	1279·07	1·3242	·94907	65
66	1302·11	1708·02	1236·42	1·3117	·94955	66
67	1247·84	1626·74	1185·27	1·3036	·94986	67
68	1193·94	1543·65	1134·57	1·2929	·95027	68
69	1156·84	1496·37	1099·29	1·2935	·95025	69
70	1149·66	1491·07	1092·31	1·2970	·95012	70
71	1165·89	1521·90	1107·36	1·3054	·94979	71
72	1192·89	1564·61	1132·72	1·3116	·94955	72
73	1213·52	1597·07	1152·10	1·3161	·94938	73
74	1187·16	1545·98	1127·70	1·3023	·94991	74
75	1181·05	1474·24	1074·85	1·3034	·94987	75

$D_{[x]}$, $N_{[x]}$, $M_{[x]}$, had been calculated to a suficient number ov places, the values ov $(uD)_x$, &c., miht hav been got with suficient accuracy by simpl subtraction; but it was found that this was not the case, and I therefore had the values ov these quantitys computed independently.

We hav seen that the l_x lives ov the $H^{M(5)}$ Tabl include $(ul)_x$ damaged lives and $l_{[x]}$ lives which ar stil select. It wil be convenient in what follos to use the symbol $(hl)_x$ somtimes, insted ov $l_{[x]}$, altering similarly the expressions for the anuitys, the asurances, and the commutation functions. It is clear that the value ov the anuitys upon the l_x mixt lives ov any age must be equal to the sum ov the anuitys on the corresponding $(hl)_x$ select lives and $(ul)_x$ damaged lives. The same thing apears from the formulas that we hav obtaind. We hav

$$(hl)_x . (h\mathbf{a})_x = (hl)_x \cdot \frac{(hN)_x}{(hD)_x} = (hl)_x \cdot \frac{(hN)_x}{(hl)_x v^x} = \frac{1}{v^x}(hN)_x .$$

Also $$(ul)_x . (u\mathbf{a})_x = (ul)_x \cdot \frac{(uN)_x}{(uD)_x} = (ul)_x \cdot \frac{(uN)_x}{(ul)_x v^x} = \frac{1}{v^x}(uN)_x .$$

$$\therefore\ (hl)_x . (h\mathbf{a})_x + (ul)_x . (u\mathbf{a})_x = \frac{1}{v^x}\{(hN)_x + (uN)_x\} = \frac{1}{v^x}N_x = l_x . \mathbf{a}_x .$$

In a similar way we may proov that $(hl)_x . (hA)_x + (ul)_x . (uA)_x = l_x . A_x$.

We hav a corresponding formula for the policy-values. Let P be the premium valued in finding the policy-value, and let ${}_n(uV)_x$ denote the value ov a policy taken out n years ago (n being >4), on a life which was then ov the age x, and is now one ov the damaged lives ov our tabl, and ov the age $x+n$. Then ${}_n(uV)_x = (uA)_{x+n} - (u\mathbf{a})_{x+n} . P$; also ${}_n(hV)_x = (hA)_{x+n} - (h\mathbf{a})_{x+n} . P$, and ${}_nV_x = A_{x+n} - \mathbf{a}_{x+n} . P$; then by means ov the relations proovd abov,

$$\begin{aligned}(hl)_{x+n} \cdot {}_n(hV)_x + (ul)_{x+n} \cdot {}_n(uV)_x &= (hl)_{x+n}\{(hA)_{x+n} - (h\mathbf{a})_{x+n} . P\} \\ &\quad + (ul)_{x+n}\{(uA)_{x+n} - (u\mathbf{a})_{x+n} . P\}\end{aligned}$$
$$= (hl)_{x+n}(hA)_{x+n} + (ul)_{x+n}(uA)_{x+n} - \{(hl)_{x+n}(h\mathbf{a})_{x+n} + (ul)_{x+n}(u\mathbf{a})_{x+n}\} P$$
$$= l_{x+n}A_{x+n} - l_{x+n}\mathbf{a}_{x+n}P = l_{x+n} \cdot {}_nV_x .$$

Hence we get $${}_n(uV)_x = \frac{l_{x+n} \cdot {}_nV_x - l_{[x+n]} \cdot {}_n(hV)_x}{l_{x+n} - l_{[x+n]}} .$$

We see that these relations hold whatever net premium is valued. A numerical exampl wil perhaps place them in a clearer liht.

If the premium valued is $\varpi_{[x]}$, then $100{}_nV_x$ wil be the average policy-value tabulated in colum (4) ov Tabl 10, and $100{}_n(hV)_x$ wil be the value tabulated in colum (5) ov the same tabl. As an illustration ov the abov theory, we wil take the case ov a policy efected at the age 35 which has been 5 years in force. We hav $\varpi_{[35]}=\cdot019915$, (Tabl 8); $(hl)_{40}=777{,}995$, $(ul)_{40}=8{,}505$, $l_{40}=786{,}500$, (Tabl 2); ${}_5V_{35}=\cdot06872$, ${}_5(hV)_{35}=\cdot05960$, (Tabl 10); $(u\mathrm{a})_{40}=1\cdot6618$, $(uA)_{40}=\cdot93608$, (Tabl 11); whence ${}_5(uV)_{35}=(uA)_{40}-(u\mathrm{a})_{40}\varpi_{[35]}$ $=\cdot93608-1\cdot6618\times\cdot019915=\cdot90299$. Hence

$$\begin{aligned}(hl)_{40}\times{}_5(hV)_{35}&=777{,}995\times\cdot05960=46{,}368\\(ul)_{40}\times{}_5(uV)_{35}&=\underline{8{,}505}\times\cdot90299=\underline{7{,}680}\\l_{40}\times{}_5V_{35}\quad&=\underline{\underline{786{,}500}}\times\cdot06872=\underline{\underline{54{,}048}}\end{aligned}$$

If the premium valued is $\varpi_{[x+1]}$, then for the mixt lives we hav the policy-value, $V_{([x+1],\,[x]+n)}$, which is tabulated in colum 8 ov the tabl; for the helthy lives we hav ${}_{n-1}(hV)_{x+1}$, (colum 9); and for the unhelthy, we hav a value which may be denoted by $(uV)_{([x+1],\,[x]+n)}$. Taking the same exampl as before, $\varpi_{[36]}=\cdot020563$, and we hav $(uV)_{([36],\,[35]+5)}=(uA)_{40}-(u\mathrm{a})_{40}\cdot\varpi_{[36]}=\cdot93608-1\cdot6618$ $\times\cdot020563=\cdot90191$. The other values can be similarly got, and we hav

$$\begin{aligned}(hl)_{40}\times{}_4(hV)_{36}\qquad&=777{,}995\times\cdot04915=38{,}238\\(ul)_{40}\times(uV)_{([36],\,[35]+5)}&=\underline{8{,}505}\times\cdot90191=\underline{7{,}670}\\l_{40}\times V_{([36],\,[35]+5)}\quad&=\underline{\underline{786{,}500}}\times\cdot05837=\underline{\underline{45{,}908}}\end{aligned}$$

We hav now to consider the $(ul)_{[x]+n}$ damaged lives containd in $l_{[x]+n}$ mixt lives, where $n<5$. If we extend the notation ov the commutation colums in the same way as we hav extended the notation for the numbers living and dying, we hav

$$\begin{aligned}(uD)_{([x]+n)+t}&=v^{x+n+t}.\,(ul)_{([x]+n)+t}=v^{x+n+t}(l_{[x]+n+t}-l_{[x+n]+t})\\&=D_{[x]+n+t}-D_{[x+n]+t}.\end{aligned}$$

But it is only necesary to deal with the case of $t=0$, when we hav $(uD)_{[x]+n}=D_{[x]+n}-D_{[x+n]}$.

By similar reasoning, $(u\mathrm{N})_{[x]+n}=\mathrm{N}_{[x]+n}-\mathrm{N}_{[x+n]}$,

and $\qquad (uM)_{[x]+n}=M_{[x]+n}-M_{[x+n]}$.

Furthermore, $(u\mathrm{a})_{[x]+n}=\dfrac{(u\mathrm{N})_{[x]+n}}{(uD)_{[x]+n}}=\dfrac{\mathrm{N}_{[x]+n}-\mathrm{N}_{[x+n]}}{D_{[x]+n}-D_{[x+n]}}$;

and $\qquad (uA)_{[x]+n}=\dfrac{(uM)_{[x]+n}}{(uD)_{[x]+n}}=\dfrac{M_{[x]+n}-M_{[x+n]}}{D_{[x]+n}-D_{[x+n]}}$.

It is also easy to see that we hav

$$(uM)_{[x]+n}=(uD)_{[x]+n}-(1-v)\,(u\mathrm{N})_{[x]+n},$$

and $$(uA)_{[x]+n}=1-(1-v)\,(u\mathrm{a})_{[x]+n}.$$

Also

$$(hl)_{[x]+n}\cdot(h\mathrm{a})_{[x]+n}=(hl)_{[x]+n}\cdot\frac{(h\mathrm{N})_{[x]+n}}{(h\mathrm{D})_{[x]+n}}=\frac{(hl)_{[x]+n}\cdot(h\mathrm{N})_{[x]+n}}{(hl)_{[x]+n}\cdot v^{x+n}}=\frac{1}{v^{x+n}}\cdot(h\mathrm{N})_{[x]+n},$$

$$(ul)_{[x]+n}\cdot(u\mathrm{a})_{[x]+n}=(ul)_{[x]+n}\cdot\frac{(u\mathrm{N})_{[x]+n}}{(u\mathrm{D})_{[x]+n}}=\frac{(ul)_{[x]+n}\cdot(u\mathrm{N})_{[x]+n}}{(ul)_{[x]+n}\cdot v^{x+n}}=\frac{1}{v^{x+n}}\cdot(u\mathrm{N})_{[x]+n};$$

$$\therefore\quad (hl)_{[x]+n}\cdot(h\mathrm{a})_{[x]+n}+(ul)_{[x]+n}\cdot(u\mathrm{a})_{[x]+n}=\frac{1}{v^{x+n}}\left\{(h\mathrm{N})_{[x]+n}+(u\mathrm{N})_{[x]+n}\right\}$$
$$=\frac{1}{v^{x+n}}\cdot\mathrm{N}_{[x]+n}=l_{[x]+n}\cdot\mathrm{a}_{[x]+n}.$$

Similarly we may sho that $(hl)_{[x]+n}\cdot(h\mathrm{A})_{[x]+n}+(ul)_{[x]+n}\cdot(u\mathrm{A})_{[x]+n}$ $=l_{[x]+n}\cdot\mathrm{A}_{[x]+n}$. From these relations it at once follos that, whatever the net premium valued, $(hl)_{[x]+n}\cdot{}_n(h\mathrm{V})_x+(ul)_{[x]+n}\cdot{}_n(u\mathrm{V})_x$ $=l_{[x]+n}\cdot{}_n\mathrm{V}_x$. When $n=$or >5, al these formulas agree with those previosly obtaind.

As a numerical exampl we wil agen take the case ov a policy efected at the age 35, but supose that it has only been one year in force. First valuing the premium $\varpi_{[35]}={\cdot}019915$, we hav (Tabl 10) ${}_1\mathrm{V}_{35}={\cdot}01589$, and ${}_1(h\mathrm{V})_{35}={\cdot}01099$; $(u\mathrm{A})_{[35]+1}$ $=\dfrac{\mathrm{M}_{[35]+1}-\mathrm{M}_{[36]}}{\mathrm{D}_{[35]+1}-\mathrm{D}_{[36]}}={\cdot}92558$, $(u\mathrm{a})_{[35]+1}=\dfrac{\mathrm{N}_{[35]+1}-\mathrm{N}_{[36]}}{\mathrm{D}_{[35]+1}-\mathrm{D}_{[36]}}=1{\cdot}9147$, and ${}_1(u\mathrm{V})_{35}=(u\mathrm{A})_{[35]+1}-(u\mathrm{a})_{[35]+1}.\varpi_{[35]}={\cdot}92558-1{\cdot}9147\times{\cdot}019915$ $={\cdot}88745$. Also, from Tabl 2, $l_{[35]+1}=816906$, $(hl)_{[35]+1}=l_{[36]}$ $=812338$, $(ul)_{[35]+1}=l_{[35]+1}-l_{[36]}=4568$. Hence

$$\begin{array}{lrcl}
(hl)_{[35]+1}\times{}_1(h\mathrm{V})_{35}= & 812338 & \times{\cdot}01099= & 8927\\
(ul)_{[35]+1}\times{}_1(u\mathrm{V})_{35}= & \underline{4568} & \times{\cdot}88745= & \underline{4054}\\
l_{[35]+1}\times{}_1\mathrm{V}_{35}\quad = & \underline{\underline{816906}} & \times{\cdot}01589= & \underline{\underline{12981}}
\end{array}$$

If the premium valued is $\varpi_{[36]}={\cdot}020563$, we have $\mathrm{V}_{([36].[35]+1)}$ $={\cdot}00495$, ${}_0(h\mathrm{V})_{36}=0$, $(u\mathrm{V})_{([36].[35]+1)}=(u\mathrm{A})_{[35]+1}-(u\mathrm{a})_{[35]+1}\cdot\varpi_{[36]}$ $={\cdot}92558-1{\cdot}9147\times{\cdot}020563={\cdot}88621$. We then get

$$\begin{array}{lrcl}
(hl)_{[35]+1}\times{}_0(h\mathrm{V})_{36} & =812338 & \times\ 0 \quad = & 0\\
(ul)_{[35]+1}\times(u\mathrm{V})_{([36].[35]+1)}= & \underline{4568} & \times{\cdot}88621= & \underline{4048}\\
l_{[35]+1}\times\mathrm{V}_{([36].[35]+1)} & =\underline{\underline{816906}} & \times{\cdot}00495= & \underline{\underline{4044}}^{*}
\end{array}$$

As another instans ov the use ov our tabls in calculating the value ov options to be exercised by the asured, let us supose that, as part ov som bisnes arangement, an offis enters into an agreement that it wil upon aplication n years hence, if a person now ov the age x shal be then alive, grant a policy on his life at the ordinary rate ov premium. If the person is alive and stil in good helth, it is clear the company wil be at no disadvantage in then granting a policy at the ordinary rate ov premium. The risk, therefore, which the company runs is, that the life wil be stil in existence at the end ov n years but hav becom damaged; and we hav seen that the probability ov this is $\frac{(ul)_{x+n}}{l_{[x]}} = \frac{l_{x+n} - l_{[x+n]}}{l_{[x]}}$. Insted ov confining ourselves to the case ov a singl contract ov this kind, it is convenient, as in most questions ov life contingencys, to supose that the company enters into $l_{[x]}$ contracts ov the kind. Then our tabl shos us that at the end ov n years ther wil be $(ul)_{x+n}$ damaged lives who wil dy of within 5 years; and the sum which the company shoud hav in hand at the outset in order to provide for these claims is $(ud)_{x+n}v^{n+1} + (ud)_{(x+n)+1}v^{n+2} + (ud)_{(x+n)+2}v^{n+3} + (ud)_{(x+n)+3}v^{n+4} + (ud)_{(x+n)+4}v^{n+5}$, and the singl premium therefore to be charged in the case ov one life is the same quantity divided by $l_{[x]}$

---

* We shoud hav got a closer agreement if, insted of taking the value ov $\mathrm{V}_{([36].[35]+1)}$ at ·00495 according to Tabl 10, we had taken the more correct value, ·004955.

$$= \frac{1}{l_{[x]}v^x}\Big((ud)_{x+n}v^{x+n+1} + (ud)_{(x+n)+1}v^{x+n+2} + (ud)_{(x+n)+2}v^{x+n+3} + (ud)_{(x+n)+3}v^{x+n+4}$$

$$+ (ud)_{(x+n)+4}v^{x+n+5}\Big) = \frac{(u\mathrm{M})_{x+n}}{\mathrm{D}_{[x]}} = \frac{(u\mathrm{D})_{x+n}}{\mathrm{D}_{[x]}} \cdot \frac{(u\mathrm{M})_{x+n}}{(u\mathrm{D})_{x+n}}$$

$= \frac{(u\mathrm{D})_{x+n}}{\mathrm{D}_{[x]}} \cdot (u\mathrm{A})_{x+n}$. The analogy ov this formula to an ordinary deferd asurance is obvios. Hitherto we hav left out ov acount the premiums which the company is to receiv. Supose for the present that this premium is P. Then the value ov the premiums to be receivd on the damaged lives wil be

$$\{(ul)_{x+n}v^n + (ul)_{(x+n)+1}v^{n+1} + (ul)_{(x+n)+2}v^{n+3} + (ul)_{(x+n)+3}v^{n+4} + (ul)_{(x+n)+4}v^{n+5}\}\mathrm{P},$$

and the chance ov the company receiving such premiums in respect ov a particular nominated life is this quantity divided by $l_{[x]}$

$$= \frac{(ul)_{x+n}v^{x+n} + (ul)_{(x+n)+1}v^{x+n+1} + (ul)_{(x+n)+2}v^{x+n+2} + (ul)_{(x+n)+3}v^{x+n+3} + (ul)_{(x+n)+4}v^{x+n+4}}{l_{[x]}v^x}$$

$$= \frac{(u\mathrm{N})_{x+n}}{\mathrm{D}_{[x]}} \cdot \mathrm{P} = \frac{(u\mathrm{D})_{x+n}}{\mathrm{D}_{[x]}} \cdot \frac{(u\mathrm{N})_{x+n}}{(u\mathrm{D})_{x+n}} \cdot \mathrm{P} = \frac{(u\mathrm{D})_{x+n}}{\mathrm{D}_{[x]}} \cdot (u\mathbf{a})_{x+n} \cdot \mathrm{P},$$

being analogos to an ordinary deferd anuity. We may use the symbols ${}_{n}(u\mathbf{a})_{[x]}$ and ${}_{n}(u\mathrm{A})_{[x]}$ to denote the deferd anuitys and asurances we hav been here considering. Combining our results, the singl premium that the company shoud receiv at the outset in consideration ov entering into the suposed contract is

$\frac{(u\mathrm{D})_{x+n}}{\mathrm{D}_{[x]}} \cdot \Big((u\mathrm{A})_{x+n} - (u\mathbf{a})_{x+n} \cdot \mathrm{P}\Big)$; and if $\mathrm{P} = \varpi_{[x+n]}$, then

$$(u\mathrm{A})_{x+n}-(u\mathbf{a})_{x+n}.\mathrm{P}=(u\mathrm{A})_{x+n}-(u\mathbf{a})_{x+n}.\varpi_{[x+n]}=(u\mathbf{a})_{x+n}\{(u\varpi)_{x+n}-\varpi_{[x+n]}\}$$
$$=1-\frac{(u\mathbf{a})_{x+n}}{\mathbf{a}_{[x+n]}}.$$

On the supposition that the risk has turnd out in favor ov the asured and agenst the company, this quantity is the value to the asured ov the policy on a damaged life at the date ov its issue, and may be denoted by the symbol $(u\mathrm{V})_{(x+n)}$. Then the singl premium that the company shoud receiv at the outset is $\frac{(u\mathrm{D})_{x+n}}{\mathrm{D}_{[x]}}\cdot(u\mathrm{V})_{(x+n)}$. The value ov an anuity-due for n years upon $[x]$ being $\frac{\mathrm{N}_{[x]}-\mathrm{N}_{x+n}}{\mathrm{D}_{[x]}}$, it follos that the annual premium payabl for n years which shoud be charged by the company as the consideration for the risk is $\frac{(u\mathrm{D})_{x+n}}{\mathrm{N}_{[x]}-\mathrm{N}_{x+n}}\cdot(u\mathrm{V})_{(x+n)}$. It must be borne in mind, however, that, if the company agree to accept this annual premium in consideration ov undertaking the risk suposed, and the asured retain the usual riht ov dropping the policy, he wil be abl to exercise an option which has not enterd into the calculation; and the company, therefore, coud not agree to accept this annual premium, unles it wer in som way protected agenst the risk ov the option being exercised to its disadvantage. Such protection woud be obtaind by making the abov annual premium an adition to the ordinary premium for an ordinary policy.

As a numerical illustration ov the abov theory, let it be required to find the net singl and annual premiums that a company shoud receiv at the outset, in consideration of entering into a contract that it wil, upon aplication 20 years hence, if a person now ov the age 30 shal be then alive, grant a policy on his life at the ordinary rate ov premium. The singl premium wil be

$$\frac{(u\mathrm{D})_{50}}{\mathrm{D}_{[30]}}\cdot(u\mathrm{V})_{(50)}=\frac{(u\mathrm{D})_{50}}{\mathrm{D}_{[30]}}\cdot\left(1-\frac{(u\mathbf{a})_{50}}{\mathbf{a}_{[50]}}\right)=\frac{2037{\cdot}1}{266022}\times\left(1-\frac{1{\cdot}9105}{13{\cdot}6346}\right)$$

$={\cdot}0076576\times{\cdot}85988={\cdot}0065846=$ about 13*s.* 2*d.* per-cent.

The annual premium wil be

$$\frac{(u\mathrm{D})_{50}}{\mathrm{N}_{[30]}-\mathrm{N}_{50}}:(u\mathrm{V})_{(50)}=\frac{2037{\cdot}1}{4784158-1298762}\times{\cdot}85988={\cdot}0005026$$

= about 1*s.* per-cent.

As a final illustration ov the method ov calculating the value ov options that may be exercised by the asured, I wil now find the formula for a short term asurance, on the supposition that al the helthy lives drop their policys at the end ov a year, leaving the offis to pay the claims upon al the damaged lives. Supose that $l_{[x]}$ persons efect insurances on their lives, and, first, that the term ov the insurance is six or more years, the efect ov which latter supposition is, that al the damaged lives at the end ov the first year wil dy during the term for which the insurance is current. Then the number ov deths in the first year wil be $d_{[x]}$, and the number ov survivers at the end ov the year wil be $l_{[x]+1}$, ov whom $l_{[x+1]}$ ar stil in good helth and, acording to our supposition, drop their policys, leaving on the company's books the damaged lives, in number, $(ul)_{[x]+1} = l_{[x]+1} - l_{[x+1]}$. Then we see that the sum which woud at the outset be suficient to provide for payment of the claims is

$$d_{[x]} \cdot v + (ud)_{[x]+1}v^2 + (ud)_{([x]+1)+1}v^3 + (ud)_{([x]+1)+2}v^4 + (ud)_{([x]+1)+3}v^5 + (ud)_{([x]+1)+4}v^6$$

$$= d_{[x]}v + (d_{[x]+1} - d_{[x+1]})v^2 + (d_{[x]+2} - d_{[x+1]+1})v^3 + (d_{[x]+3} - d_{[x+1]+2})v^4 + (d_{[x]+4} - d_{[x+1]+3})v^5 + (d_{[x]+5} - d_{[x+1]+4})v^6 = (M_{[x]} - M_{[x+1]}) \div v^x.$$

By a similar proces ov reasoning it can be proovd that, if P is the premium paid by each ov the $l_{[x]}$ lives, then the value ov al the premiums receivabl on the suppositions made, is $(N_{[x]} - N_{[x+1]})P \div v^x$; and equating the value ov the premiums to the value ov the claims, we get $P = \dfrac{M_{[x]} - M_{[x+1]}}{N_{[x]} - N_{[x+1]}}$.

Next let us supose that al the helthy lives drop their policys at the end ov the second year, the original term ov the insurance being 7 or more years, so that al the damaged lives wil dy within the term ov the insurance. Then it may be proovd by a proces ov reasoning strictly analogos to the foregoing that the annual premium to provide for the claims is $\dfrac{M_{[x]} - M_{[x+2]}}{N_{[x]} - N_{[x+2]}}$; and in general, if the helthy lives drop their policys at the end ov the *t*th year, the original term ov the insurance being $t+5$ years or more, then the annual premium that wil provide for the claims is $\dfrac{M_{[x]} - M_{[x+t]}}{N_{[x]} - N_{[x+t]}}$. The form ov this expression shos us that the numerator corresponds to the claims among the $l_{[x]}$ lives under deduction ov the claims among the $l_{[x+t]}$ lives who ar stil select after the laps ov t years; that is to say, to al the claims during the t years and to the claims among the damaged lives thereafter, and the denominator corresponds to the premiums payabl in respect ov the same lives.

The formula wil take a different shape when n, the original term ov the insurance, $< t+5$. Supose, for instans, that the original term ov the insurance is n years, and that the helthy lives withdraw at the end ov the tth year; then it is easy to see that the sum which is necesary to provide for the claims during the first t years is $(M_{[x]}-M_{[x]+t}) \div v^x$, and the sum necesary to provide for the claims arising from the deths ov damaged lives in the folloing $n-t$ years is

$$(M_{[x]+t}-M_{[x+t]}-M_{[x]+n}+M_{[x+t]+n-t}) : v^x.$$

Adding these together the total sum necesary to provide for the claims is

$$(M_{[x]}-M_{[x+t]}-M_{[x]+n}+M_{[x+t]+n-t}) \div v^x.$$

Similarly, if P is the premium payabl in respect ov each insurance, the value ov al the premiums at the outset wil be

$$(N_{[x]}-N_{[x+t]}-N_{[x]+n}+N_{[x+t]+n-t})\,P \div v^x;$$

and equating these quantitys, we get

$$P=\frac{M_{[x]}-M_{[x+t]}-M_{[x]+n}+M_{[x+t]+n-t}}{N_{[x]}-N_{[x+t]}-N_{[x]+n}+N_{[x+t]+n-t}}.$$

Here the terms ov the numerator, $M_{[x]}-M_{[x]+n}$, correspond to the whole ov the claims that arise out ov $l_{[x]}$ persons in n years, and the other terms, $M_{[x+t]}-M_{[x+t]+n-t}$, correspond to the claims during the last $(n-t)$ years out ov the lives which ar stil select at the end ov t years; and the difference between these is clearly the whole ov the claims for t years and the claims in respect ov the damaged lives during the other $n-t$ years. Similarly for the denominator.

Next, let us supose that policys ar granted on the haf-premium system, so that at the end ov five years the premium is dubld, remaining thereafter constant; and let us further supose that at the end ov the five years al the helthy lives withdraw. Then reasoning as before, the number ov lives asured being $l_{[x]}$, the sum necesary to provide for the claims during the five years is $(M_{[x]}-M_{x+5}) \div v^x$; and the sum necesary to provide for the claims among the damaged lives, after the expiry ov five years, is $(M_{x+5}-M_{[x+5]}) \div v^x$. Therefore the total sum necesary to provide for the claims is $(M_{[x]}-M_{[x+5]}) \div v^x$. If P is the premium payabl during the first five years, which is increast to 2P subsequently, the value ov the premiums receivabl wil be

$$\frac{N_{[x]}-N_{x+5}}{v^x}\cdot P+\frac{N_{x+5}-N_{[x+5]}}{v^x}\cdot 2P=\frac{N_{[x]}+N_{x+5}-2N_{[x+5]}}{v^x}\cdot P.$$

Equating the two quantitys, we get $P = \dfrac{M_{[x]} - M_{[x+5]}}{N_{[x]} + N_{x+5} - 2N_{[x+5]}}$.

This quantity is the premium that wil exactly sufice to meet the claims acording to the suppositions we hav made; but it dos not follo that it wil be the proper premium to charge. In order to decide as to this, we must calculate the increasing premium in the ordinary way by the formula $P' = \dfrac{M_{[x]}}{N_{[x]} + N_{x+5}}$. Then, if $P < P'$, we lern that we may safely charge the premium P'; but if $P > P'$, the latter premium wil not be suficient to met the risk during the first five years. We miht then charge the premium P during the five years; but it woud be unnecesary to charge 2P for the remainder ov life, and the proper cours woud be to charge for the remainder ov life the premium P'' determind by the equation $P(N_{[x]} - N_{x+5}) + P''.N_{x+5} = M_{[x]}$.

Tabl 12 (p. 440) contains the results ov a number ov calculations made by the preceding formulas for quinquennial ages at entry from 15 to 70.

Colum (2) shos the premium for a singl year's insurance calculated in the ordinary way.

Colums (3) (4) (5) relate to insurances for a term ov three years. In colum (3) is givn the singl premium for a three years' insurance; in colum (4), the annual premium calculated in the ordinary way; and in colum (5), the annual premium calculated on the supposition that al the helthy lives withdraw at the end ov the first year. It wil be notist that the latter premium is much greater than the ordinary in every instans except for age 15.

The next five colums relate to insurances for a term of seven years. Colum (6) shos the singl premium; colum (7) the annual premium calculated in the ordinary way; and colums (8) (9) and (10) the annual premiums calculated on the supposition that al the helthy lives withdraw after 1, 3, and 5 years respectivly.

Colum (11) shos the net premium that wil exactly meet the risk on the asumtions that the premium is dubld at the end ov five years, and that al the helthy lives then withdraw; while colum (12) shos the quantity already tabulated in the erlier part ov this paper as the "first year's risk". It wil be observd that this quantity is very nearly equal to the premium in colum (8), but is thro'out a litl les, the reason being that the net premium payabl for a whole-life policy is greater than that upon a seven years' policy, and therefore the policy-value is les.

The figurs in the tabl seem to me deserving ov very careful study, as they hav a most important bearing upon varios points ov life insurance finans. They wil, for instans, greatly asist the actuary who is considering as to the propriety ov retaining or adopting the system ov haf-credit policys. The most obvios leson which they teach is that, altho the risk ov deth during the first insurance year is very smal, and insurances limited to a singl year miht therefore be safely granted at a very lo rate ov premium; yet, when the term ov the insurance is extended, even only to three years, the option which can then be exercised by the asured, ov dropping or continuing the policy at the end ov the first year, renders a very much larger premium necesary. Another, and perhaps even more important conclusion, is, that insurances upon yung lives ov about 20, shoud be accepted with great caution, as at that age the value ov the option that can be exercised by the asured is very large. It is a somwhat difficult task to decide what weit shoud be givn to these results in calculating offis premiums for short-term insurances. The premiums for age 20 ar greater than those for ages 25, 30, 35; but I supose that no actuary wil be prepared to publish in his prospectus a lower rate ov premium for age 30 than for age 20; and it is probably suficient to charge the same premium. After full consideration, it seems to me that the premiums containd in the folloing tabl woud fairly meet al the requirements ov the case, and miht wel be adopted in practis; and I shoud be glad to lern the opinions ov my brother actuarys upon this point.

Proposed Yearly Offis Premiums for the Insurance ov £100 *on a Select Life for a Term ov Years.*

Age.	One Year.	Three Years.	Five Years.	Seven Years.	Ten Years.	Age.
	£ s. d.	£ s. d.	£ s. d.	£ s. d.	£ s. d.	
15	0 16 6	1 1 6	1 2 0	1 2 6	1 3 9	15
20	0 16 6	1 1 6	1 2 0	1 2 6	1 3 9	20
25	0 16 6	1 1 6	1 2 0	1 2 6	1 3 9	25
30	0 16 6	1 1 6	1 2 0	1 2 6	1 3 9	30
35	0 16 6	1 1 6	1 3 1	1 4 6	1 5 9	35
40	0 18 2	1 2 10	1 5 7	1 7 6	1 9 10	40
45	1 0 9	1 6 3	1 10 2	1 13 2	1 16 5	45
50	1 4 8	1 12 0	1 17 3	2 1 3	2 5 11	50
55	1 10 11	2 1 6	2 9 1	2 14 7	3 1 7	55
60	2 0 9	2 17 3	3 8 9	3 16 6	4 5 6	60
65	2 15 2	4 2 9	4 17 2	5 7 2	6 0 11	65
70	3 15 10	5 18 2	7 1 9	7 17 6	8 14 10	70

Tabl 12.—*Singl and Annual Premiums for the Insurance ov* 100 *on a Select Life for the Terms specified.*

Age x	Assurance for One Year. — Single Premium.	Assurance for Three Years.			Assurance for Seven Years.					Premium payabl for first five years, on the sup-positions that at the end ov five years the Premium is dubld and al the Helthy Lives withdraw.	First Year's Risk ataching to the Grant ov an ordinary Whole-Life Policy, as explaind in the erlier part ov this paper.	Age x
		Single Premium.	Annual Premium.		Single Premium.	Annual Premium.						
			Ordinary.	On the Assumption that all the Helthy Lives withdraw after One Year.		Ordinary.	On the Assumption that all the Helthy Lives withdraw after					
							One Year.	Three Years.	Five Years.			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
15	·28855	1·0251	·3563	·2952	3·3829	·5487	·2952	·3817	·5151	·5146	·2945	15
20	·48073	2·1686	·7559	·9935	5·4382	·8922	1·0674	1·0424	1·0003	1·0131	1·0640	20
25	·43269	1·9043	·6633	·8544	4·8228	·7890	·9215	·8925	·8670	·8756	·9161	25
30	·43284	1·8424	·6417	·7911	4·8766	·7975	·8428	·8780	·8643	·9021	·8360	30
35	·47110	1·9640	·6843	·8234	5·3571	·8776	·9569	·9661	·9481	·9753	·9450	35
40	·52873	2·1879	·7629	·9092	6·0522	·9941	1·0414	1·0727	1·0664	1·1239	1·0258	40
45	·61545	2·5589	·8932	1·0556	7·5023	1·2386	1·3063	1·3461	1·3280	1·4023	1·2783	45
50	·75978	3·2164	1·1246	1·3248	9·5389	1·5877	1·6045	1·6677	1·6843	1·7455	1·5632	50
55	·99987	4·3586	1·5286	1·8116	12·9323	2·1828	2·0613	2·1971	2·2968	2·3366	2·0064	55
60	1·38475	6·2786	2·2128	2·6694	18·3686	3·1745	2·9569	3·1244	3·3295	3·3035	2·8662	60
65	1·97112	9·4275	3·3482	4·1808	25·5215	4·5685	4·1802	4·4438	4·7509	4·7046	4·0568	65
70	2·81723	13·7263	4·9312	6·0996	36·4928	6·8988	6·0996	6·6751	7·1804	7·0672	5·8932	70

Key to the Notation uzed in the preceding Paper.

The subscripts $_{[x]}$, $_{[x+1]}$, ... $_{[x+n]}$, &c., denote that the symbols to which they ar atacht, relate to lives which ar respectivly ov the ages x, $x+1$, ... $x+n$, &c., and ar select (or helthy) lives.

The subscript $_{[x]+n}$ denotes that the lives to which the symbol relates ar ov the age $x+n$, and wer select at the age x: see p. 393.

The subscripts $_{[x-1]+1}$, $_{[x-2]+2}$, ... $_{[x-t]+t}$, denote that the lives ar now ov the age x, and wer select at the ages $x-1$, $x-2$, . . . $x-t$, respectivly: see pp. 394, 400–404.

The letter *h* prefixt to a symbol, as in $(hl)_x$, $(hd)_x$, $(h\mathrm{a})_x$, denotes that the lives to which the symbol relates ar select (or *h*elthy): see p. 431.

The symbols $(hl)_{[x]+n}$, $(h\mathrm{a})_{[x]+n}$, &c., relate to lives which wer select at the age x, and ar now ov the age $x+n$, and stil select.

The same is the case with $_n(h\mathrm{V})_x$ or $_n(h\mathrm{V})_{[x]}$: see pp. 414, 431.

The letter *u* prefixt to a symbol denotes that the lives to which it relates ar the damaged (or *u*nhelthy) lives ov our tabl. When the subscript is $_{[x]+n}$, the lives wer select at the age x.

For further explanations, the reader is referd to the folloing pages:—

$\mathrm{D}_{[x]+n}$, $\mathrm{N}_{[x]+n}$, $\mathrm{M}_{[x]+n}$: see p. 395.—$\mathrm{A}_{[\overset{1}{x}]y}$: see p. 397.

$\mathrm{V}_{([x+1],\,[x]+n)}=1-\mathrm{a}_{[x]+n}\div\mathrm{a}_{[x+1]}$: see p. 414.

$(ul)_x$, $(ul)_{(x)+t}$, $(ud)_{(x)+t}$: see p. 423.

$(ul)_{(x-t)+t}$, $(ud)_{(x-t)+t}$: see p. 398.

$(ul)_{[x]+n}$, $(ul)_{([x]+n)+t}$, $(ud)_{([x]+n)+t}$: see p. 425.

$(uq)_{[x]+n}(=(ud)_{[x]+n}\div(ul)_{[x]+n})$, $(uq)_{([x]+n)+t}$: see p. 427.

$(u\mathrm{D})_x$, $(u\mathrm{M})_x$, $(u\mathrm{N})_x$, $(u\mathrm{a})_x$, $(u\mathrm{A})_x$: see p. 429.

$(u\mathrm{D})_{[x]+n}$, $(u\mathrm{D})_{([x]+n)+t}$, $(u\mathrm{M})_{[x]+n}$, $(u\mathrm{N})_{[x]+n}$, $(u\mathrm{a})_{[x]+n}$, $(u\mathrm{A})_{[x]+n}$: see p. 432.

$_n(u\mathrm{V})_x$: see p. 431.—$(u\mathrm{V})_{([x+1],\,[x]+n)}$: see p. 432.

$_n(u\mathrm{a})_x$, $_n(u\mathrm{A})_x$: see p. 434.—$(u\mathrm{V})_{(x+n)}$: see p. 435.

DISCUSSION.

The PRESIDENT (Mr. A. H. Bailey.)—I do not think it is possible for anyone to follow a paper of this kind during its reading, and I have had no time to study it. What has passed through my mind is this: how is the practical business of life assurance to be carried on, if it is to be subject to the nicety and the refinements which Mr. Sprague seeks to introduce? In my small way I have tried to simplify the practice as much as possible. The broad principle on which we proceed is that, while individual life is uncertain, yet among a number of lives it is subject to a law; and on that broad basis we have been able to carry on, with reasonable success, a system of finance which has worked fairly well. But if we are to have a series of tables for every age at entry and every duration of policy, why I do not know how it is possible to work such a system. Further, I am sceptical as to selection operating to the extent which Mr. Sprague would have us suppose. It is true, theoretically, that in the surrender of a policy the assured is able to exercise an option against the company. But, practically, in the great majority of cases, the surrender of a policy depends upon pecuniary circumstances which have nothing whatever to do with the state of health, and, even if it had, I believe that men generally are exceedingly bad judges of their own prospects of longevity. No doubt where policies are in the hands of third parties, to some small extent the holder does exercise this option. Mr. Sprague says: "It is no new suggestion that the lives which apply to surrender their policies will, on the average, be in a better state of health than those who continue their policies." I do not think it is true that those who surrender must therefore be taken as selected lives, and the values of their policies calculated accordingly. And when Mr. Sprague adds, "I submit that by means of my tables we can estimate the liability of a life office under its policies more correctly than has ever been possible hitherto", I think that it is vain to aim at this extreme accuracy. I remember the collection of the facts upon which the Institute Tables were based. Mr. Sprague was one of the members of the committee; pains and thought were bestowed by all of us upon them, and the majority came to the conclusion that there seemed, after about the lapse of five years from the period of entry, to be a line of demarcation, and that it was worth while to have separate tables formed for the lives which had been assured for five years at least. It was considered that that would represent, not scientifically, but sufficiently for practical use, the effect of mortality after selection had worn off. I remember Mr. Sprague was in a minority; he strongly objected to this determination, and finally decided that, if the Institute as a body would not do the further work he desired, he would do it himself. This he has accomplished, and I am sure that when he asks his professional brethren to give a friendly reception to these tables, they will do so. He says, again, "Before the subject of selection can be thoroughly understood, it will be essential to obtain information as to the rate of mortality among select lives, none of which withdraw from observation. Three sources occur to me whence the necessary information might be obtained. One of these I have mentioned in former papers, namely, the experience among the Government annuitants. These annuitants, when nominated, are probably, on the average, a very select body of lives." Well, I am not quite sure that that is the case. I know that Government annuities are frequently bought in pursuance of directions in wills, without any reference to health. Then the second source is that of the Peerage Families—the idea being that those who marry are healthy men. I am not at all clear about that. If these men are considered eligible husbands, the state of their health is a secondary consideration. Upon the whole, I do not think it would be possible for any gentleman in this room to deal with such an elaborate paper as this without devoting a good deal of time to its study.

Mr. BADEN—Mr. President, like yourself, I feel that, having regard to the importance of the subject, and the great pains with which Mr. Sprague has treated it, I have not had time to consider the question sufficiently to make any criticism on the paper. But the feeling that has been uppermost in my mind during the reading of this paper has been a feeling of regret that the practice of assurance, as it now exists, should render the working out of such a paper as this, if not a necessity, at all events a desirable thing. We cannot ignore the fact that a great change has passed over the practice of life assurance since the time when the Institute Tables were framed; we have now come to a practical system of life assurance, in which there is so much expenditure made for the attainment of business, and such a heavy loading and heavy expense falls upon the first years of assurance, that it becomes a matter almost of necessity to calculate with every degree of finesse the amount of risk we are running year by year, after the first years are passed. We may deplore the fact, but we must look it in the face. Therefore I cannot help thinking that the paper is one of the greatest value, enabling us to meet the necessity, which will be forced upon us on all sides, of determining the way in which offices are going, and to form an estimate of the effect which that practice,—unfortunately, as I deem it,—is having upon the finance of life assurance companies.

Mr. FINLAISON—Mr. Sprague has raised an enormous superstructure on the foundation of the Institute Experience, and our thanks are due to him for the way in which he has worked upon it. Whether the foundation will entirely bear the superstructure, I am, however, not prepared to say. We have also to thank him for the new and happy notation which he has invented in certain parts of this paper, and in which he has shown his usual skill. I think that the broad principles of the paper are satisfactorily proved; for instance, that the annuities and net premiums of the H^M Table at the younger ages are not those which are properly deducible from the data, by reason of the abnormal vitality of the entrants at the older ages during the first few years after insurance. The experience of the Government life annuitants, analyzed under each age at entry, is already in existence. How soon it will be possible to publish it I will not venture to say, as that will depend upon my other engagements. But it does not quite cover the same ground as is covered by the H^M Table, for the bulk of assured lives enter under 50, but annuities are not usually purchased until people attain the age of 55 or 60 years. Therefore the experience of annuitants is a supplement, and does not run parallel to that of assured lives. The annuitant experience is generally in the direction of the results shown in the paper; but while a capability of selection among annuitants appears to exist up to the oldest ages observed, the mortality for the year of age in which an annuity is purchased, is not as small as that shown in Mr. Sprague's adjusted results. The unadjusted annual mortality per thousand of the Government Male Life Annuitants is as follows:—

	Years elapsed from Date of Purchase.					
Ages.	0	1	2	3	4	5 and upwards.
55–59	19	20	23	28	28	26
60–64	20	27	31	27	30	32
65–69	44	36	34	34	47	50
70–74	43	54	65	63	72	73
75–79	48	79	84	88	89	110

The actual number of deaths from which the ratios are computed being as undermentioned:—

	0	1	2	3	4	5 and upwards.
55–59	21	32	34	39	36	225
60–64	32	60	62	47	46	409
65–69	45	60	60	64	91	873
70–74	33	63	76	72	81	1275
75–79	26	65	66	68	67	1487

At 80–84, where the entrants are few, the annual rate is 104 per thousand in the year of purchase; but as much as 156 in the thousand for those who entered at 75–79. These figures do not, I think, bear out Mr. Sprague's remark that, at the beginning and the end of the table, the effect of selection wears out sooner; but they corroborate the conclusion of Mr. Sprague—that the older the age, the more capability there is for selection.

Mr. Walford—I regard this paper as aiming after scientific exactitude, and I think that that was its true origin. I remember very well indeed a most interesting conversation at Scheveningen, near the Hague, as far back as 1869, between Mr. Sprague and Mr. Sheppard Homans, on this very question of the effect of selection and the mode of illustrating that effect, and the practical values which might result. Mr. Baden has, perhaps, hit upon one of the most important uses to which the table may be applied; yet it is but just to say, those uses have become more apparent since the time—twelve years ago—of which I am speaking. I feel perfectly sure that those uses were not in the mind of the author when he first began. When one comes to speak of scientific exactitude, we find in all branches of scientific enquiry how much, of late years, exactitude has been striven after; and it is, unquestionably, the tendency of the age amongst scientific men that they should, each in their own department, endeavour to improve the mechanism or instruments by which they work. After saying that, one cannot help also saying that the finer the instruments used, the more necessary it is they be used only by those who thoroughly understand them. And it would seem to me that in this very paper—and it will be understood that I say this without any sort of desire to disparage it—it does suggest the refinement of processes which one cannot help feeling may in unskilful hands lead to absolute danger—danger, I mean, of Mr. Sprague's theories and teachings being misapplied. There is one little table near the end of the paper—"Proposed yearly office premiums for the insurance of £100 on a select life for a term of years"—I am disposed to think that the little table embodies in a large degree the teachings which run through the paper and the tables which it contains. I will not say more upon this paper than this—it places at our disposal much knowledge; and knowledge is power.

Mr. GEORGE KING—This paper seems to me a most remarkable one. The chief point that strikes me is the conscientious way in which Mr. Sprague has worked out all his results. He takes nothing for granted, and he is never afraid of trouble. I think that in both these respects we can all learn a lesson from him. If a thing is worth doing at all, it is worth doing well. Further, his great skill in dealing with very difficult problems is also worthy of admiration. I am sorry, Mr. President, to dissent from one or two remarks which you made at the beginning of the discussion. You seem to take exception to the niceties which Mr. Sprague has introduced into our science; and you seem, further, to think that too much is made of what has been called selection. Now, I hardly imagine that Mr. Sprague would intend these niceties to be brought into every office calculation, but he means only that they are of great value in helping us to get at the broad principles on which our office calculations should be based. There is one little table which illustrates what I now mean—that giving the premiums for half-premium-rate policies. You remarked, sir, that we do our business on broad averages, and that, if we make a profit on it, on the whole, that, one may say, is almost enough. But I think that we ought to go further, and try to make a profit on each class of our business. It is not sufficient merely to take broad averages for all classes in the aggregate. We ought to take the average for each class. Mr. Sprague has investigated, in a very interesting way, the half-premium rates, and he proves that our ordinary formulas for these are defective. It does not follow that we should adopt his method of calculating the rates for such policies; but I think that it does follow that we ought to look at our ordinary calculations, and see how far we are likely to be led astray by them. Then, as to selection, there is one question I should like to ask. If the dropping of policies by healthy lives is not the cause of the rapid increase of mortality, and of the mortality increasing beyond the normal mortality of an ordinary population, what is the cause? I do not understand that Mr. Sprague is trying to make much of this selection. He is merely investigating that which unquestionably exists. As to Mr. Sprague's object in this paper, I think, if we may judge from his previous writings on the subject, it is intended to be of use to us principally in valuations, the other uses being subsidiary. Valuations by his select tables, although there is a table for each age at entry, need not be more laborious than valuations by the combined H^{M} and $H^{M(5)}$ Tables. If policies have been in force five years or more, we have merely to value Mr. Sprague's premiums by the $H^{M(5)}$ annuities. If policies have been in force for less than five years, it is true that a different course must be followed. It would be easier to value each policy separately; but unfortunately the Act of 1870 renders that course inadmissible, because we are asked, although not compelled, to give the value of the sum assured, and the value of the premium, separately. However, I do not think that even in the first five years there need be any very insuperable difficulty in valuing by Mr. Sprague's table.

After five years the valuation is as easy as any other valuation in use. With regard to the result of Mr. Sprague's tables on a valuation, he says that he would like some actuary to investigate the point for him. In the paper which some time ago I read before the Institute, certain formulas were given which very much facilitate an investigation of the kind. I have made a simple calculation, which occupied not very many minutes, using formula 4, *Journal*, vol. xx, p. 258, and I find that after 5 years Mr. Sprague's table gives a reserve for my model office of £24,436—while the H^M reserve is 20,268. After fifty years Mr. Sprague's reserve is £632,050. The H^M and $H^{M(5)}$ is £638,680, or nearly £7,000 more; and the H^M reserve is £625,308; so that Mr. Sprague's tables come almost exactly half-way between the H^M pure and simple and the combined H^M and $H^{M(5)}$. Mr. Sprague has overlooked the fact that I gave similar figures at 3 per-cent interest when his last paper was read on this subject. They appear in the *Journal*, vol. xxi, p. 245. I have only one remark more to make, and that is in regard to the notation. The notation which Mr. Sprague has adopted is admirable. The facility with which it was read by the Secretary, I think, showed that it was a very graphic notation; but I would suggest that a key to it should be appended to the paper, because it is rather troublesome, if one wants to read perhaps a single paragraph, to hunt all over the paper for the meaning of the symbols. Further, I believe that the majority of the calculations are at 4 per-cent, but it would be extremely valuable to have the rate of interest inserted in the heading of each table, and I would therefore suggest that that should be done before the paper takes its place in the *Journal*.

Mr. BERRIDGE—There is one point to which I should like to draw the attention of the meeting, especially as it illustrates the use of these select tables; and that is the question of surrender values. Without going into Mr. Sprague's refinement as to the initial expenses and using the premium at the next higher age, I think that, if you compare column 2, which is the value by the H^M Table, with column 5, which is the value by Mr. Sprague's table, purely, you will see that the differences at the shorter periods of duration are much larger in proportion to the values than at the longer ones. Although I do not agree with Mr. Sprague that all the surrendered policies are healthy lives, but, on the contrary, agree with our chairman that, to a great extent, those surrenders are brought about by necessity, we do make a deduction from the surrender value on the ground that the life may be healthy. I think that our practice of giving a constant percentage, whatever the duration of the policy may be, is wholly wrong. At some of the younger ages in these tables the differences would be 20 per-cent, and they would drop down to 4 or 5; whereas in practice I think we rather reverse the process, more particularly when we agree to give a certain percentage which rather runs up the value of policies of shorter durations as against the older ones.

Mr. Sprague's Reply.

It seems to me premature to attempt to answer the question put by Mr. Bailey,—how will the practice of life insurance be altered by the results of my investigations? Time only can furnish a trustworthy answer to this question. I claim to have greatly extended and improved the theory of life insurance, and to have furnished actuaries with a most powerful means for prosecuting further investigations; but the full practical application of my theory still remains to be worked out. In the meantime, some most important practical conclusions may be drawn from the figures contained in the preceding paper; for instance, it is clear that the premiums for the insurance of young lives, of say, 20 to 25, should be higher than those given by the ordinary tables, and the surrender values of their policies less. I note that Mr. Bailey is not satisfied that, in the surrender and lapse of policies, an option is exercised by the assured, to the disadvantage of the office. It is quite true, as he says, that in the great majority of cases the surrender of a policy depends upon pecuniary circumstances, which have nothing whatever to do with the state of health; but this is in no way inconsistent with the exercise of the above-mentioned option. Looking at the matter practically, it is to be borne in mind that the proportion of damaged lives is very small, being (according to my tables) only 1 per-cent of the total number at the age of 35. If, then, the pecuniary circumstances of 100 lives of 35 are such as to suggest to them the propriety or necessity of surrendering their policies, it is probable that one of them will be a damaged life, and that, while the 99 healthy lives will surrender their policies, the 100th—the damaged life—will use every endeavour, by obtaining a loan from the office, or by appealing to friends, to keep the policy in force for the benefit of his family; or, if this is impossible, will endeavour to sell the policy to some speculative person who is willing to give a higher price than the office surrender value, on the chance of making a profit by the early dropping of the damaged life. In these circumstances, it is not the 99 healthy persons who exercise their option to the disadvantage of the office, but the 100th—the damaged life; and it is to me quite incredible that any individual's personal experience, however extensive, can be sufficient to prove that such option is not exercised to the disadvantage of the offices by the holders of policies on damaged lives. There can be no doubt, I think, that the option is exercised (more or less) to the disadvantage of every life office,—that, when large numbers of lapses and surrenders take place, the policies are (with few exceptions) on lives that are still healthy, so that an undue proportion of damaged lives is left behind among the policies that remain on the company's books; and the larger the number of lapses and surrenders, the greater will be the injurious effect upon the office, as evidenced by an increase in the rate of mortality, in the manner illustrated in the table on page 428 of my paper. Mr. Bailey expresses a doubt as to the correctness of my statement, "that the men who marry may, on the average, be considered as select lives at the date of marriage." This leads me to conjecture that he must have overlooked the remarkable statistics published by Dr. Stark, and reprinted in the number of this *Journal* for July last. Dr. Stark shows (*J.I.A.* xxii, 247) that at the ages of 20 to 25 the rate of mortality among husbands and widowers is only ·680 per-cent, while for bachelors it is 1·200 per-cent; at ages 25 to 30 it is ·820 per-cent for the former, against 1·432 per-cent for the latter; and, according to his statistics, the rate of mortality among bachelors is higher at all ages, although the proportionate difference is not so great at higher ages as at those above quoted. These figures appear to me to prove beyond all question that the married men are, on the average, select lives at the date of marriage; and this conclusion is confirmed by the statistics derived from the British Peerage families. I proved (*J.I.A.* xxi, 432) that at the age of 25 the probability of dying in a year was only ·39 per-cent for married men, against 1·30 for bachelors; while at the age 30 the probability is ·65 per-cent for married men, against 1·13 for bachelors. Looking at these figures I cannot doubt that, when the rate of mortality among recently-married men of all ages shall be investigated, it will be found to be very light. The figures given by Mr. Finlaison, as obtained by him from the experience of the Government annuitants, are extremely interesting, and demonstrate —what is doubted by Mr. Bailey—that selection has a very great influence upon the mortality of this class of lives, although apparently not so much at the outset as among assured lives. Mr. Bailey has referred to the discussions which took place among the committee appointed to superintend the preparation of the Institute Life Tables; and I accept his statement of the facts, although the circumstances had escaped my own memory; but I think he will see, when he comes to examine my paper more carefully, that it is not a consequence of my resolution which he quotes. That resolution led me to write the paper (*J.I.A.* xv, 328) *On the Rate of Mortality among Insured Lives as influenced by the Duration of the Insurance*, but I have not up to the present time seen my way to complete that investigation satisfactorily. I have therefore been led to adopt, at all events provisionally, the conclusion of the majority of the committee, and to assume that the effect of selection has practically worn out after the lapse of 5 years, so that we may assume the $H^{M(5)}$ Table to be a fair representation of the mortality among insured lives after that time. I have therefore constructed a set of tables which join on smoothly to the $H^{(M5)}$ Table after the lapse of 5 years or less, and I trust that these tables will be found practically useful in life office valuations. In the course of preparing these tables, and putting together a paper in explanation of them, I have arrived at what appear to me to be very important improvements in the theory of life contingencies, and I have, without delay and without reserve, placed these before my professional brethren. The improvements in the theory may be considered quite apart from their application to the Institute Mortality Experience, and I should be glad to see them applied to other data, such as those of Mr. J. A. Higham and the late Mr. Spens. The subject of "selection" is very far indeed from being exhausted, and I hope others will assist me in cultivating what still seems to be a very promising field of enquiry.

HOME AND FOREIGN INTELLIGENCE.

Friendly Societies in New Zealand.

We have received from Mr. F. W. Frankland a copy of the Registrar's 3rd Annual Report, and extract from it the following passages, which, we believe, will interest our readers:—

1.—BUSINESS OF THE REGISTRY OFFICE.

I. Department Work during the Year ended 30th June 1880.

The work of framing regulations, and the publication and issue of model tables of contributions, referred to in the last report, has not recurred during the past year. Unless any changes in details are required, or additional tables of contributions for special purposes (*e.g.*, for annuities, instead of sick payments, after certain ages), the work, from its very nature, once done, remains.

The Transference on to Cards of Information given in the Returns of Sickness and Mortality.—This work is now kept up annually in respect of members who were sick, died, or seceded during the year. The original transcription of all the members' names, &c., on cards having been once effected, the annual labour is not extreme; and, by dealing with the particulars of sickness and mortality annually, the work that would otherwise be thrown upon the office at the end of the quinquennium is largely reduced. A large amount of time has, however, been occupied in the careful examination of the quinquennial returns.

8. Notice of Cancellation.

The Loyal Goldsborough Lodge, pronounced insolvent by its valuer (Mr. Leslie), desiring to amalgamate with the Loyal Waimea Lodge, of the same district, a mutual agreement was arrived at, and a *de facto* amalgamation took place. On the Registrar being informed of this, he intimated to the Goldsborough Lodge that, until the provisions of section 22 of the Act were complied with, the amalgamation would be illegal, and the trustees of the Goldsborough Lodge could not be relieved of their responsibility for the funds transferred. It was found impracticable, under the peculiar circumstances of the case, to comply with these provisions, and the difficulty was obviated by a transfer (by the method known as "clearance") of all members of the Goldsborough Lodge to the Waimea Lodge. The former had thus ceased to exist through lack of members, and section 11 (1, *b*) of the Act became applicable. Notice of cancellation of registry was immediately given to the lodge; and effect will be given to this on the 23rd August next in the event of there being no valid objection previously urged.

II.—WORKING OF THE AMENDMENT ACT OF 1878.

The Registrar has to report that, except in regard to two of its provisions, the Amendment Act has so far worked very smoothly and well. The information required under section 4 has been in general furnished—sometimes with evidence of cheerfulness, and certainly not with greater reluctance than the information required under section 13 of the principal Act. In two particulars only has special reluctance been shown in a few instances—the furnishing of the names of members, and the furnishing of imformation respecting the wives of members.

It is the 7th section of the Act which appears to have given greatest trouble. It has done so in two ways, and in each case the trouble has proceeded solely from the Provincial District of Otago. The Odd Fellows and the Foresters of that district are both affected by this section, though in different ways. The former had since 1874 adopted the practice of crediting all interest beyond 4 per-cent to the Management Funds of their lodges; the latter have been in the habit of paying their court surgeon and other medical expenses out of the Sick and Funeral Funds of the courts. Both these practices became illegal by the Amendment Act of 1878. Yet the returns for 1879 show that in five out of the six registered courts of the United Otago District of Foresters from which returns have been received, the malpractice was continued during that year; while in the case of the Manchester Unity of Odd Fellows, although the law appears to have been obeyed by nearly all the lodges, and the district rules have been remodelled so as to comply with the Act, a petition has been presented to Parliament praying that this special prohibition should be rescinded. The Foresters' courts were communicated with on the subject of their breach of the law; but, as the District had recently registered rules which would have the effect of preventing a repetition of the malpractice, no prosecution was instituted. In the case of the Odd Fellows a letter was sent to the District Secretary (8th August 1879) which, on account of its containing information useful to all societies, it may be desirable to quote *in extenso*:—

"SIR,—It having come under my notice that a petition is being circulated amongst the various lodges* of the Manchester Unity of Odd Fellows, with the object of obtaining an alteration of the law in respect to the appropriation of interest earned by the benefit funds of friendly societies, I deem it my duty to address to you a short explanation of the grounds on which this portion of the law is based, and to request you kindly to circulate this explanation amongst the members of the various lodges.

"The rate of interest earned by an investment consists always of three parts, one part representing the remuneration paid by the borrower for the use of capital, the second an insurance premium to cover the risk of depreciation or loss, and the third a compensation for other disadvantages or inconveniences attending the investment. The first of these is a nearly constant quantity in a given country at a given time, and is the rate for the time being at which money could be borrowed, on absolute security and with perfect facilities for realization by the lender. In most English-speaking communities it is but slightly below the rate at which the Government can borrow. The excess of the actual interest over this minimum rate is therefore always, on the average, proportionate to the counterbalancing disadvantages attending the investment. Where and in as far as these disadvantages consist in imperfect security, the portion of the interest which represents insurance against risk, must in all cases be reserved to meet the future liabilities of the fund, as the contributions are calculated not only on the assumption that a certain rate of interest shall annually accrue, but also on the assumption that there is absolute security for the principal. No provision whatever is made in the tables for the possibility that losses may be sustained. Now, although the portion of the rate of interest which represents insurance against risk, is in all cases in excess of the mathematical premium which would be required to neutralize losses on an average of transactions, and is in fact determined by what is called the *moral*, as contrasted with the *mathematical*, weight of the risk to the average lender; although therefore the investors at the high rates of interest will on an average certainly obtain a larger profit in the long run than the investors on Government security; yet it is with the moral rather than with the mathematical weight of the risk* that benefit clubs, and especially small ones, are concerned, and it therefore behoves them to reserve at least a large portion of the interest earned, even where it is at a rate greatly in excess of that assumed in the calculation of the tables.

---

* That is, amongst the lodges of the Otago District of Odd Fellows only. The other districts are perfectly sound in their practice on this head.

"It is obvious from these considerations that where a high rate of interest is obtained, the portion which may be regarded as pure profit requires great nicety for its determination, and cannot be ascertained by any hard-and-fast rule, such as would be established by appropriating to management expenses the whole excess over a fixed minimum rate.

"To appropriate an excess of interest at the rate of, say, 6 per-cent, arising from the rent of offices, would be a transaction standing on a very different footing, as regards safety, from an appropriation of the extra ½ per-cent on Post Office Savings' Bank deposits. The point is one, therefore, which can only be satisfactorily dealt with at a valuation.

"There is, however, in the present case another reason why the rate of interest assumed in a valuation or in the construction of tables, should be somewhat lower than the rate which we may expect will be actually realized. This reason is contained in the fact that the tables do not include any provision for the possibility of adverse fluctuations in the society's experience of sickness or mortality—fluctuations which are likely to be very considerable in the case of small lodges, especially in regard to the amount of protracted sickness. Hence, if a society were to enter on a given period of, say, five years with exactly enough funds in hand to meet its liabilities according to the tables, and were throughout the five years to appropriate all interest beyond 4 per-cent to the Management Fund, it follows that, apart from the profits accruing from secessions, there would be an equal likelihood of a surplus or a deficiency at their close. Now, the existence of a surplus is not a matter to be deprecated; but a deficiency is a serious disaster. To permit the two results to be equally probable is quite an inadmissible course. Life insurance offices regard it as an axiom that there ought never to be a deficiency at a valuation. One counteracting element has already been alluded to—namely, the profit arising from secession; but, as a further and more perfect safeguard, it is desirable that the interest actually earned should exceed the interest assumed, so that the excess may compensate for an unfavourable sickness experience, should such be the society's lot. It is not an infrequent thing for life insurance offices to realize a rate of interest exceeding by 2 per-cent or 3 per-cent the rate assumed in valuations and in the computation of the premiums. Thus the Australian Widows' Fund was recently valued at 4½ per-cent, at a time when the annual interest was about 8½ per-cent on the aggregate funds.

---

* That is to say, the possibility of a given loss is a disadvantage which far more than counterbalances the equal possibility of an equal amount of gain.

"Such are the general principles which forbid the application to management expenses of interest earned by a benefit fund. But in the case of the Otago District of the Manchester Unity of Odd Fellows, there are circumstances which would make such an appropriation, were it to be continued, especially disastrous. The rate of contribution which has hitherto obtained, is one which all actuaries are agreed in regarding as totally inadequate to provide the benefits promised. This rate, it is understood, will continue to be charged to all members initiated prior to the 1st July 1879. Even as regards new members, the contributions to be charged are considerably lower than the minimum scale which it is considered prudent for societies to adopt. A rate of interest much higher than 4 per-cent would be needed to make these contributions adequate; hence it would be a most pernicious thing to make 4 per-cent the maximum rate obtainable for the sick and funeral fund. Lastly, although no lodges of the district have as yet been valued, a comparison of the funds in hand with the duration of the lodges, the number of members, and the nature of the benefits, will in many cases afford *primâ facie* evidence against solvency."

IV.—THE RETURNS OF SICKNESS AND MORTALITY FOR THE QUINQUENNIUM 1873-77.

When the last report was presented, the whole of the data contained in these returns had been transferred to cards, a specimen of which was shown on page 6 of that report. Since then the data have been subjected to a careful examination, with the view partly of rectifying errors as far as possible, but chiefly of excluding from the experience table which it is proposed to construct, all data of a demonstrably erroneous or even of a suspicious nature. This work, which was entrusted to Mr. Frankland, the Actuary, is nearly completed. The result has been the exclusion from the proposed experience of the returns furnished by a considerable number of lodges and courts. The number of lives which the residue of data will embrace may be stated approximately at from 7,000 to 8,000. This will doubtless be equivalent to between 20,000 and 30,000 years of life—an experience about equal in extent to that on which Mr. Ansell's Friendly Societies Table was based. As soon as the weeding process is completed, everything will be ready for the tabulation of results. It is proposed to exhibit the rate of sickness and mortality at each age of life, and, if time permits, the influence of duration of membership on both, so as to illustrate the effect of the selection which members exercise against the societies through discontinuances. The data are of course far too scanty to bring out a reliable law of deterioration of average health, but, if positive results are shown, it will go far to rebut the argument against the use of English data in the construction of premiums, derived from the low rate of sickness which obtains in New Zealand societies. Apart from the altogether vulgar error of those who rely on a low rate of sickness without taking into consideration the ages of the members, there is the subtler fallacy which leaves out of account the effect of recency of admission. From conversation with Mr. Templeton, F.I.A., on the subject of his Victorian experience on this head, Mr. Frankland anticipates that this influence of selection or recency of admission, taken in conjunction with the secession of the healthier members, will be found to be considerable. There is of course another feature which tends to bring out a fictitious appearance of salubrity in the experience of New Zealand societies, namely, the feature noticed in a foot-note to page 10 of the last report, the occasional forbearance of members in good circumstances to claim the allowances to which they are entitled under the rules. From replies to circulars issued by the Registrar, it appears that the saving thus effected has been in many lodges considerable; and it can hardly be doubted, when the economic progress of the colony is taken into account, that this cause of apparently-low sickness rates, will not operate as strongly in the future as it has done in the past. It would be difficult, however, if not impossible, to eliminate the influence of this factor from the data.

Other points of great importance are the recency of admission, or otherwise, of the members, and the proportion of persons on the books who are either suffering from sickness likely to be permanent, or who have in the past experienced very heavy sickness, and whose expectation of life is therefore lower than the normal expectation for their several ages, while the amount of sickness per annum to be expected by them during the remainder of life is higher than the normal amount. These considerations have, in the opinion of Mr. Frankland, not yet met with the degree of attention they merit at the hands of actuaries. The influence of past sickness on the expectation of future sickness and mortality is, in his opinion, a field for research as vast and important as the influence of medical selection on mortality has proved to be. It is manifest that a life office, if it could at the time of a valuation ascertain the state of health of its policyholders, would be able to form a more accurate estimate of its liabilities than it can do when it knows only their ages and the duration of their policies. This advantage, necessarily denied to a life office, is open to a friendly society, because it has official cognizance of the sickness experienced by its members in the past; and hence Mr. Frankland anticipates that in the future it will be possible to do much towards individualizing the liabilities of these societies, and that even now a beginning in this direction should be made. For example, if an undue proportion of members are found to be suffering from sickness likely to be permanent, he holds it to be imperative, especially in the case of a small society, to make an extra reserve in respect of such members—that is, a reserve beyond that which the mere *ages* of the members would indicate.

It has been the duty of the Registrar during the past year to decline to recommend the Government to approve the appointment of a gentleman as valuer to one of the Otago societies, as it appeared, from the nature of the qualifications which were represented as fitting him to perform the duties, that he had entirely misapprehended the true nature of a valuer's functions, and had confused them with those of an accountant. As an illustration of the kind of knowledge which gentlemen must possess, in order to be qualified for the position of valuers, it may be well to quote the queries which were proposed to this gentleman as a test of fitness. They were the following:—

"1. What particulars would you think it necessary, generally, to ask for from a lodge or court as premisses of your valuation?

"2. By what principles would you be guided in the selection of a table or tables of sickness and mortality, and of a rate or rates of interest?

"3. What view do you take of the mortality to be expected by New Zealand societies as compared with similar societies in England? How would this view affect your estimate of the *sickness* liabilities?

"4. What do you consider to be the present state of our knowledge of the liability to sickness after the age of seventy?

"5. Whether do you regard occupation or locality as having the more important bearing on the liabilities of a society? State in what way you conceive that the influence of each should be allowed for.

"6. Besides acute sickness, involving temporary incapacity for work, members contract for allowances in *chronic* sickness. What view do you take of this latter risk, considered in its bearing on the funds of a society?

"7. On what principles would you deal with a surplus disclosed by the valuation?"

CORRESPONDENCE.

SUGGESTION AS TO STATISTICS OF INDIVIDUAL SICKNESS.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—I venture to bring under your notice an idea with which I have been much impressed recently, and which, I would fain hope, may be the germ of important contributions to the theory of sickness insurance. The idea which has occurred to me, is that it might be advisable to compile statistics from the returns furnished by friendly societies, which would have the effect of showing the extent to which estimates of the probabilities of future sickness and mortality might be legitimately influenced by the past sickness experiences of the several individuals to which those estimates relate. In other words, statistics might be compiled which would show, for the individual member, the influence of past sickness on the expectation of future sickness and mortality. Compilers of sickness experience tables have hitherto confined themselves, I believe, to an exposition of the average amount of sickness experienced at each age of life, first, by members, of either sex, *en masse;* and secondly, by members following particular occupations or classes of occupation (*e.g.*, light labour with exposure to weather, heavy labour with exposure to weather, &c.), or residing in localities falling under such categories as "rural districts", "town districts", "city districts". I am not aware of any attempt to construct "select" or "analyzed" sickness tables showing the influence upon the sickness at each age, of aught save sex, occupation, and locality. Yet the investigations which have been made in regard to the influence of selection upon mortality, resulting in the "select" or "analyzed" mortality tables of Messrs. Sprague, King, and others, —not to speak of the most conspicuous of all analyzed tables, the $H^{M(5)}$,—would readily suggest the propriety of an investigation into the influence of original medical selection and of subsequent discontinuances, upon the sickness experience of friendly societies, resulting in the compilation of "analyzed" sickness tables to exhibit the connection between sickness and duration of membership. So far, the analogy between life insurance offices and friendly societies would appear to be perfect. But in the case of the friendly society a new element, capable of numerical measurement, offers itself as a basis of computation. *The friendly society has official cognizance of the past sickness experience of its members.* In the case of each of its insurers, it knows not only his age and the duration of his policy, but also the amount, distribution in time, and qualitative nature of the sickness experienced by him since he first became a "free member". It has therefore data respecting the health of its members better capable of numerical expression than those embodied in the proposal papers of a life policyholder. These data are not, in the early stages of membership, as valuable as those contained in the report of a medical referee (indeed, their value *commences* by being *nil*, since at first there is *no* past sickness experience to go upon); but their value is constantly increasing, whereas the value of the medical report is continually waning as the policy grows older; and, above all, the data possessed by the friendly society are partly* capable of definite numerical measurement. I conceive, therefore, that it would be possible to construct "analyzed" sickness and mortality tables of great value, showing the sickness and mortality experienced at each age by separate sets of members classified according to the character of their previous sickness experience. Thus, one might make a beginning by constructing two tables, showing the sickness (and mortality) experienced at each age from 25 onwards, by those who had suffered in passing from age 20 to 25—(1) less than half, and (2) more than double, the expected sickness. Then one would have two tables, showing the sickness (and mortality) experienced at each age from 30 onwards, by those who had suffered (1) less than half, and (2) more than double, the expected sickness during the preceding decennium; and so on to the end of life.

---

* The *amount* and the *distribution in time* of the member's past sickness is capable of definite measurement, though of course the *qualitative nature* of his sickness is not.

It might seem as if the suggestions here ventured upon were over-refinements, possessing only a theoretical interest; but there are reasons which induce me to believe that the investigation I have sketched the outlines of, is at any rate of more pressing importance than that which would be the analogue, in the science of sickness assurance, of those researches on the influence of selection on mortality which have so deservedly assumed prominence in the more honoured science of life assurance. The smallness of the membership of the majority of friendly societies, renders it of vital importance, I think, at a valuation, to ascertain whether there is an undue proportion of members likely to experience an excessive sickness or to die early. *Roughly*, the past sickness is already taken into account by valuers in estimating the liabilities of a society; but such an investigation as I have suggested would enable us to replace the present crude process by one of a more scientific nature.

From a theoretical point of view, also, the subject appears to me of immense interest. When the statistical researches had been carried to a state of ideal perfection, the s_x (sickness experienced in passing from age x to age $x+1$) and the p_x, q_x, of our calculations, would all have to be multiplied into a factor derived from the past sickness experience of the individual. Or, replacing the view of things which regards the changes in the elements of life contingency calculations as a series of annual jerks, by the more scientific view which regards those changes as constituting a continuous process, we may say that the "proportion of sick" and the "force of mortality" at any instant of age, as derived from tables based on experience *en masse*, must each be multiplied by a factor derived from the past sickness experience of the individual. The general mathematical form of each factor is, I think, rigorously determinable on *à priori* grounds. It must, I think, be an exponential function of the duration of the attack, and also some function of the time that has elapsed since the attack (the value of the function probably getting nearer to unity, *i.e.* becoming less operative in the calculation, as the attack recedes into the past). Where there has been more than one attack, the factor will be a product of several separate exponential expressions, one corresponding to each attack.

I have the honour to be, Sir,
Your most obedient servant,

FREDERICK W. FRANKLAND,
Govt. Actuary for Friendly Societies.

Wellington, New Zealand,
21 *August* 1880.

INDEX TO VOL. XXII.

A.

Actuarial Notes, *see* King, G., and Wilding, R.

Annuity on the last *r* Survivors of *m* Lives $\left(a^{r}_{\overline{wxyz\ldots(m)}}\right)$. G. King on a General Expression for its Value, 293.

Arithmometer. D. Carment on its Application to the Construction of Tables of the Values of Endowment Assurance Policies, 368.

B.

Baden (A.), Remarks on the Estimate of the Liability of a Life Office under its Policies, 43.

Bailey (A. H.), Remarks on the Estimate of the Liability of a Life Office under its Policies, 43.

——— His Opening Address as President, 381.

Bonuses—how Earned and how Distributed, A. Hewat, 286.

Boole's Finite Differences, R. Wilding on the Factorial Notation in, 60.

C.

Carment (D.), On the Application of the Arithmometer to the Construction of Tables of the Values of Endowment Assurance Policies, 368.

Construction of Life Tables. Major-Gen. J. C. Hannyngton on Mr. Gray's Methods, 136.

——— Tables of Endowment Assurance Policy-Values, D. Carment on the Application of the Arithmometer to the, 368.

D.

Death Rates among Infants in Scotland, W. Robertson on, 140.

——— of Males and Females, Dr. J. Stark's Remarks on the Influence of Marriage on the, 233.

Declined Lives, Register of, 7.

De Forest (E. L.), On the Formula for the Force of Mortality, 231.

E.

Employees of Life Offices, Scheme for the Encouragement of Life Insurance among the, 365.

Encouragement of Life Insurance among the Employees of Life Offices, Scheme for the, 365.

Endowment Assurance Policies. D. Carment on the Application of the Arithmometer to the Construction of Tables of their Values, 368.

——— Premiums. Table by H. A. Smith of Annual and Single Premiums for Assurances payable at Ages 20 to 75 for Ages at Entry 15 to 60, according to Institute H^{M} 3 per-cent, 143.

F.

Factorial Notation of Boole's Finite Differences, R. Wilding on the, 60.

Factor Tables. J. W. L. Glaisher on the Calculation of Logarithms to a large number of places by their means, 346.

Fire Insurance. A Theory of Statistics, by T. Miller, 103.

Force of Mortality, Formula for the, E. L. De Forest, 231.

FOREIGN INTELLIGENCE:—
  Frankland (F. W.). Suggestion as to Statistics of Individual Sickness, 454.
  Friendly Societies in New Zealand.—Extracted from Registrar's 3rd Annual Report, 419.

G.

Glaisher (J. W. L.), Note on the Calculation of Logarithms to a large number of places by means of Factor Tables, 346.
Graduation of Mortality Tables. Notes on the Practical Application of Mr. Makeham's Formula. G. King and G. F. Hardy, 191.
——— Observations by J. Sorley, with special reference to the Conditions under which certain Methods are to be preferred, 309.
——— with reference to the Weight of the Observations, Professor C. F. McCay on the, 24.
Gray (P.). His Methods of Constructing Life Tables, Major-Gen. J. C. Hannyngton on, 136.
——— Note on Mr. Glaisher's Calculation of Logarithms to a large number of places by means of Factor Tables, 350.

H.

Hannyngton (Major-Gen. J. C.). On Mr. Gray's Methods of Constructing Life Tables, 136.
Hardy (G. F.). On the Practical Application of Mr. Makeham's Formula to the Graduation of Mortality Tables, 191.
Hewat (A.). Bonuses—how Earned and how Distributed, 286.
Higham (J. A.). Remarks on the Estimate of the Liability of a Life Office under its Policies, 43.

I.

Infant Mortality in Scotland, W. Robertson on, 140.
Institute of Actuaries. A Suggestion by C. Walford, towards adding a new Feature of Usefulness to the, 1.

INSTITUTE OF ACTUARIES—
  Preliminary Examination Paper, 1879, 63; 1880, 298.
  Intermediate ditto, 1879, 65; 1880, 300.
  Final ditto, 1880, 302.

INSTITUTE OF ACTUARIES *(contd.)*—
  Proceedings of the Institute, Session 1878–9, 67; 1879–80, 304.
  Annual Report, 1879, 69; 1880, 305.
  Income and Expenditure for the Year ending 31 March 1879, and Balance Sheet 31 March 1879, 72.
  Ditto for the Year ending 31 March 1880, and Balance Sheet 31 March 1880, 308.
  List of all Members who have passed the Examinations of the Institute, 73.
  Opening Address by the President, A. H. Bailey, 381.

Interpolation. Explanation by T. B. Sprague of a New Formula, 270.
Issue of a Marriage hitherto Childless, T. B. Sprague on the Probability that there will hereafter be, 117.

K.

King (G.). Actuarial Notes—
  I. On a general expression for the Value of an Annuity on the last *r* Survivors of *m* Lives $\left(a^{\,r}_{\overline{wxyz\ldots(m)}}\right)$, 293.
  II. On Reversionary Annuities, 296.
——— On the Practical Application of Mr. Makeham's Formula to the Graduation of Mortality Tables, 191.

L.

Lapses in the Eighth Year, T. B. Macaulay on, 134.
Law of Mortality. Can it be represented in a Mathematical Form? T. E. Young, 139.
Law Report. Judgment of the Master of the Rolls in the case of The London Assurance *v.* Mansel, 20.
Liability of a Life Office under its Policies. How should it be estimated? Being Remarks by J. A. Higham, A. H. Bailey, A. Baden, and T. B. Sprague, 43.
Logarithms. J. W. L. Glaisher on the calculation of Logarithms to a large number of places by means of Factor Tables, 346.
Lubbock's Formula. T. B. Sprague on the Application of Lubbock's Formula of Summation to find the Value of Annuities and Assurances, 55.

M.

Macaulay (T. B.). On Retirants in the Eighth Year, 134.
Macfarlane (A.). On the Tradition Question in Probabilities, 356.
Makeham's Formula. G. King and G. F. Hardy on its Practical Application to the Graduation of Mortality Tables, 191.
Market Value of an Annuity, R. Wilding on the, 61.
Marriage among Widowers, T. B. Sprague on the Rate of, 77.
——— among Widows and Spinsters, T. B. Sprague on the Rates of, 352.
——— entered into by a Man above Forty. T. B. Sprague on the Probability that it will be fruitful, 359.
——— hitherto Childless, T. B. Sprague on the Probability that there will hereafter be Issue of, 117.
——— its Influence on the Death Rates of Males and Females. Abridged from Dr. J. Stark's Letters to the Registrar-General of Births, Deaths, and Marriages in Scotland, 233.
McCay (Prof. C. F.) On the Adjustment of Mortality Tables with reference to the Weight of the Observations, 24.
Medical Examiners (Local). C. Walford on the Position of Life and other Insurance Associations in relation to them. Is it satisfactory? 153.
Miller (T.). Fire Insurance, a Theory of Statistics, 103.

N.

New Zealand Friendly Societies, 449.

O.

Original Tables. D. Carment. A Table of the Values of Endowment Assurance Policies payable at ages 50, 55, 60, and 65, for ages at entry 20 to 40, and duration of policy 1 to 44 years; Institute H^M 3 per-cent, 372.
——— H. A. Smith. Annual and Single Premiums for Endowment Assurances, payable at ages 20 to 75, for ages at entry 15 to 60; Institute H^M 3 per-cent, 143.

P.

Policy in the year 1721, Form of Life, 248.
Policy-Value and the Rate of Interest. W. G. Walton on the relation between them, 135.
Policy-Value. R. Teece on the Reserve that should be made for Policies on Recently Assured Lives, 250.
——— What is a Policy legally worth to the Insured? Reprinted from the *New York Insurance Monitor*. 132.
Policy-Values in Terms of Premiums, R. Wilding on Formulas for, 131.
Probabilities. A. Macfarlane on the Tradition Question, 356.

R.

Reversionary Annuities. G. King, 293.
Robertson (W.). On Death Rates among Infants in Scotland, 140.

S.

Select Mortality Tables to be employed in combination with the Institute $H^{M(5)}$ Table, T. B. Sprague on the Construction and Use of a Series of, 391.
Sickness. Suggestion by F. W. Frankland as to Statistics of Individual, 454.
Smart's Interest Tables, R. Wilding on, 129.
Smith (H. A.) A Table of Annual and Single Premiums for Endowment Assurances, payable at ages 20 to 75 for ages at entry 15 to 60, according to Institute H^M 3 per-cent, 143.
Sorley (J.). Observations on the Graduation of Mortality Tables; with special reference to the Conditions under which certain Methods are to be preferred, 309.
Sprague (T. B.). Remarks on the Estimate of the Liability of a Life Office under its Policies, 43.
——— On the Application of Lubbock's Formula of Summation to find the Value of Annuities and Assurances, 55.
——— On the Rate of Remarriage among Widowers, 77.
——— On the Probability that there will hereafter be Issue of a Marriage hitherto Childless, 117.
——— Explanation of a New Formula for Interpolation, 270.
——— Rates of Marriage among Widows and Spinsters, 352.
——— Note on the Probability that a Marriage entered into by a Man above Forty will be fruitful, 359.
——— On the Construction and Use of a Series of Select Mortality Tables, to be employed in combination with the Institute of Actuaries $H^{M(5)}$ Table. Part II—Construction (*continued*); Part III—Use. 391.

Stark (Dr. J.). Extracts from his Letters to the Registrar-General of Births, Deaths, and Marriages in Scotland, on the Influence of Marriage on the Death Rates of Males and Females, 233.

Summation. T. B. Sprague on the Application of Lubbock's Formula to find the Value of Annuities and Assurances, 55.

T.

Teece (R.). On the Reserve that should be made for Policies on Recently Assured Lives, 250.

Tradition Question in Probabilities. A. Macfarlane, 356.

W.

Walford (C.). A Suggestion towards adding a new Feature of Usefulness to the Institute of Actuaries, 1.

——— On the Position of Life and other Insurance Associations in relation to their Local Medical Examiners. Is it satisfactory? 153.

Walton (W. G.). Relation between the Value of a Policy and the Rate of Interest, 135.

Widowers, T. B. Sprague on the Rate of Remarriage among, 77.

Widows and Spinsters, Rates of Marriage among, T. B. Sprague on the, 352.

Wilding (R.). Actuarial Notes—
  I. On the Factorial Notation in Boole's Finite Differences, 60.
  II. On the Market Value of an Annuity, 61.
  III. Smart's Interest Tables, 129.
  IV. Formulas for Policy-Values in Terms of Premiums, 131.

Y.

Young (T. E.). Can the Law of Mortality be represented in a Mathematical Form? 139.

END OF VOL. XXII.

C. & E. Layton, Printers, 150, Fleet Street; and 12, Gough Square.

acB

THE NEW YORK PUBLIC LIBRARY
REFERENCE DEPARTMENT

This book is under no circumstances to be taken from the Building

NOV 24 1926		

form 410